A Dynamic Distributive Law

Yde Venema
Universiteit van Amsterdam
staff.science.uva.nl/~yde

August 10, 2007
Coalgebraic Logic Workshop
Oxford

(Largely based on joint work with Marta Bilkova, Clemens Kupke, Alexander Kurz, Alessandra Palmigiano, Luigi Santocanale)

Overview

- Introduction: reorganizing modal logic
- A modal distributive law
- A game-theoretical perspective
- Uniform interpolation
- Automata
- Axiomatizing ∇
- A coalgebraic generalization
- Concluding remarks

Overview

- Introduction: reorganizing modal logic
- A modal distributive law
- A game-theoretical perspective
- Uniform interpolation
- Automata
- Axiomatizing ∇
- A coalgebraic generalization
- Concluding remarks

The Cover Modality ∇

- Define the language ML of standard modal logic by

$$\varphi \ ::= \ p \mid \neg p \mid \bot \mid \top \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond \varphi \mid \Box \varphi$$

The Cover Modality ∇

- Define the language ML of standard modal logic by

$$\varphi \ ::= \ p \mid \neg p \mid \bot \mid \top \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond \varphi \mid \Box \varphi$$

- Given set Φ of formulas, define

$$\nabla \Phi := \Box \bigvee \Phi \wedge \bigwedge \Diamond \Phi$$

(here $\Diamond \Phi := \{ \Diamond \varphi \mid \varphi \in \Phi \}$)

The Cover Modality ∇

- Define the language ML of standard modal logic by

$$\varphi ::= p \mid \neg p \mid \bot \mid \top \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond \varphi \mid \Box \varphi$$

- Given set Φ of formulas, define

$$\nabla \Phi := \Box \bigvee \Phi \wedge \bigwedge \Diamond \Phi$$

(here $\Diamond \Phi := \{\Diamond \varphi \mid \varphi \in \Phi\}$)

- History: (predicate logic), Fine, Moss, Abramsky, Walukiewicz

The Cover Modality ∇

- Define the language ML of standard modal logic by

$$\varphi ::= p \mid \neg p \mid \bot \mid \top \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond \varphi \mid \Box \varphi$$

- Given set Φ of formulas, define

$$\nabla \Phi := \Box \bigvee \Phi \wedge \bigwedge \Diamond \Phi$$

(here $\Diamond \Phi := \{\Diamond \varphi \mid \varphi \in \Phi\}$)

- History: (predicate logic), Fine, Moss, Abramsky, Walukiewicz
- Define the language ML_{∇} by

$$\varphi ::= p \mid \neg p \mid \bot \mid \top \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \nabla \Phi$$

Semantics

Fix a Kripke model $\mathbb{S} = \langle S, R, V \rangle$.

$\mathbb{S}, s \Vdash \nabla\Phi$ iff for all $t \in R[s]$ there is a $\varphi \in \Phi$ with $\mathbb{S}, t \Vdash \varphi$ and for all $\varphi \in \Phi$ there is a $t \in R[s]$ with $\mathbb{S}, t \Vdash \varphi$

Semantics

Fix a Kripke model $\mathbb{S} = \langle S, R, V \rangle$.

$\mathbb{S}, s \Vdash \nabla\Phi$ iff for all $t \in R[s]$ there is a $\varphi \in \Phi$ with $\mathbb{S}, t \Vdash \varphi$
and for all $\varphi \in \Phi$ there is a $t \in R[s]$ with $\mathbb{S}, t \Vdash \varphi$

Call a relation Z full on two sets A and B if $\forall a \in A \exists b. \in B Zab$ and $\forall b \in B \exists a. \in A Zab$.

Semantics

Fix a Kripke model $\mathbb{S} = \langle S, R, V \rangle$.

$\mathbb{S}, s \Vdash \nabla\Phi$ iff for all $t \in R[s]$ there is a $\varphi \in \Phi$ with $\mathbb{S}, t \Vdash \varphi$
and for all $\varphi \in \Phi$ there is a $t \in R[s]$ with $\mathbb{S}, t \Vdash \varphi$

Call a relation Z full on two sets A and B if $\forall a \in A \exists b. \in B Zab$ and $\forall b \in B \exists a. \in A Zab$.

