Coalgebraic Logics: A Computer Science Perspective

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Part I: Coalgebraic Logics: Motivation and Some Results

A Computer Science View

Coalgebraic Logics: Describe *computational phenomena* with *modal logics*

• State Transition Systems	$\longrightarrow$ Hennessy-Milner Logic
• Probabilistic Effects	$\longrightarrow$ Probabilistic Modal Logic
• Games	$\longrightarrow$ Coalition Logic
• Ontologies . . .	$\longrightarrow$ Description Logic . . .

Logical Aspects

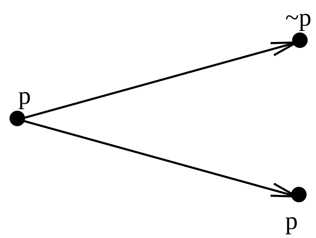
- completeness
- complexity
- cut elimination
- interpolation . . .

Computer Science Aspects

- *Genericity:* development of uniform proofs/algorithms/tools?
- *Modularity:* synthesis of complex systems from simple building blocks

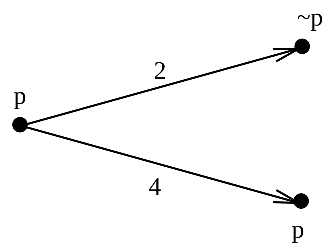
A Cook's Tour Through Modal Semantics

Kripke Models.



$$C \to \mathcal{P}(C) \times \mathcal{P}(A)$$

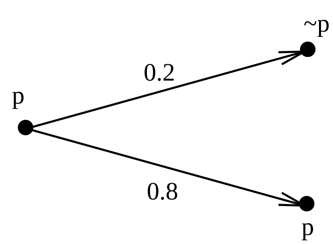
Multigraphs.



$$C \to \mathcal{B}(C) \times \mathcal{P}(A)$$

$$\mathcal{B}(X) = \{f : X \to \mathbb{N} \mid \mathrm{supp}(f) \text{ finite}\}$$

Probabilistic Systems.



$$C \to \mathcal{D}(C) \times \mathcal{P}(A)$$

$$\mathcal{D}(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\}$$

Unifying Feature: Coalgebraic Semantics

All examples are instances of **Coalgebras**

$$(C, \gamma : C \to TC)$$

where $T : \mathsf{Set} \to \mathsf{Set}$ is an endofunctor, the *signature functor*.

(Dually, T*-algebras* are pairs $(A, \alpha : TA \to A)$)

Intuition.

- *coalgebras* are generalised transition systems
- *morphisms* of coalgebras are generalised p-morphisms

Computer Science Concerns

- *Genericity:* Prove things once and for all, *parametric in* T
- *Modularity:* Construct complex functors from simple ingredients

Coalgebraic Semantics of Modal Logics

Given: $T : \mathsf{Set} \to \mathsf{Set}$

Question: What's the “right” logic for T-coalgebras?

- should generalise well-known cases, e.g. K, probabilistic/graded modal logic, coalition logic
- theory should be *parametric* in T

$\rightsquigarrow$ *uniform* theorems that apply to a large class of logics

Semantically: What's a modal operator, or: what is $[\![\Box\phi]\!]$?

Moss' Coalgebraic Logic I

Kripke Frames: $C \to \mathcal{P}(C)$

Concrete Syntax

$$\frac{}{\bot \in L} \qquad \frac{\phi, \psi \in L}{\phi \to \psi \in L} \qquad \frac{\Phi \in \mathcal{P}(L)}{\nabla\Phi \in L}$$

Modal Semantics

$$c \models \nabla\Phi \iff (\gamma(c), \Phi) \in \mathcal{P}(\models)$$

Abstract Syntax:

$$L \cong F(L) = 1 + L^2 + \mathcal{P}(L)$$

Algebraic Semantics

$$\begin{tikzcd} F(L) \arrow[r] \arrow[d, "i"'] & F(\mathcal{P}(C)) \arrow[d, "\hat{\gamma}"'] \\ L \arrow[r, "[\![\cdot]\!]"'] & \mathcal{P}(C) \end{tikzcd}$$

$$\nabla\Phi = \Box \bigvee \Phi \wedge \Diamond\Phi$$

T-coalgebras: $C \to T(C)$

Concrete Syntax

$$\frac{}{\bot \in L} \qquad \frac{\phi, \psi \in L}{\phi \to \psi \in L} \qquad \frac{\Phi \in TL}{\nabla\Phi \in L}$$

