Copower functors

H. Peter Gumm

Philipps-Universität Marburg

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Functor properties

Relevant properties

- standard
- separating
- connected
- bounded
  - finitary

Preservation properties

- weak pullbacks
  - preimages
    - weak kernel pairs
  - wide pullbacks
    - intersections
    - finite ones can be assumed

General program

Functor properties $\longleftrightarrow$ Coalgebraic structure theory

Functors preserve ...

- $\mathbb{P}$
  - weak (wide) pullbacks
  - intersections
  - *not* bounded
- $\mathbb{F}$
  - weak pullbacks
  - finite intersections
- $X^2 - X + 1$
  - intersections
  - *not* preimages
  - *not* kernel pauirs
- $(-)^3_2$ and $\mathbb{P}_k$, $k > 2$
  - preimages
  - *not* kernel pairs
- $\mathbb{Z}[-]$ (bags with credit)
  - kernel pairs
  - *not* preimages

Fuzzy sets and bags

- Purpose
  - Provide parametrized class of functors
  - tune parameters for desired properties
  - start with standard examples
    - $\mathbb{P}(-) = 2^{-}$, subfunctor: $\mathbb{P}_\omega(-) = 2^{-}_{\omega}$
- 2 is . . .
  - . . . a complete semilattice $\mathcal{L}$
  - . . . a commutative monoid $\mathcal{M}$
- Generalizing yields two types of functors
  - $\mathcal{L}^X := \{\sigma : X \to \mathcal{L}\}$
  - $\mathcal{M}^X_\omega := \{\sigma : X \to \mathcal{M} \mid \sigma(x) = 0_{a.e.}\}$

$\mathcal{L}$- fuzzy sets

$\mathcal{L}$ a complete $\bigvee$-semilattice, define $\mathcal{L}^X := \{\sigma : X \to \mathcal{L}\}$

- For $f : X \to Y$
  - $\mathcal{L}^f(\sigma) = \lambda y. \bigvee\{\sigma(x) \mid f(x) = y\}$
- $\mathcal{L}^{(-)}$ is a *Set*-functor
- $\mathcal{L}$-coalgebras are $\mathcal{L}$- valued relations
- $\mathcal{L}$ preserves
  - preimages
  - intersections
- $\mathcal{L}$ weakly preserves kernel pairs $\iff$ $x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x \wedge y_i)$

$\mathcal{M}$-bags

$\mathcal{M}$ commutative monoid, $\mathcal{M}_\omega^X := \{\sigma : X \to \mathcal{M} \mid \sigma(x) = 0_{a.e.}\}$

- For $f : X \to Y$
  - $\mathcal{M}_\omega^f(\sigma) = \lambda y. \sum\{\sigma(y) \mid f(y) = x\}$
- *finite* bags, multiplicities from $\mathcal{M}$
  - $\mathbb{N}$: standard bags
  - $\mathbb{Z}$ bags “with credit”
- $\mathcal{M}$-coalgebras: $\mathcal{M}$-valued relations

x	y	a_1
z	u	a_2
b_1	b_2	$=$

Theorem (HPG, T.Schröder)

- $\mathcal{M}_\omega^f$ *preserves preimages* $\iff$ $\mathcal{M}$ *is positive.*
- $\mathcal{M}_\omega^f$ *weakly pres. kernel pairs* $\iff$ $\mathcal{M}$ *is refinable.*

Common generalization

- $\mathcal{M}$ commutative monoid
  - image finiteness essential,
  - commutativity, unit element
- $\mathcal{L}$ complete semilattice
  - idempotency essential
  - zero element

$$\mathcal{M}^f_\omega(\sigma)(y) := \sum_{f(x)=y} \sigma(x)$$

$$\mathcal{L}^f(\sigma)(y) := \bigvee_{f(x)=y} \sigma(x)$$

Observe

$$\mathcal{M}^X_\omega \cong \coprod_{x\in X} \mathcal{M}$$

$$\mathcal{L}^X \cong \prod_{x\in X} \mathcal{L} \cong \coprod^{\mathfrak{S}}_{x\in X} \mathcal{L}$$

