Coalgebra and Modal Logic

Notes from a Research Program

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my sincere thanks to Alexander Kurz

My Goals

I looked at the other titles/abstracts while preparing this talk, and also the TANCL program.
My goals are to present

- ⋆ a big picture on the whole subject and beyond.
- ⋆ a somewhat-detailed look at a problem area involving interactions with measure theory and probability.
- ⋆ another somewhat-detailed look at an application area: revisiting modal weak completeness theorems.

Problems, problems

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It's not going to last, anyways.

The big picture

algebra	coalgebra
initial algebra	final coalgebra
least fixed point	greatest fixed point
congruence relation	bisimulation equivalence relation
Foundation Axiom	Anti-Foundation Axiom
iterative conception	coiterative conception
equational logic	modal logic
recursion: map out of an initial algebra	corecursion: map into a final coalgebra
useful in syntax	useful in semantics
construct	observe
bottom-up	top-down

On the set theory connection

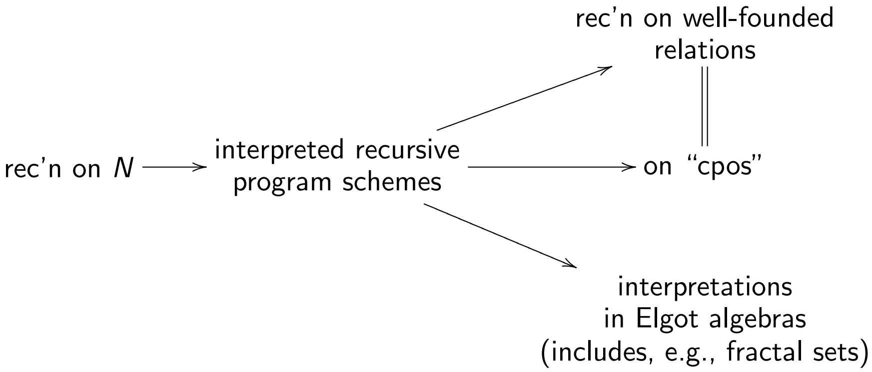
Foundation Axiom	Anti-Foundation Axiom
iterative conception	coiterative conception

Theorem (Turi; Turi and Rutten; implicit in Aczel)

The Foundation Axiom is equivalent to the assertion that the universe V together with $id : \mathcal{P}V \to V$ is an initial algebra of $\mathcal{P}$ on the category of classes.
The Anti-Foundation Axiom is equivalent to the assertion that the universe V together with $id : V \to \mathcal{P}V$ is a final coalgebra of $\mathcal{P}$ on the category of classes.

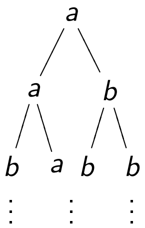
On coalgebraic treatments of recursion

recursion: map out of an initial algebra	corecursion: map into a final coalgebra



Where did coalgebraic logic come from?

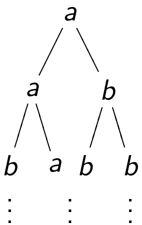
Let's consider the functor on sets $F(w) = \{a, b\} \times w \times w$.
The final coalgebra F^* consists of infinite binary trees such as



A (finitary) logic to probe coalgebras of F

$$\varphi \in L : \quad a \mid b \mid \text{left} : \varphi \mid \text{right} : \varphi \mid$$

An example



Here are some formulas satisfied by our tree:

a

left : a

right : left : b

It's easy in this case to see that the trees correspond to certain theories (sets of formulas) in this logic.

It is not so easy to connect the logic back to the functor $F(w) = \{a, b\} \times w \times w$.

Another try

We are dealing with $F(w) = \{a, b\} \times w \times w$.
Let's try the least fixed point of F

$$L = \{a, b\} \times L \times L.$$

Another try

We are dealing with $F(w) = \{a, b\} \times w \times w$.
Ok, it's empty.
Let's try the least fixed point plus a trivial sentence to start:

$$L = (\{a, b\} \times L \times L) + \{true\}.$$

Or, we could add a conjunction operation, with $\bigwedge \emptyset = \text{true}$.
Either way, we get formulas like

$$\langle b, \langle a, true, true \rangle, \langle a, true, true \rangle \rangle$$
$$\langle a, true, \langle b, \langle a, true, true \rangle, \langle a, true, true \rangle \rangle \rangle$$