Modal Semantics

$$c \models \nabla\Phi \iff (\gamma(c), \Phi) \in T(\models)$$

Abstract Syntax:

$$L \cong F(L) = 1 + L^2 + T(L)$$

Algebraic Semantics

$$\begin{tikzcd} F(L) \arrow[r] \arrow[d, "i"'] & F(\mathcal{P}(C)) \arrow[d, "\hat{\gamma}"'] \\ L \arrow[r, "[\![\cdot]\!]"'] & \mathcal{P}(C) \end{tikzcd}$$

Need: F-algebra structure $F(\mathcal{P}(C)) \to \mathcal{P}(C)$

Moss' Coalgebraic Logic II

Algebraic Semantics of Coalgebraic Logic:

$$\begin{tikzcd} 1 + L^2 + TL \arrow[r] \arrow[d, "i"'] & 1 + (\mathcal{P}C)^2 + T(\mathcal{P}C) \arrow[d, "{[\perp, \rightarrow, \hat{\gamma}]}"] \\ L \arrow[r, "\llbracket \cdot \rrbracket"'] & \mathcal{P}(C) \end{tikzcd}$$

where $\hat{\gamma}$: $\underbrace{T(\mathcal{P}C) \xrightarrow{\delta} \mathcal{P}(TC)}_{\text{distributive law}} \xrightarrow{\gamma^{-1}} \mathcal{P}(C)$

Representation Theorem: $\sum_n A_n \times X^n \twoheadrightarrow TX$, e.g. $X \xrightarrow{M} TX$

gives algebraic semantics of **Unary Modalities:**

$$\underbrace{\mathcal{P}(C) \xrightarrow{M} T(\mathcal{P}C) \xrightarrow{\delta} \mathcal{P}(TC)}_{\text{unary modality}} \xrightarrow{\gamma^{-1}} \mathcal{P}(C)$$

Coalgebraic Semantics of Modal Logics

Structures for T coalgebras determine the semantics of modal operators: they

assign a *nbhd frame translation* or, equivalently, a *predicate lifting*

$$[\![M]\!] : TC \to \mathcal{P}\mathcal{P}(C) \qquad [\![M]\!] : \mathcal{P}(C) \to \mathcal{P}(TC)$$

to every modal operator M of the language, parametric in C.

Together with a T-coalgebra (C, γ) this gives a

neighbourhood frame *boolean algebra with operator*

$$C \xrightarrow{\gamma} TC \xrightarrow{[\![M]\!]} \mathcal{P}\mathcal{P}(C) \qquad \mathcal{P}(C) \xrightarrow{[\![M]\!]} \mathcal{P}(TC) \xrightarrow{\gamma^{-1}} \mathcal{P}(C)$$

Induced **Coalgebraic Semantics** $[\![\phi]\!] \subseteq C$ of a modal formula

from a *modal perspective* equivalent *algebraic viewpoint*

$$c \in [\![M\phi]\!] \text{ iff } [\![\phi]\!] \in [\![M]\!] \circ \gamma([\![\phi]\!]) \qquad c \in [\![M\phi]\!] \iff \gamma(c) \in [\![M]\!]([\![\phi]\!])$$

Examples

Neighbourhood Frames, i.e. coalgebras $C \to \mathcal{P}\mathcal{P}(C)$

$$[\![\Box]\!] = \mathrm{id} : \underbrace{\mathcal{P}\mathcal{P}(C)}_{TC} \to \mathcal{P}\mathcal{P}(C)$$

(identical nbhd frame translation)

Kripke Frames, ie. coalgebras $C \to \mathcal{P}(C)$

viewed as neighbourhood frames

$$\begin{aligned} [\![\Box]\!] : \overbrace{\mathcal{P}(C)}^{TC} &\to \mathcal{P}\mathcal{P}(C) \\ c &\mapsto \{c' : c' \supseteq c\} \end{aligned}$$

via boolean algebras with operators

$$\begin{aligned} [\![\Box]\!] : \mathcal{P}(C) &\to \mathcal{P}\overbrace{\mathcal{P}(C)}^{TC} \\ c &\mapsto \{c' : c' \subseteq c\} \end{aligned}$$

Probabilistic Transition Systems , i.e. coalgebras $C \to \mathcal{D}C$

$$\begin{aligned} [\![L_p]\!] : \mathcal{P}(C) &\to \mathcal{P}\overbrace{\mathcal{D}(C)}^{TC} \\ c &\mapsto \{\mu : C \to [0,1] : \mu(c) \geq p\} \end{aligned}$$

(algebraic perspective)

Genericity I: Expressivity

Easy, but important: Coalgebraic Logics are bisimulation invariant.