Theorem (Moss)

$\mathbb{S}, s \Vdash \nabla\Phi$ iff the satisfaction relation $\Vdash$ is full on $R[s]$ and Φ

Semantics

Fix a Kripke model $\mathbb{S} = \langle S, R, V \rangle$.

$\mathbb{S}, s \Vdash \nabla\Phi$ iff for all $t \in R[s]$ there is a $\varphi \in \Phi$ with $\mathbb{S}, t \Vdash \varphi$
and for all $\varphi \in \Phi$ there is a $t \in R[s]$ with $\mathbb{S}, t \Vdash \varphi$

Call a relation Z full on two sets A and B if $\forall a \in A \exists b. \in B Zab$ and $\forall b \in B \exists a. \in A Zab$.

Theorem (Moss)

$\mathbb{S}, s \Vdash \nabla\Phi$ iff the satisfaction relation $\Vdash$ is full on $R[s]$ and Φ

iff there is a $Z \subseteq \Vdash$ which is full on $R[s]$ and Φ

Reorganizing Modal Logic

Conversely, express $\Box$ and $\Diamond$ in terms of ∇

$$
\begin{aligned}
\Diamond\varphi &\equiv \nabla\{\varphi, \top\} \\
\Box\varphi &\equiv \nabla\varnothing \vee \nabla\{\varphi\}.
\end{aligned}
$$

Reorganizing Modal Logic

Conversely, express $\Box$ and $\Diamond$ in terms of ∇

$$\begin{aligned} \Diamond\varphi &\equiv \nabla\{\varphi, \top\} \\ \Box\varphi &\equiv \nabla\varnothing \vee \nabla\{\varphi\}. \end{aligned}$$

Theorem The languages ML and ML_∇ are effectively equi-expressive.

Overview

- Introduction: reorganizing modal logic
- A modal distributive law
- A game-theoretical perspective
- Uniform interpolation
- Automata
- Axiomatizing ∇
- A coalgebraic generalization
- Concluding remarks

A modal distributive law

A modal distributive law

Theorem For any sets Φ, Φ' of formulas,

$$\nabla\Phi \wedge \nabla\Phi' \equiv \bigvee_{Z \in \Phi \bowtie \Phi'} \nabla\{\varphi \wedge \varphi' \mid (\varphi, \varphi') \in Z\},$$

where $\Phi \bowtie \Phi'$ is the set of full relations between Φ and Φ'.

A modal distributive law

Theorem For any sets Φ, Φ' of formulas,

$$\nabla\Phi \wedge \nabla\Phi' \equiv \bigvee_{Z \in \Phi \bowtie \Phi'} \nabla\{\varphi \wedge \varphi' \mid (\varphi, \varphi') \in Z\},$$

where $\Phi \bowtie \Phi'$ is the set of full relations between Φ and Φ'.

Proof of '$\Rightarrow$': Suppose $\mathbb{S}, s \Vdash \nabla\Phi \wedge \nabla\Phi'$.

A modal distributive law

Theorem For any sets Φ, Φ' of formulas,

$$\nabla\Phi \wedge \nabla\Phi' \equiv \bigvee_{Z \in \Phi \bowtie \Phi'} \nabla\{\varphi \wedge \varphi' \mid (\varphi, \varphi') \in Z\},$$

where $\Phi \bowtie \Phi'$ is the set of full relations between Φ and Φ'.

Proof of '$\Rightarrow$': Suppose $\mathbb{S}, s \Vdash \nabla\Phi \wedge \nabla\Phi'$.

Define $Z \subseteq \Phi \times \Phi'$ as

$$Z := \{(\varphi, \varphi') \mid \mathbb{S}, t \Vdash \varphi \wedge \varphi' \text{ for some } t \in R[s]\}.$$

A modal distributive law

Theorem For any sets Φ, Φ' of formulas,

$$\nabla\Phi \wedge \nabla\Phi' \equiv \bigvee_{Z \in \Phi \bowtie \Phi'} \nabla\{\varphi \wedge \varphi' \mid (\varphi, \varphi') \in Z\},$$

where $\Phi \bowtie \Phi'$ is the set of full relations between Φ and Φ'.