Semantics

We want to define $t \models \varphi$ for t a tree and $\varphi \in L$.
Note that $\models \subseteq F^* \times L$.
We treat this as an object, applying F to it.
In fact, we also have

$$\begin{array}{ll} \pi_1 : \models \to F^* & \pi_2 : \models \to L \\ F\pi_1 : F(\models) \to F^* & F\pi_2 : F(\models) \to F(L) \hookrightarrow L \end{array}$$

$$\begin{array}{lll} t \models \langle a, \varphi, \psi \rangle & \text{iff} & (\exists u, v) t = \langle a, u, v \rangle \& (\langle u, \varphi \rangle \in \models) \& (\langle v, \psi \rangle \in \models) \\ & \text{iff} & (\exists x \in F(\models) \qquad\qquad x \text{ is } \langle a, \varphi, \psi \rangle \\ & & F\pi_1(x) = t, \\ & & \text{and } F\pi_2(x) = \langle a, \varphi, \psi \rangle \end{array}$$

What are we trying to do?

$$\frac{\text{Modal logic}}{???} \quad = \quad \frac{\text{the functor } K(a) = \mathcal{P}(a) \times \mathcal{P}(\text{AtProp})}{\text{an arbitrary (?) functor } F}$$

The logic ??? should be interpreted on all coalgebras of F.
It should characterize points in (roughly) the sense that

> points in a coalgebra have the same L theory
> iff they are bisimilar
> iff they are mapped to the same point in the final coalgebra

What has been done?

The first paper constructed logics L_F from functors F and gives semantics so that

$$\frac{\text{the } \nabla \text{ fragment}}{L_F} = \frac{\text{the functor } K}{\text{a functor } F \text{ meeting some conditions}}$$

But L_F often has an unfamiliar syntax, and in general one needs an infinitary boolean operations.
There's no logical system around.
(In fact, it was only this year that Palmigiano and Venema axiomatized the ∇ fragment. of standard modal logic.)

What has been done?

A more influential line of work constructs logics L_F so that

$$\frac{\text{standard modal logic}}{L_F} = \frac{\text{the functor } K}{\text{a functor } F \text{ which is polynomial in } \mathcal{P}_{fin}}$$

Here we have nicer syntaxes, and complete logical systems.
The class of functors is smaller, but it contains everything of interest.
The logics are not constructed just from the functors.
This is the result of many people's work, including Rößiger, Kurz, Pattinson, Jacobs, and others.

Beyond the known

Suppose we liked the Kripke semantics and then asked *where did modal logic come from?*
This line of work would suggest an answer; compare with van Benthem's Theorem.
In addition, it would give many other logical languages and systems with similar features.

Points in the final coalgebra of F "are" the L_F theories of all points in all coalgebras.
So if we have some independent reason to consider L_F, we can use it to study the final coalgebra, *or to get our hands on it in the first place*.

One such case concerned *universal Harsanyi type spaces*, a semantic modeling space originating in game theory.

The category *Meas*

A measurable space is a pair $M = (M, \Sigma)$, where M is a set and Σ is a σ-algebra of subsets of M.
Usually Σ contains all singletons $\{x\}$, but this is not needed here.
A morphism of measurable spaces $f : (M, \Sigma) \to (N, \Sigma')$ is a function $f : M \to N$ such that for each $A \in \Sigma'$, $f^{-1}(A) \in \Sigma$.
This gives a category which is often called *Meas*.
Meas has products and coproducts.

The functor Δ on *Meas*

A probability measure on M is a σ-additive function $\mu : \Sigma \to [0,1]$ such that $\mu(\emptyset) = 0$, and $\mu(M) = 1$.
There is an endofunctor $\Delta : \mathit{Meas} \to \mathit{Meas}$ defined by:

$$\Delta(M) \text{ is the set of probability measures on } M$$

endowed with the σ-algebra generated by
$\{B^p(E) \mid p \in [0,1], E \in \Sigma\}$, where

$$B^p(E) \quad = \quad \{\mu \in \Delta(M) \mid \mu(E) \geq p\}.$$

Here is how Δ acts on morphisms.
If $f : M \to N$ is measurable, then for $\mu \in \Delta(M)$ and $A \in \Sigma'$, $(\Delta f)(\mu)(A) = \mu(f^{-1}(A))$. That is, $(\Delta f)(\mu) = \mu \circ f^{-1}$.