Hennessy-Milner Property:

Bisimulation coincides with logical equivalence over *image finite* transition systems.

- what is *image finite* for T-coalgebras?
- *additional condition(s)* on the logic (e.g. exclude empty set of operators)

Theorem (P, 2001)

If T is *ω-accessible* and the modal structure is *separating*, i.e. for predicate liftings

$$TC \ni t \mapsto \{[\![M]\!](c) : c \subseteq C, M \text{ modal op}\}$$

is injective, then the induced logic has the Hennessy-Milner property.

Theorem (Schroeder, 2005)

Admitting polyadic modalities, the structure that comprises *all* predicate liftings is separating.

Genericity II: Completeness

Deduction for Coalgebraic Logics: propositional logic plus a set $\mathcal{R}$ of

one-step rules ϕ/ψ: ϕ propositional, ψ clause over $Ma, a \in V$

Intuition. Rules axiomatise those nbhd frames that come from coalgebras

One Step Derivability of χ (propositional over $\{Mx : x \subseteq X\}$) over a set X

- $TX \models \chi$ defined inductively by $[\![Mx]\!] = [\![M]\!](x)$
- $\mathcal{R}X \vdash \chi$ iff $\{\psi\sigma : X \models \phi\sigma, \phi/\psi \in \mathcal{R}\} \vdash_{\mathsf{PL}} \chi$

$\mathcal{R}$ is one-step sound (complete) if $TX \models \chi$ whenever (only if) $\mathcal{R}X \vdash \chi$

Theorem (P, 2003, Schroeder 2006)

Soundness and weak completeness are implied by their one-step counterparts.

Theorem (Schroeder 2006)

The set of axioms that is one-step sound is one-step complete.

Genericity III: Complexity

Shallow Model Construction for T-coalgebras: inductively strip off modalities

$$
\begin{array}{ccc}
\forall \phi/\psi \in R.\psi\sigma \to \chi \implies \neg\phi\sigma \text{ satisfiable} & & \text{Countermodel of } \phi\sigma\text{'s} \\
\Uparrow & & \Downarrow \\
\neg\chi \text{ satisfiable} & & \text{Countermodel of } \chi
\end{array}
$$

Crucial Requirement is **Resolution Closure** of $\mathcal{R}$:

derivable consequences are derivable using a *single* rule.

Theorem. (Schroeder/P, 2006)

If $\mathcal{R}$ is resolution closed and rule matching is in NP, then satisfiability is in PSPACE.

Example. K, KD, Coalition Logic, GML, PML, Majority Logic are in PSPACE.

Construction of Resolution Closed Sets

Example: K axiomatised by rules

$$\frac{a}{\Box a} \qquad \frac{a \wedge b \to c}{\Box a \wedge \Box b \to \Box c}$$

Rule Resolution:

$$\frac{a \wedge b \to c}{\Box a \wedge \Box b \to \Box c} \qquad \frac{c \wedge d \to e}{\Box c \wedge \Box d \to \Box e}$$

Resolving the conclusions at c

$$\frac{(a \wedge b \to c) \wedge (c \wedge d \to e)}{\Box a \wedge \Box b \wedge \Box d \to \Box e}$$

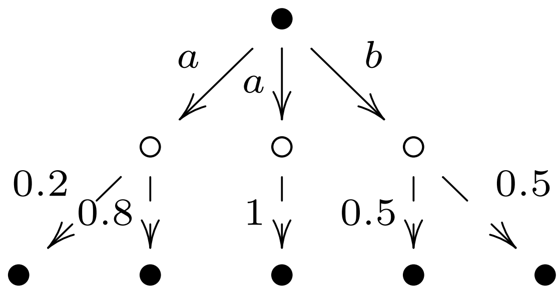
Eliminating c from the premise:

$$\frac{a \wedge b \wedge d \to e}{\Box a \wedge \Box b \wedge \Box d \to \Box e}$$

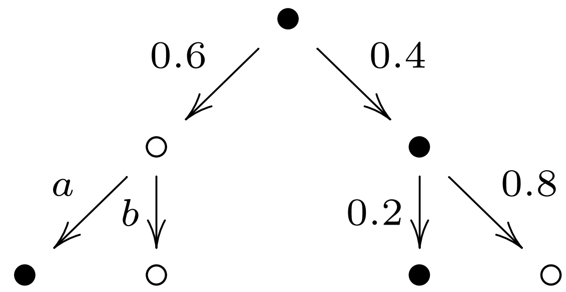
(This converges to a cut-free sequent-calculus . . .)