The copower functor

Given

- category $\mathfrak{C}$ and $\mathcal{A} \in \mathfrak{C}$
- copowers of $\mathcal{A}$ exist in $\mathfrak{C}$
- $U : \mathfrak{C} \to \mathcal{S}et$ any (forgetful) functor

$$\mathcal{A}_{\mathfrak{C}}[X] := U(\coprod_{x \in X}^{\mathfrak{C}} \mathcal{A})$$

For any map $f : X \to Y$ let

$$\begin{tikzcd}
\mathcal{A}_{\mathfrak{C}}[X] \arrow[rr, "\mathcal{A}_{\mathfrak{C}}[f]"] & & \mathcal{A}_{\mathfrak{C}}[Y] \\
& \mathcal{A} \arrow[ul, "e_x"] \arrow[ur, "e_{f(x)}"'] &
\end{tikzcd}$$

Theorem

$\mathcal{A}_{\mathfrak{C}}[-]$ is a $\mathcal{S}et$-endofunctor.

Free product functor

$\mathfrak{V}$ variety of algebras, $\mathcal{A} \in \mathfrak{V}$
$\mathcal{A}_{\mathfrak{V}}[X] \approx$ free product

- $\mathcal{M}_{\mathfrak{Mc}}[X] = \mathcal{M}_{\omega}^{X}$
- $\mathcal{L}_{\mathfrak{S}}[X] = \mathcal{L}^{X}$

Which properties of $\mathcal{A}$ and $\mathfrak{V}$ guarantee . . .

- . . . weak pullback preservation
  - . . . image preservation
  - . . . weak kernel preservation

$$\begin{tikzcd}
F_{\mathfrak{V}}(A \times X) \arrow[r, two heads, "\pi_\theta"'] & \color{red}{\mathcal{A}_{\mathfrak{V}}[X]} \arrow[dd, dashed] \\
A \times X \arrow[u, hook] & \\
\mathcal{A} \arrow[u, "e_x"'] \arrow[ruu, "\epsilon_x"'] \arrow[r, "g_x"'] & Q
\end{tikzcd}$$

Weak pullback preservation for $\mathcal{A}_{\mathfrak{V}}[-]$

- $\mathfrak{Sl}$: Complete semilattices $\iff$ $x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \wedge y_i)$
- $\mathfrak{Mc}$: Commutative monoids $\iff$ positive and refinable
- $\mathfrak{M}$: Monoids $\iff$ positive and *equidivisible*
- $\mathfrak{Sg}$: Semigroups $\iff$ *equidivisible*.

Equidivisibility: Given $a \cdot b = c \cdot d$, there exists k such that

$$\underbrace{\overbrace{a \cdot k}^{c} \cdot d}_{b} \quad \text{or} \quad \underbrace{\overbrace{c \cdot k}^{a} \cdot b}_{d}$$

Product refinement

Refinable

$$\mathcal{A}\times\mathcal{B}\cong\mathcal{C}\times\mathcal{D} \iff \begin{array}{ccc|c} \mathcal{U}_0 & \times & \mathcal{U}_1 & \mathcal{A} \\ \times & & \times & \times \\ \mathcal{V}_0 & \times & \mathcal{V}_1 & \mathcal{B} \\ \hline \mathcal{C} & \times & \mathcal{D} & \cong \end{array}$$

Product refinement

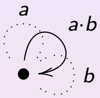
Equidivisible

$$\mathcal{A} \times \mathcal{B} \cong \mathcal{C} \times \mathcal{D} \iff \begin{array}{ccc|c} \mathcal{A} & & & \mathcal{A} \\ \times & & & \times \\ \mathcal{K} & \times & \mathcal{D} & \mathcal{B} \\ \hline \mathcal{C} & \times & \mathcal{D} & \cong \end{array} \quad \text{or} \quad \begin{array}{ccc|c} \mathcal{C} & \times & \mathcal{K} & \mathcal{A} \\ & & \times & \times \\ & & \mathcal{B} & \mathcal{B} \\ \hline \mathcal{C} & \times & \mathcal{D} & \cong \end{array}$$

Theorem

1. *Equidivisible* **semigroups** *are refinable.*
2. *Any two product decompositions have a common refinement*