A connection

For each $p \in [0,1]$, B^p may be regarded as a predicate lifting.
B^p takes measurable subsets of each space M to measurable subsets of ΔM.
It is natural in the sense that if $f : M \to N$, then the diagram below commutes:

$$\begin{tikzcd}
\mathcal{P}_{meas}(N) \arrow[r, "B^p_N"] \arrow[d, "f^{-1}"'] & \mathcal{P}_{meas}(\Delta N) \arrow[d, "(\Delta f)^{-1}"] \\
\mathcal{P}_{meas}(M) \arrow[r, "B^p_M"'] & \mathcal{P}_{meas}(\Delta M)
\end{tikzcd}$$

Universal Harsanyi type spaces

I am not going to say what Harsanyi type spaces are.
They are "multi-player" versions of coalgebras of

$$F(M) \quad = \quad \Delta(M \times S),$$

where S is a fixed space.
The universal space "is" a final coalgebra.

Prior work

Much of the prior work on this topic used the final sequence

$$1 \xleftarrow{\;!\;} F1 \xleftarrow{\;F!\;} FF1 \longleftarrow \cdots$$

But in this category, the functors involved usually don't preserve the colimits.

So the literature primarily considered subcategories of *Meas* where one had additional results (Kolmogorov's Theorem).

An alternative approach was initiated by Heifetz and Samet: see "Topology-free typology of beliefs" *Journal of Economic Theory*, 1998.

Their work essentially used coalgebraic modal logic(!)

So it was not so hard to believe that it would generalize.

The Measurable Polynomial Functors

The class of *measure polynomial functors* is the smallest class of functors on *Meas* containing the identity, the constant functor M for each measurable space M and closed under products, coproducts, and Δ.

Theorem (with Ignacio Viglizzo 2004)

Every MPF has a final coalgebra.

The point for this talk is that the proof used developments in coalgebraic modal logic and also was related to the point of this talk.
Especially important was the work of Rößiger (1999,2001) and Jacobs (2001).

Ingredients

For a measure polynomial functor T, we define a finite set $\mathrm{Ing}(T)$ of functors by the following recursion:
For the identity functor, $\mathrm{Ing}(Id) = \{Id\}$;
for a constant space M, $\mathrm{Ing}(M) = \{M, Id\}$,
$\mathrm{Ing}(U \times V) = \{U \times V\} \cup \mathrm{Ing}(U) \cup \mathrm{Ing}(V)$,
and similarly for $U + V$;
$\mathrm{Ing}(\Delta S) = \{\Delta S\} \cup \mathrm{Ing}(S)$.
We call $\mathrm{Ing}(T)$ the set of *ingredients of* T.
Each measure polynomial functor T has only finitely many ingredients.

Example

Let $[0, 1]$ be the unit interval of the reals, endowed with the usual Borel σ-algebra, and $T = [0, 1] \times (\Delta X + \Delta X)$. Then

$$\mathrm{Ing}(T) = \{Id, [0, 1], \Delta Id, \Delta Id + \Delta Id, [0, 1] \times (\Delta Id + \Delta Id)\}.$$

The syntax of L_T

$$\mathsf{true}_S : S \qquad \frac{A \subseteq M \text{ measurable or a singleton}}{A : M}$$

$$\frac{\varphi : S \quad \psi : S}{\varphi \wedge \psi : S} \qquad \frac{\varphi : U \quad \psi : V}{\langle \varphi, \psi \rangle_{U \times V} : U \times V}$$

$$\frac{\varphi : U}{\mathsf{inl}_{U+V}\varphi : U + V} \qquad \frac{\varphi : V}{\mathsf{inr}_{U+V}\varphi : U + V}$$

$$\frac{\varphi :: S, p \in [0,1]}{B^p\varphi : \Delta S} \qquad \frac{\varphi : T}{[\mathsf{next}]\varphi : Id}$$

The notation $\varphi :: S$ means that for every constant functor $M \in \mathsf{Ing}(T)$, every subformula of φ of sort M is a *measurable* set.