Modularity

Example. Combining Probabilities and Non-Determinism



Simple Segala Systems



Alternating Systems

Coalgebraic Interpretation

$$C \to \mathcal{P}(A \times \mathcal{D}(C))$$

$$C \to \mathcal{P}(A \times C) + \mathcal{D}(C)$$

Semantics of Combination. Functor Composition – ingredients represent features.

Logic Combinations. Mimic Functor Composition

Logics for Combined Systems

Simple Segala Systems: $C \to \mathcal{P}(A \times \mathcal{D}(C))$

$\mathcal{L}_n \ni \phi ::= \top \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid \Box_a \psi$ (nondeterministic formulas; $\psi \in \mathcal{L}_u$, $a \in A$)

$\mathcal{L}_u \ni \psi ::= \top \mid \psi_1 \wedge \psi_2 \mid \neg\psi \mid L_p \phi$ (probabilistic formulas; $\phi \in \mathcal{L}_n$, $p \in [0,1] \cap \mathbb{Q}$).

Alternating Systems: $C \to \mathcal{P}(A \times C) + \mathcal{D}(C)$

$\mathcal{L}_o \ni \rho ::= \top \mid \rho_1 \wedge \rho_2 \mid \neg\rho \mid \phi + \psi$ (alternating formulas; $\phi \in \mathcal{L}_u$, $\psi \in \mathcal{L}_n$)

$\mathcal{L}_u \ni \phi ::= \top \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid L_p \rho$ (probabilistic formulas; $\rho \in \mathcal{L}_o$, $p \in [0,1] \cap \mathbb{Q}$)

$\mathcal{L}_n \ni \psi ::= \top \mid \psi_1 \wedge \psi_2 \mid \neg\psi_2 \mid \Box_a \rho$ (nondeterministic formulas; $\rho \in \mathcal{L}_o$, $a \in A$)

Semantics by Example: given $\gamma : C \to \mathcal{P}(A \times C) + \mathcal{D}(C)$

- ($\mathcal{L}_o$) $[\![\phi + \psi]\!] = \gamma^{-1}([\![\phi]\!] + [\![\psi]\!]) \subseteq C$
- ($\mathcal{L}_u$) $[\![L_p \rho]\!] = [\![L_p]\!]([\![\rho]\!]) \subseteq \mathcal{D}C$
- ($\mathcal{L}_n$) $[\![\Box_a \rho]\!] = [\![\Box_a]\!]([\![\rho]\!]) \subseteq \mathcal{P}(A \times C)$

Modularity I: Expressivity

Features: Basic Building Blocks comprising

- an endofunctor $F : \mathsf{Set}^n \to \mathsf{Set}$
- typed modal operators $M : i_1, \dots, i_k$
- predicate liftings $[\![M]\!] : \mathcal{P}(X_1) \times \cdots \times \mathcal{P}(X_k) \to \mathcal{P}F(X_1, \dots, X_k)$

Example 1: Uncertainty

- $\mathcal{D} : \mathsf{Set} \to \mathsf{Set}$
- $L_p : 1$ ($p \in [0,1] \cap \mathbb{Q}$)
- $[\![L_p]\!]$ as before

Example 2: Binary Choice

- $\coprod : \mathsf{Set}^2 \to \mathsf{Set}$
- $+ : 1, 2$
- $[\![+]\!] : (x, y) \mapsto x + y$

Theorem (Cirstea, 2000)

The logic associated with any combination of features that are ω-accessible and separating has the Hennessy-Milner property.