A category with one object

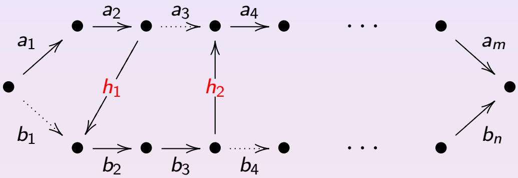


- Semigroup $\mathcal{S}$: one-object-category
    - Elements of $\mathcal{S}$ are morphisms
    - Composition is multiplication
- *Equidivisibility* is categorically:
    - diagonal property

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "c"'] & \bullet \arrow[d, "b"] \\
\bullet \arrow[r, "d"'] \arrow[ur, dashed, "k"'] & \bullet
\end{tikzcd}$$

Refinement

Given $a_1 \cdot a_2 \cdot \ldots \cdot a_m = p = b_1 \cdot b_2 \cdot \ldots \cdot b_n$,



Theorem

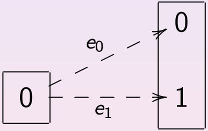
Any two product decompositions have a common refinement.

$$p = \underbrace{a_1 \cdot a_2 \cdot \rlap{$\overbrace{}^{a_3}$} h_1}_{b_1} \cdot b_2 \cdot b_3 \cdot \underbrace{h_2 \cdot a_4 \cdot \ldots}_{b_4} \cdot \ldots$$

Copower functors are almost universal

What is special about copower functors ?

F faithful:
$$\begin{tikzcd} Y^X \arrow[r, tail, "F"'] & FY^{FX} \end{tikzcd}$$



F faithful

$\Longleftrightarrow$ $Fe_0 \neq Fe_1$

Theorem

Every faithful $\mathcal{S}$et-functor F has a representation $F(-) \cong \mathcal{A}_{\mathfrak{C}}[-]$ with $\mathcal{A} \in \mathfrak{C}$ for some (non-full) subcategory $\mathfrak{C}$ of Set.

Coalgebraic logic

Formulae

$$\begin{array}{lll} \phi & :: & \textbf{true} \mid \neg\phi \mid \bigwedge_{i\in I} \phi_i \\ & \mid & \ldots\ \textit{modalities}\ \ldots \end{array}$$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "\llbracket\phi\rrbracket"'] & 2 \end{tikzcd}$$

- semantics: $\llbracket\phi\rrbracket : A \to 2$
- $x \models \phi \;:\Longleftrightarrow\; \llbracket\phi\rrbracket(x) = 1$
- $x \approx y \;:\Longleftrightarrow\; \forall\phi. x \models \phi \iff y \models \phi$
- f definable $:\Longleftrightarrow\; \exists\phi. f = \llbracket\phi\rrbracket$
- … U-definable $\Longleftrightarrow\; \exists\phi. f_{|U} = \llbracket\phi\rrbracket_{|U}$

Coalgebraic logic

Formulae

$$\begin{array}{lcl} \phi & :: & \mathbf{true} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \\ & \mid & \ldots\ \textit{modalities}\ \ldots \end{array}$$

- semantics: $[\![\phi]\!] : A \to 2$
- $x \models \phi \;:\Longleftrightarrow\; [\![\phi]\!](x) = 1$
- $x \approx y \;:\Longleftrightarrow\; \forall\phi.\, x \models \phi \iff y \models \phi$
- f definable $:\Longleftrightarrow\; \exists\phi.f = [\![\phi]\!]$
- … U-definable $\iff \exists\phi.f_{|U} = [\![\phi]\!]_{|U}$

$$\begin{tikzcd} U \arrow[r, hook] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "{[\![\phi]\!]}"'] & 2 \end{tikzcd}$$

Fact ($\bigwedge$ vs. $\wedge$)

1. $\bigwedge$:: *f is definable* $\iff$ *f respects $\approx$.*
2. $\wedge$:: *f is U-definable for each $U \subseteq_{fin} A$* $\iff$ *f respects $\approx$.*

Pattinson-Schröder Logic

- Formulae

$$\begin{aligned} \phi \quad :: \quad & \mathbf{true} \mid \neg\phi \mid \bigwedge_{i\in I} \phi_i \\ \mid \quad & {\color{red}[w]\phi} \ \text{ for each } w \in F(2) \end{aligned}$$