The semantics

Let $c : X \to TX$ be a coalgebra of T.
The semantics assigns to each $S \in \mathsf{Ing}(T)$ and each $\varphi : S$ a subset $[\![\varphi]\!]^c_S \subseteq SX$.

$$
\begin{array}{lcl}
[\![\mathsf{true}]\!]^c_S & = & SX \\
[\![A]\!]^c_M & = & A \\
[\![\varphi \wedge \psi]\!]^c_S & = & [\![\varphi]\!]^c_S \cap [\![\psi]\!]^c_S \\
[\![\langle \varphi, \psi \rangle]\!]^c_{U \times V} & = & [\![\varphi]\!]^c_U \times [\![\psi]\!]^c_V \\
[\![\mathsf{inl}\, \varphi]\!]^c_{U+V} & = & \mathsf{inl}([\![\varphi]\!]^c_U) \\
[\![\mathsf{inr}\, \varphi]\!]^c_{U+V} & = & \mathsf{inr}([\![\varphi]\!]^c_V) \\
[\![B^p \varphi]\!]^c_{\Delta S} & = & B^p([\![\varphi]\!]^c_S) \\
[\![[\mathsf{next}]\varphi]\!]^c_{Id} & = & c^{-1}([\![\varphi]\!]^c_T)
\end{array}
$$

Coalgebra morphisms preserve the semantics

That is, if $f : b \to c$ is a morphism of coalgebras $b : X \to TX$ and $c : Y \to TY$, and if $\varphi : S$, then

$$(Sf)^{-1}([\![\varphi]\!]^c_S) = [\![\varphi]\!]^b_S.$$

Theories occurring in nature

For each coalgebra $c : X \to TX$ and each $x \in SX$, we define

$$d^c_S(x) \quad = \quad \{\varphi : S \mid x \in [\![\varphi]\!]^c_S\}.$$

We call each such set $d^c_S(x)$ a *satisfied theory*.

The *canonical sets* S^* for $S \in \mathsf{Ing}(T)$

by $S^* = \{d^c_S(x) \mid x \in SX$ for some coalgebra $c : X \to TX\}$.

the sets $|\varphi|_S$

$|\varphi|_S = \{s \in S^* \mid \varphi \in s\}$.
$\varphi \in d^c_S(x)$ iff $d^c_S(x) \in |\varphi|_S$.

The *canonical spaces* S^* for $S \in \mathsf{Ing}(T)$

Each S^* is a measurable space, via the σ-algebra generated by the family of sets $|\varphi|_S$ for $\varphi :: S$.

The main work

There are maps as shown in blue below

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "d^c_{Id}"'] & TX \arrow[d, "d^c_T"] \arrow[dr, "Td^c_{Id}"] & \\ Id^* \arrow[r, "{[\mathrm{next}]^{-1}}"'] & T^* \arrow[r, "r_T"'] & T(Id^*) \end{tikzcd}$$

and then $Id^*, r_T \circ [\mathrm{next}]^{-1}$ is a final coalgebra of T.

I'm skipping all the hard stuff.

The Dynkin $\lambda - \pi$ Lemma is used, for example.

A PS to this part

My newly-finished Ph.D. student Chunlai Zhou has axiomatized the logic of Harsanyi types spaces.
His work is finitary and improves on earlier systems (Heifetz & Mongin, Meier).
His work makes essential use of linear programming.

Summary so far

We built final coalgebras from the satisfied theories in independently-motivated logics.
This strengthens the motivation for both the logics and the final coalgebras.

Third part: modal weak completeness

One of the goals of this TANCL workshop is to investigate treatments of logics that go beyond rank 1 axiomatizations.
My contribution here is a coalgebraic re-working of the basic *weak completeness results* for various standard modal logics.

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One of the goals of this TANCL workshop is to investigate treatments of logics that go beyond rank 1 axiomatizations.

My contribution here is a coalgebraic re-working of the basic *weak completeness results* for various standard modal logics.

I have to confess that my work here originally had different motivations:

I wanted to teach the weak completeness results to students who lacked the background to really understand maximal consistent sets and filtration.

Also, I wanted a more "semantic" method than tableaux.

The ∇ fragment of modal logic

We start with a set AtProp of atomic sentences.

$$\varphi \in \mathcal{L}_\nabla : \qquad p \mid \neg p \mid \varphi \wedge \psi \mid \nabla S \text{ for } S \subseteq \mathcal{L}_\nabla \mid$$

The semantics is

$$w \models \nabla S \quad \text{iff} \quad \begin{array}{l} \text{every } y \leftarrow x \text{ satisfies some } \varphi \in S \\ \text{and every } \varphi \in S \text{ is satisfied by some } y \leftarrow x \end{array}$$

So ∇S "abbreviates" $\bigwedge_{\varphi \in S} \Diamond\varphi \wedge \Box \bigvee_{\varphi \in S} \varphi$.