Proof of '$\Rightarrow$': Suppose $\mathbb{S}, s \Vdash \nabla\Phi \wedge \nabla\Phi'$.

Define $Z \subseteq \Phi \times \Phi'$ as

$$Z := \{(\varphi, \varphi') \mid \mathbb{S}, t \Vdash \varphi \wedge \varphi' \text{ for some } t \in R[s]\}.$$

Claim 1: Z is full on Φ and Φ'.

A modal distributive law

Theorem For any sets Φ, Φ' of formulas,

$$\nabla\Phi \wedge \nabla\Phi' \equiv \bigvee_{Z \in \Phi \bowtie \Phi'} \nabla\{\varphi \wedge \varphi' \mid (\varphi, \varphi') \in Z\},$$

where $\Phi \bowtie \Phi'$ is the set of full relations between Φ and Φ'.

Proof of '⇒': Suppose $\mathbb{S}, s \Vdash \nabla\Phi \wedge \nabla\Phi'$.

Define $Z \subseteq \Phi \times \Phi'$ as

$$Z := \{(\varphi, \varphi') \mid \mathbb{S}, t \Vdash \varphi \wedge \varphi' \text{ for some } t \in R[s]\}.$$

Claim 1: Z is full on Φ and Φ'.

Claim 2: $\mathbb{S}, s \Vdash \nabla\{\varphi \wedge \varphi' \mid (\varphi, \varphi') \in Z\}$.

Reorganizing Modal Logic: proposition part

- Fix (finite) set X of proposition letters

Reorganizing Modal Logic: proposition part

- Fix (finite) set X of proposition letters
- Define local description connective $\odot$: given set $P \subseteq \mathsf{X}$, put

$$\odot P := \bigwedge_{p \in P} p \wedge \bigwedge_{p \in \mathsf{X} \setminus P} \neg p$$

Reorganizing Modal Logic: proposition part

- Fix (finite) set X of proposition letters
- Define local description connective $\odot$: given set $P \subseteq \mathsf{X}$, put

$$\odot P := \bigwedge_{p \in P} p \wedge \bigwedge_{p \in \mathsf{X} \setminus P} \neg p$$

- Conversely, for every $q \in \mathsf{X}$, have

$$q \equiv \bigvee_{q \in P} \odot P$$

Reorganizing Modal Logic: proposition part

- Fix (finite) set X of proposition letters
- Define local description connective $\odot$: given set $P \subseteq \mathsf{X}$, put

$$\odot P := \bigwedge_{p \in P} p \wedge \bigwedge_{p \in \mathsf{X} \setminus P} \neg p$$

- Conversely, for every $q \in \mathsf{X}$, have

$$q \equiv \bigvee_{q \in P} \odot P$$

Proposition ML is effectively equi-expressive with the language given by

$$\varphi ::= \odot P \mid \bigvee \Phi \mid \bigwedge \Phi \mid \nabla \Phi$$

Coalgebraic Modal Logic

Define distributed conjunction $\bullet$:

$$P \bullet \Phi := \odot P \wedge \nabla \Phi$$

Coalgebraic Modal Logic

Define distributed conjunction $\bullet$:

$$P \bullet \Phi := \odot P \wedge \nabla \Phi$$

Define the language CML by

$$\varphi ::= \bigvee \Phi \mid \bigwedge \Phi \mid P \bullet \Phi$$

Coalgebraic Modal Logic

Define distributed conjunction $\bullet$:

$$P \bullet \Phi := \odot P \wedge \nabla \Phi$$

Define the language CML by

$$\varphi ::= \bigvee \Phi \mid \bigwedge \Phi \mid P \bullet \Phi$$

Conversely, express

$$\odot P \equiv P \bullet \varnothing \vee P \bullet \{\top\} \qquad \nabla \Phi \equiv \bigvee_{P \subseteq X} P \bullet \Phi$$