- Semantics
  - $x \models [w]\phi \ :\Longleftrightarrow \ F[\phi](\alpha(x)) = w$

$$\begin{tikzcd} {} \arrow[dr, dotted, "x"] & & \\ & A \arrow[r, "[\![\phi]\!]"] \arrow[d, "\alpha"'] & 2 \\ & FA \arrow[r, "F\ [\![\phi]\!]"] & F2 \\ & & {} \arrow[u, dotted, "w"'] \end{tikzcd}$$

Stability: $\nabla \subseteq \approx$

- $\models$ is homomorphism stable

$$\varphi : \mathcal{A} \to \mathcal{B} \implies (\forall x \in A . x \models \phi \iff \varphi(x) \models \phi)$$

- Proof by formula induction

$$\begin{tikzcd}
{} \arrow[dr, dotted, "x"] & & & & {} \arrow[dl, dotted, "\varphi(x)"'] \\
& A \arrow[rr, "\varphi"] \arrow[d, "\alpha"] & & B \arrow[d, "\beta"] & \\
& FA \arrow[rr, "F\varphi"] \arrow[dr, "F\ [\![\phi]\!]^A"'] & & FB \arrow[dl, "F\ [\![\phi]\!]^B"] & \\
& & F2 & & \\
& & {} \arrow[u, dotted, "w"] & &
\end{tikzcd}$$

Completeness: $\approx \subseteq \nabla$

- Assume: F separating
- define coalgebra on $A/\approx$ so that $\pi_\approx$ is a homomorphism
- $x \models [w_x]\phi_f$ but $y \models [w_y]\phi_f$

$$\begin{tikzcd}
{} \arrow[dr, dotted, "x"] \arrow[dr, dotted, "y"'] & & & \\
& A \arrow[r, two heads, "\pi_\approx"] \arrow[rr, bend left, "\llbracket \phi_f \rrbracket"] \arrow[d, "\alpha"] & A/\approx \arrow[r, "f"] \arrow[d, dashed] & 2 \\
& F(A) \arrow[r, "F\pi_\approx"] \arrow[rr, bend right, "F \llbracket \phi_f \rrbracket"'] & F(A/\approx) \arrow[r, "Ff"] & F(2) \\
& & & {} \arrow[u, dotted, "w_x"] \arrow[u, dotted, "w_y"']
\end{tikzcd}$$

Finitary conjunctions/disjunctions

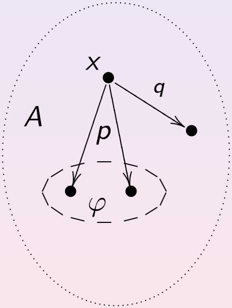
- If F is *finitary*, then finite conjunctions suffice:
  - $\exists U \subseteq_{fin} X$ with $\alpha(x), \beta(x) \in F(U)$
  - $f \circ \pi_{\approx}$ definable relative to U
    - $x \models [w_x]\phi_f$ and $y \models [w_y]\phi_f$

$$\begin{tikzcd}
 & {} \arrow[d, dotted, shift right, "x"'] \arrow[d, dotted, shift left, "y"] & & \\
U \arrow[r, hook, "\iota_U"] & A \arrow[r, two heads, "\pi_\approx"] \arrow[rr, bend left, "\llbracket \phi_f \rrbracket"] \arrow[d, "\alpha"] \arrow[dl, dotted, shift right] \arrow[dl, dotted, shift left] & A/\approx \arrow[r, "f"] \arrow[d, dashed] & 2 \\
F(U) \arrow[r, "F\iota_U"] & F(A) \arrow[r, two heads, "F\pi_\approx"] \arrow[rr, bend right, "F \llbracket \phi_f \rrbracket"'] & F(A/\approx) \arrow[r, "Ff"] & F(2) \\
 & & & {} \arrow[u, dotted, shift left, "w_x"] \arrow[u, dotted, shift right, "w_y"']
\end{tikzcd}$$

Modal logic for Copower functors

- $\mathcal{M}$ a commutative monoid
- $\mathcal{M}[X]$ = X-bags, multiplicities from $\mathcal{M}$
  - $\mathcal{M}[X]$ separates points $\checkmark$
  - $\mathcal{M}[X]$ finitary $\checkmark$

$$\begin{array}{lcl} \phi & :: & \mathit{true} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \\ & \mid & [p,q]\phi, \quad \text{where } p, q \in \mathcal{M} \end{array}$$