Which modal sentences are the smartest?

Let $\mathcal{A}$ be any Kripke model. Fix a number n. For every $a \in A$ and every h, we define the sentence φ_a^h. The definition is by recursion on h (simultaneously for all $a \in A$) as follows:

$$\varphi_a^0 = \bigwedge\{p : a \models p\} \wedge \bigwedge\{\neg p : a \models \neg p\}.$$

Given φ_b^h for all $b \in A$, we define

$$\varphi_a^{h+1} \quad = \quad \nabla\{\varphi_b^h : a \to b\} \wedge \varphi_a^0.$$

Each φ_a^h belongs to $\mathcal{C}_{h,n}$.
The idea is that φ_a^n gives us as much information as possible about the points reachable from a in $\leq h$ steps.

Height and Order

We define the *height* and *order* of an arbitrary sentence φ of modal logic.
The height measures the maximum nesting depth of boxes, and the order gives the largest subscript on any atomic proposition occurring.
For example,

$$\begin{array}{rcl} ht(\Diamond p_3 \wedge \Box\Diamond p_2) & = & 2 \\ ord(\Diamond p_3 \wedge \Box\Diamond p_2) & = & 3 \end{array}$$

$$\mathcal{L}_{h,n} \quad = \quad \{\varphi : ht(\varphi) \leq h, ord(\varphi) \leq n\}.$$

The sets $C_{h,n}$

We define the sets $C_{h,n}$ of *canonical sentences of height h and order n* as follows:
$C_{0,n}$ = the complete conjunctions of order n.
$C_{h+1,n}$ is the collection of sentences of the form $\nabla S \wedge \hat{T}$, where

$$
\begin{aligned}
S &\subseteq C_{h,n} \\
T &\subseteq \{p_1, \ldots, p_n\} \\
\hat{T} &= (\textstyle\bigwedge_{p_i \in T} p_i) \wedge (\textstyle\bigwedge_{p_i \notin T} \neg p_i)
\end{aligned}
$$

In other words, $\alpha \in C_{h+1,n}$ is of the form

$$
(\bigwedge_{\psi \in S} \Diamond \psi) \ \wedge \ (\Box \bigvee S) \ \wedge \ (\bigwedge T) \ \wedge \ (\bigwedge_{p_i \notin T} \neg p_i)
$$

for some $S \subseteq C_{h,n}$ and some $T \subseteq \{p_1, \ldots, p_n\}$.

Examples: $C_{0,1}$ and $C_{1,1}$

$C_{0,1} = \{p_1, \neg p_1\}$. Henceforth we drop the subscript.
$C_{1,1} = \{\alpha_1, \ldots, \alpha_8\}$, where

$$
\begin{array}{lcl@{\qquad}lcl}
\alpha_1 & = & \nabla\emptyset \wedge p & \alpha_5 & = & \nabla\{\neg p\} \wedge p \\
\alpha_2 & = & \nabla\emptyset \wedge \neg p & \alpha_6 & = & \nabla\{\neg p\} \wedge \neg p \\
\alpha_3 & = & \nabla\{p\} \wedge p & \alpha_7 & = & \nabla C_{0,1} \wedge p \\
\alpha_4 & = & \nabla\{p\} \wedge \neg p & \alpha_8 & = & \nabla C_{0,1} \wedge \neg p
\end{array}
$$

Note that $\nabla\emptyset \equiv \Box\text{false}$.
$C_{0,2}$ has $2^2 = 4$ elements. $C_{1,2}$ has $2 \times 2^4 = 32$ elements.
And $C_{2,2}$ has $2 \times 2^{32} = 8,589,934,592$ elements.