Coalgebraic Modal Logic

Define distributed conjunction $\bullet$:

$$P \bullet \Phi := \odot P \wedge \nabla \Phi$$

Define the language CML by

$$\varphi \ ::= \ \bigvee \Phi \ \mid \ \bigwedge \Phi \ \mid \ P \bullet \Phi$$

Conversely, express

$$\odot P \equiv P \bullet \varnothing \vee P \bullet \{\top\} \qquad \nabla \Phi \equiv \bigvee_{P \subseteq X} P \bullet \Phi$$

Proposition The languages ML and CML are effectively equi-expressive.

Modal Distributive Normal Forms

Modal Distributive Normal Forms

- Define the language CML$^-$ by

$$\varphi ::= \bigvee \Phi \mid P \bullet \Phi$$

Theorem The languages ML and CML$^-$ are effectively equi-expressive.

Modal Distributive Normal Forms

- Define the language CML$^-$ by

$$\varphi ::= \bigvee \Phi \mid P \bullet \Phi$$

Theorem The languages ML and CML$^-$ are effectively equi-expressive.

Proof via modal distributive law for $\bullet$:

$$(P \bullet \Phi) \wedge (P' \bullet \Phi') \equiv \begin{cases} \bigvee \varnothing \quad (= \bot) & \text{if } P \neq P' \\ \bigvee_{Z \in \Phi \bowtie \Phi'} P \bullet \{\varphi \wedge \varphi' \mid (\varphi, \varphi') \in Z\} & \text{if } P = P' \end{cases}$$

Overview

- Introduction: reorganizing modal logic
- A modal distributive law
- A game-theoretical perspective
- Uniform interpolation
- Automata
- Axiomatizing ∇
- A coalgebraic generalization
- Concluding remarks

Game semantics for ML

Position	Player	Legitimate moves
$(\varphi_1 \vee \varphi_2, s)$	$\exists$	$\{(\varphi_1, s), (\varphi_2, s)\}$
$(\varphi_1 \wedge \varphi_2, s)$	$\forall$	$\{(\varphi_1, s), (\varphi_2, s)\}$
$(\Diamond\varphi, s)$	$\exists$	$\{(\varphi, t) \mid t \in R[s]\}$
$(\Box\varphi, s)$	$\forall$	$\{(\varphi, t) \mid t \in R[s]\}$
$(\bot, s)$	$\exists$	$\varnothing$
$(\top, s)$	$\forall$	$\varnothing$
$(p, s), s \in V(p)$	$\forall$	$\varnothing$
$(p, s), s \notin V(p)$	$\exists$	$\varnothing$
$(\neg p, s), s \notin V(p)$	$\forall$	$\varnothing$
$(\neg p, s), s \in V(p)$	$\exists$	$\varnothing$

Game semantics for ML_∇

Position	Player	Legitimate moves
$(\varphi_1 \vee \varphi_2, s)$	$\exists$	$\{(\varphi_1, s), (\varphi_2, s)\}$
$(\varphi_1 \wedge \varphi_2, s)$	$\forall$	$\{(\varphi_1, s), (\varphi_2, s)\}$
$(\nabla\Phi, s)$	$\exists$	$\{Z \subseteq S \times Fmas \mid Z \in \Phi \bowtie R[s]\}$
$Z \subseteq S \times Fmas$	$\forall$	$\{(s, \varphi) \mid (s, \varphi) \in Z\}$
$(\bot, s)$	$\exists$	$\varnothing$
$(\top, s)$	$\forall$	$\varnothing$
$(p, s), s \in V(p)$	$\forall$	$\varnothing$
$(p, s), s \notin V(p)$	$\exists$	$\varnothing$
$(\neg p, s), s \notin V(p)$	$\forall$	$\varnothing$
$(\neg p, s), s \in V(p)$	$\exists$	$\varnothing$