- $x \models [p,q]\phi \iff$
  - $p = \sum\{m \mid x \xrightarrow{m} y \models \phi\}$
  - $q = \sum\{m \mid x \xrightarrow{m} y \models \neg\phi\}$

Separating Functors

- F arbitrary functor, $\kappa \in \mathit{Card}$
  - … before we started with X^κ
  - … approximated F by F_κ 's
- now start with κ^X …
  - represent $F(X)$ by all κ-patterns
  - F separating $\iff$ injective

$$\begin{tikzcd}
F\kappa \times X^\kappa \arrow[rr, "\eta_X"'] \arrow[dr] & & FX \\
& F_\kappa \arrow[ur, hook] &
\end{tikzcd}$$

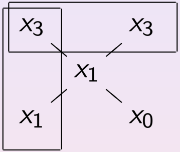
$$\begin{tikzcd}
FX \arrow[r, tail] & F\kappa^{\kappa^X}
\end{tikzcd}$$

Fact

F is κ-separating $\iff$ F is a subfunctor of some A^{κ^X}

Intuition useful for Functors

- Functors are **generalized containers**
  - $F(1)$: shapes
  - $F(2)$: $0-1$-patterns
  - $F(3)$: $\ldots$
  - $F(X)$: $\ldots$
- Shapes are “independent”
  - $F = \sum_{i \in I} F_i$
    - where $|F_i(1)| = 1$.

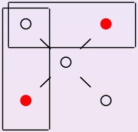


Which functors are determined by their $\kappa-$patterns ?

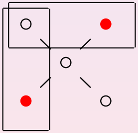
Intuition for $A, a \models [p]\phi$

- Let $w \in F(2)$
- Assume $\alpha(a) = u \in F(A)$
    - replace all places x in u by :
    - $[\![\phi]\!](x) = \begin{cases} ● & \text{if } x \models \phi \\ \circ & \text{if } x \not\models \phi \end{cases}$
- $A, a \models [w]\phi :\Longleftrightarrow F\ [\![\phi]\!]\alpha(a) = w$

$w =$



$a \xmapsto{\alpha}$



Recovering elements

- Define “support” of $c \in F(X)$
    - $\|c\| := \bigcap\{U \subseteq X \mid c \in F(U)\}$
    - ok, if F preserves intersection
        - e.g. for finite containers
- Complications
    - $U \subseteq X \not\Rightarrow F(U) \subseteq F(X)$
    - possibly $c \notin F(\|c\|)$
- Always works:
    - $\mu(c) = \{U \mid c \in F\iota[F(U)]\}$
      where $\iota : U \subseteq X$
    - $\mu(c)$ is a filter on X

$$\begin{tikzcd}
 & F(X) & X \\
 & F(U) \arrow[u] & U \arrow[u, hook, "\iota"'] \\
c \arrow[ruu, dotted, "\in"] \arrow[ru, dotted, "?"'] & \vdots &
\end{tikzcd}$$

Membership

Transformation $\mu : F \overset{\cdot}{\longrightarrow} \mathbb{F}$

- not necessary natural
- but subcartesian
    - largest subcartesian transformation

Theorem

μ is natural $\iff$ F preserves preimages

$$\begin{tikzcd}
F(X) \arrow[r, "\mu_X"] & \mathbb{F}(X) \\
F(U) \arrow[u] \arrow[r, "\mu_U"'] & \mathbb{F}(U) \arrow[u]
\end{tikzcd}$$

Conclusion

1. Copower functor useful for custom made examples
    - Generalize powerset functor and finite-bag-functor
    - Two parameters to play with
        - algebra $\mathcal{A}$, category $\mathfrak{C}$
2. Natural logics, easy to describe
    - $a \models [p, q]\phi$
3. Functors, in general, are generalized containers
    - Shapes $= F(1)$, independent
    - 0-1-patterns $= F(2) =$ modalities
    - *Element* filter: $\mu : F(X) \rightarrow \mathbb{F}(X)$
    - preserve preimages iff cannot lose elements.

Thanks

$$T^{h}@_{n}x$$