The models $\mathcal{C}_{h,n}(L)$

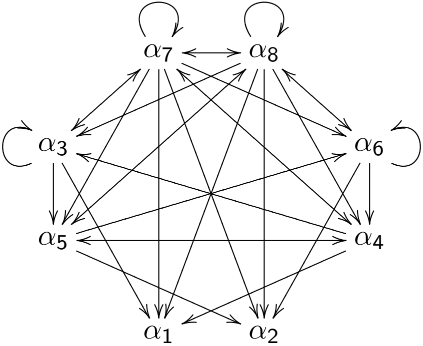
Let L be a normal modal logic.
We define $(\mathcal{C}_{h,n}(L), \rightsquigarrow)$, the *canonical model of consistent sentences of* L *of height* h *and order* n:

- The points of $\mathcal{C}_{h,n}(L)$ are the elements of $C_{h,n}$ which happen to be consistent in the logic L.
- $\alpha \rightsquigarrow \beta$ iff $\alpha \wedge \Diamond\beta$ is consistent in L. This comes from the Kozen-Parikh completeness theorem for PDL.
- $\alpha \models p$ iff $\vdash \alpha \rightarrow p$ in L.

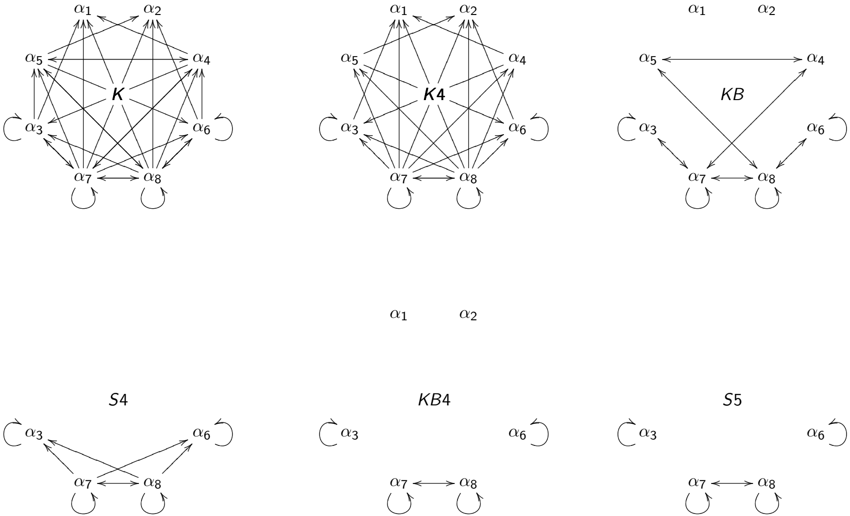
This part of the definition is what we are exploring here.

$\mathcal{C}_{1,1}(K)$

The points satisfying p are exactly those on the left side of the figure: α_1, α_3, α_5, and α_7.



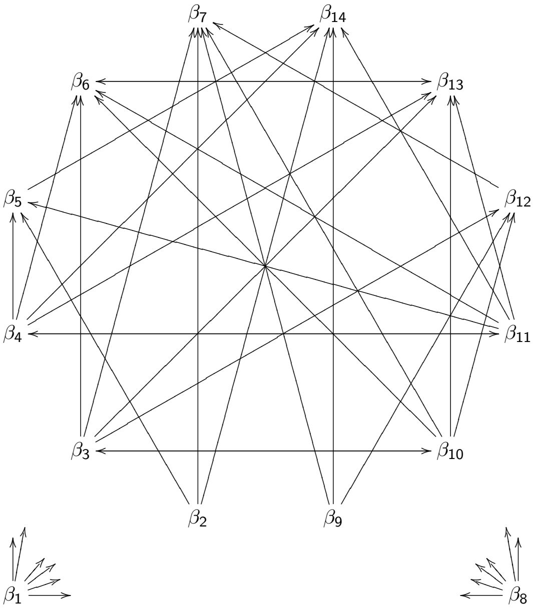
A sentence φ in one proposition p and of height 1 is valid iff φ holds at all points of the model above.


K
α_1 α_2 α_3 α_4 α_5 α_6 α_7 α_8
$K4$
α_1 α_2 α_3 α_4 α_5 α_6 α_7 α_8
KB
α_1 α_2 α_3 α_4 α_5 α_6 α_7 α_8
$S4$
α_3 α_6 α_7 α_8
$KB4$
α_1 α_2 α_3 α_6 α_7 α_8
$S5$
α_3 α_6 α_7 α_8