Modularity II: Completeness and Complexity

Deduction for combined logics: Extend features with *typed* one-step rules

Example 1: Uncertainty

$$\frac{\sum_{1\leq i\leq n} r_i a_i \geq k : 1}{\bigvee_{1\leq i\leq n} \mathrm{sgn}(r_i) L_{p_i} a_i}$$

(plus side conditions)

Example 2: Binary Choice

$$\frac{(\bigwedge_{i=1}^m \alpha_i \rightarrow \bigvee_{j=1}^n \beta_j) : 1 \quad (\bigwedge_{i=1}^m \gamma_i \rightarrow \bigvee_{j=1}^n \delta_j) : 2}{\bigwedge_{i=1}^m (\alpha_i + \gamma_i) \rightarrow \bigvee_{j=1}^n (\beta_j + \delta_j)} \; (m, n \geq 0)$$

Deduction for Combined Logics: type correct application of deduction rules

Theorem. (Cirstea/P, 2003)

One-step completeness of all features implies weak completeness of combinations.

Theorem. (Schroeder/P, 2007)

Satisfiability for combined logics is in PSPACE provided rule matching for all features is in NP.

Part II: Extensions and Open Problems

Frequently Asked Questions

Coalgebraic Completeness Theorem

Referee: This is nice, but can you also do S4?

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DP: Not yet – $\Box p \to \Box\Box p$ is not rank 1!

So maybe it's time to go beyond rank 1 . . .

Frame Conditions

Recall: Coalgebraic Logics can always be axiomatised by rank-1 axioms.

Our Setting : Rank-1 axioms $\mathcal{A}$ + Frame Conditions Φ, i.e.

- $\mathcal{A}$ is rank-1, sound and complete w.r.t. *alll* T-coalgebras
- Φ is a set of additional axioms (not neccessarily rank 1), e.g. T or 4.

Kripke Frame Analogy. $\mathcal{A} = \mathsf{K}$ and e.g. $\Phi = 4$

Semantical Consequence and Deduction

- $T\Phi \models \phi$ iff $C \models \phi$ whenever $C \models \Phi$, for all T-coalgebras C
- $\mathcal{A}\Phi \vdash \phi$ iff ϕ is derivable from $\mathcal{A} \cup \Phi$

Question. For which ϕ do we have completeness, i.e. $T\Phi \models \phi \implies \mathcal{A}\Phi \vdash \phi$?

Partial Answers and Open Questions

Frame Completeness. $T\Phi \models \phi \implies \mathcal{A}\Phi \vdash \phi$ holds, for example, if

- if Φ is a collection of *positive* formulas
- if Φ is any collection of rank 0/1 formulas (e.g. T)
- if $\Phi = 4$ or $\Phi = T, 4$

Open Questions.

- semantical characterisation of admissible frame conditions?
- syntactical characterisation? Sahlquist completeness theorem?

Proof Theory

Observation I. Rule Resolution *seems* to lead to sequent calculus presentations, but:

Observation II. General Rule Premises are of the form

$$\bigwedge_{J \subseteq I} \Big(\bigwedge_{j \in J} a_j \to \bigvee_{j \notin J} b_j \Big)$$

Open Questions

- can we *systematically* derive sequent calculi?
- are they cut-free?
- and have interpolation and/or subformula properties?

Decidability and Complexity

Decidability via finite models: by-product of completeness via fmp

Challenge Question: Complexity

In a setting *without* frame conditions . . .

Semantically

- coalgebraic *shallow models*
- based on *extended rulesets*

Syntactically

- cut-free sequent calculus
- induced by extended rulesets

Ruleset extension is algorithmic: *resolution closure*

Open Questions

- is resolution closure meaningful outside rank-1, and when? (yes, e.g. for S4)
- does either the syntactical or the semantic method extend?

Fixpoint Formulas

Application Pull. Reasoning about *ongoing behaviour:* safety and liveness

Language Extension: **flat fixpoint formulas**

$$M^*\phi \equiv \nu x.\phi \wedge Mx \quad \text{and} \quad M_*\phi \equiv \mu x.\phi \vee Mx$$

(many possible variations)

New (Fixpoint) Axioms, e.g.

$$\mathsf{F} \equiv M^*p \to p \wedge MM^*p \qquad (p \wedge M^*(p \to Mp)) \to M^*p$$

Trivial Theorem:

$\mathcal{A}\mathsf{S} \vdash \phi \implies T \models \phi$ if $\mathcal{A}$ is sound w.r.t. all T-coalgebras

Hard Problem: Completeness.

Even Harder Problem: Complexity

Any Answers?