$\mathcal{C}_{2,1}(S4)$

$$
\begin{array}{rclrcl}
\beta_2 & = & \nabla\{\alpha_3,\alpha_6,\alpha_7\} \wedge p & \beta_1 & = & \nabla\{\alpha_3,\alpha_6,\alpha_7,\alpha_8\} \wedge p \\
\beta_3 & = & \nabla\{\alpha_3,\alpha_7,\alpha_8\} \wedge p & & & \\
\beta_4 & = & \nabla\{\alpha_6,\alpha_7,\alpha_8\} \wedge p & & & \\
\beta_5 & = & \nabla\{\alpha_6,\alpha_7\} \wedge p & & & \\
\beta_6 & = & \nabla\{\alpha_7,\alpha_8\} \wedge p & & & \\
\beta_7 & = & \nabla\{\alpha_3\} \wedge p & \beta_8 & = & \nabla\{\alpha_3,\alpha_6,\alpha_7,\alpha_8\} \wedge \neg p \\
\beta_9 & = & \nabla\{\alpha_3,\alpha_6,\alpha_8\} \wedge \neg p & & & \\
\beta_{10} & = & \nabla\{\alpha_3,\alpha_7,\alpha_8\} \wedge \neg p & & & \\
\beta_{11} & = & \nabla\{\alpha_6,\alpha_7,\alpha_8\} \wedge \neg p & & & \\
\beta_{12} & = & \nabla\{\alpha_3,\alpha_8\} \wedge \neg p & & & \\
\beta_{13} & = & \nabla\{\alpha_7,\alpha_8\} \wedge \neg p & & & \\
\beta_{14} & = & \nabla\{\alpha_6\} \wedge \neg p & & &
\end{array}
$$

These are the elements of $C_{2,1}$ consistent in $S4$. The structure as always is given by $\beta_i \rightsquigarrow \beta_j$ iff $\beta_i \wedge \Diamond\beta_j$ is consistent in $S4$.


β_7
β_{14}
β_6
β_{13}
β_5
β_{12}
β_4
β_{11}
β_3
β_{10}
β_2
β_9
β_1
β_8

Some properties

For each h and n, $\mathcal{C}_{h,n}$ is a finite subset of $\mathcal{L}_{h,n}$.

Lemma

Let $\chi \in \mathcal{L}_{h,n}$ and $\alpha \in \mathcal{C}_{h,n}$. Then in K,

$$\text{either} \vdash \alpha \to \chi \text{ or else} \vdash \alpha \to \neg\chi.$$

Lemma

$\vdash \bigvee \mathcal{C}_{h,n}$.
And for $\alpha \neq \beta$,

$$\vdash \alpha \to \neg\beta.$$

More properties

Lemma

The following hold for all h and n:

1. *If $KT \leq L$, $\mathcal{C}_{h,n}(L)$ is reflexive.*
2. *If $KD \leq L$, $\mathcal{C}_{h,n}(L)$ is serial.*
3. *If $KB \leq L$, $\mathcal{C}_{h,n}(L)$ is symmetric.*

(One interesting failure is that if $L = K$ with $\Diamond\varphi \to \Box\varphi$, then $\mathcal{C}_{h,n}$ is not a partial function.)

Lemma (Existence Lemma)

Let $\psi \in \mathcal{L}_{h,n}$, let φ be arbitrary, and suppose that $\varphi \wedge \Diamond\psi$ is consistent in L. Then there is some $\alpha \in \mathcal{C}_{h,n}(L)$ such that $\varphi \wedge \Diamond\alpha$ is consistent in L, and $\vdash \alpha \to \psi$ in K.

Easy weak completeness results

Lemma (Truth Lemma for $\mathcal{C}_{h,n}(L)$)

For all $\alpha \in \mathcal{C}_{h,n}(L)$ and all $\psi \in \mathcal{L}_{h,n}$,

$$(\mathcal{C}_{h,n}(L), \alpha) \models \psi \quad \textit{iff} \quad \vdash \alpha \to \psi \textit{ in } K.$$

Theorem

We have the following completeness/decidability results:

1. *KT for reflexive models. (T is $\Box\varphi \to \varphi$.)*
2. *KD for serial models. (D is $\Diamond$true.)*
3. *KB for symmetric models. (B is $\varphi \to \Box\Diamond\varphi$.)*
4. *etc.*

With a trick, one can also get the result for partial functions.

Completeness for classes of transitive models

Lemma

$\mathcal{C}_{h,n}(K4)$ is transitive.

Theorem

$K4$ is complete for transitive models. Other results for all combinations of B, D, T, 4, and 5.

Completeness for classes of transitive models, continued

Theorem ($K4McK = K$ with the McKinsey axioms $\Box\Diamond\varphi \to \Diamond\Box\varphi$.)

$\mathcal{C}_{h,n}(K4McK)$ is transitive, and each point has a successor with at most one successor.
Thus $K4McK$ is weakly complete for this class.

Theorem ($KL = K$ with the Löb axioms $\Box(\Box\varphi \to \varphi) \to \Box\varphi$.)

$\mathcal{C}_{h,n}(KL)$ is transitive and converse well-founded
Thus KL is weakly complete for this class.

COMPLETENESS FOR CLASSES OF TRANSITIVE MODELS, CONTINUED

THEOREM ($K4McK = K$ WITH THE MCKINSEY AXIOMS $\Box\Diamond\varphi \to \Diamond\Box\varphi$.)

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Thus KL is weakly complete for this class.

OPEN QUESTION

If $K4 \leq L$, then is $(\mathcal{C}_{h,n}(L), \rightsquigarrow)$ transitive?

Connections to older work

Recall that $\mathcal{L}_{h,n}$ is the (infinite) set of modal formulas of height $\leq h$ and of order $\leq n$.
Let $Can(L)$ be the canonical model of a logic L.
Consider the equivalence on $Can(L)$ induced by $\mathcal{L}_{h,n}(L)$.

Theorem

$\mathcal{C}_{h,n}(L)$ is isomorphic to the minimal filtration of $Can(L)$.

Fine defined a model $\mathfrak{C}_{h,n}$ in connection with K4.

Theorem

$\mathcal{C}_{h,n}(K4) \cong \mathfrak{C}_{h,n}$.

$K\Box^*$

$$\begin{array}{ll} \text{Mix} & \Box^*\varphi \to (\varphi \wedge \Box\Box^*\varphi) \\ \text{Induction} & (\varphi \wedge \Box^*(\varphi \to \Box\varphi)) \to \Box^*\varphi \end{array}$$

We build $\mathcal{C}_{h,n}(K\Box^*)$ the same way we built $\mathcal{C}_{h,n}$ except that we use

$$\begin{array}{l} (\bigwedge_{\psi\in R} \Diamond\psi) \ \wedge \ (\Box \bigvee R) \ \wedge \\ (\bigwedge_{\psi\in S} \Diamond^*\psi) \ \wedge \ (\Box^* \bigvee S) \ \wedge \\ (\bigwedge T) \ \wedge \ (\bigwedge_{p_i\notin T} \neg p_i) \end{array}$$

and we also are only interested in sentences of this form which are consistent in $K\Box^*$.
The *ht* function works as before, except we also say that $ht(\Box^*\varphi) = 1 + ht(\varphi)$.
The analogs of general Lemmas on $\mathcal{C}_{h,n}$ hold.

Completeness for $K\Box^*$

Lemma

Let $\alpha, \beta \in \mathcal{C}_{h,n}(K\Box^*)$ and $\Diamond^*\varphi \in \mathcal{L}_{h,n}$. Suppose that $\alpha \rightsquigarrow \beta$ and $\vdash \beta \to \Diamond^*\varphi$. Then $\vdash \alpha \to \Diamond^*\varphi$ as well.

Lemma

Let $X \subseteq \mathcal{C}_{h,n}(K\Box^*)$ be closed under $\rightsquigarrow$. Then $\vdash \bigvee X \to \Box^* \bigvee X$.

Lemma (Truth Lemma for $\mathcal{C}_{h,n}(K\Box^*)$)

For all $\alpha \in \mathcal{C}_{h,n}(K\Box^*)$ and all $\psi \in \mathcal{L}_{h,n}$, $(\mathcal{C}_{h,n}(K\Box^*), \alpha) \models \psi$ iff $\vdash \alpha \to \psi$ in $K\Box^*$.

Theorem

$K\Box^*$ is complete and decidable.

Concluding Summary/Questions

1. Coalgebraic versions of modal logic are connected to exploration of other issues.
2. One can construct a final coalgebra by taking as the carrier the satisfied theories in an associated logic.
3. One can prove modal weak completeness/decidability results using models built from sentences in the same logics.

Issues

The definition of the models $\mathcal{C}_{h,n}$ is not as *principled* as one would like, especially if we are to generalize these models to coalgebraic settings.
Even more, why does all this work?