Strategic normal forms

- 'static' distributive law:

$$\varphi \wedge (\psi_1 \vee \psi_2) \quad \equiv \quad (\varphi \wedge \psi_1) \vee (\varphi \wedge \psi_2)$$

$$\forall\exists \qquad \exists\forall$$

Strategic normal forms

- 'static' distributive law:

$$\varphi \wedge (\psi_1 \vee \psi_2) \quad \equiv \quad (\varphi \wedge \psi_1) \vee (\varphi \wedge \psi_2)$$

$\forall\exists$ $\exists\forall$

- modal distributive law:

$$\nabla\Phi \wedge \nabla\Phi' \quad \equiv \quad \bigvee_{Z \in \Phi \bowtie \Phi'} \nabla\{\varphi \wedge \varphi' \mid (\varphi, \varphi') \in Z\}$$

$\forall\exists\forall$ $\exists\exists\forall\forall$

Overview

- Introduction: reorganizing modal logic
- A modal distributive law
- A game-theoretical perspective
- Uniform interpolation
- Automata
- Axiomatizing ∇
- A coalgebraic generalization
- Concluding remarks

Bisimulation Quantifiers

Bisimulation Quantifiers

- Fix set X of proposition letters
- Syntax: if φ is a formula, then so is $\tilde{\exists} p.\varphi$
- Semantics:

$$\mathbb{S}, s \Vdash \exists p.\varphi \text{ iff } \mathbb{S}', s' \Vdash \varphi \text{ for some } \mathbb{S}', s' \leftrightarroweq_p \mathbb{S}, s,$$

where $\leftrightarroweq_p$ denotes bisimilarity wrt $X \setminus \{p\}$-formulas.

Bisimulation Quantifiers

- Fix set X of proposition letters
- Syntax: if φ is a formula, then so is $\tilde{\exists} p.\varphi$
- Semantics:

$$\mathbb{S}, s \Vdash \exists p.\varphi \text{ iff } \mathbb{S}', s' \Vdash \varphi \text{ for some } \mathbb{S}', s' \leftrightarroweq_p \mathbb{S}, s,$$

where $\leftrightarroweq_p$ denotes bisimilarity wrt $X \setminus \{p\}$-formulas.

- Example: $\tilde{\exists} p(\Diamond p \wedge \Diamond \neg p) \equiv \Diamond \top$.

Bisimulation Quantifiers & Uniform interpolation

Proposition Let φ, ψ be modal formulas, p not occurring in ψ. Then

- $\varphi \models \tilde{\exists} p.\varphi$
- $\varphi \models \psi$ iff $\tilde{\exists} p.\varphi \models \psi$

Bisimulation Quantifiers & Uniform interpolation

Proposition Let φ, ψ be modal formulas, p not occurring in ψ. Then

- $\varphi \models \tilde{\exists} p.\varphi$
- $\varphi \models \psi$ iff $\tilde{\exists} p.\varphi \models \psi$

Corollary ('Uniform Interpolation')
Let φ, χ be formulas with

$$\varphi \models \psi.$$

Assume $\mathsf{Var}(\varphi) \setminus \mathsf{Var}(\psi) = \{p_1, \dots, p_n\}$.
Then

$$\varphi \models \tilde{\exists} p_1 \cdots p_n.\varphi \models \psi.$$

Uniform interpolation of ML

Theorem Modal logic has uniform interpolation.

Proof sketch

Uniform interpolation of ML

Theorem Modal logic has uniform interpolation.

Proof sketch

$\tilde{\exists}$ is definable in CML^{-}, and hence in ML:

Uniform interpolation of ML

Theorem Modal logic has uniform interpolation.

Proof sketch

$\tilde{\exists}$ is definable in CML^{-}, and hence in ML:

- $\tilde{\exists} p(\varphi \vee \psi) \equiv \tilde{\exists} p.\varphi \vee \tilde{\exists} p.\psi$

Uniform interpolation of ML

Theorem Modal logic has uniform interpolation.

Proof sketch

$\tilde{\exists}$ is definable in CML^{-}, and hence in ML:

- $\tilde{\exists} p(\varphi \vee \psi) \equiv \tilde{\exists} p.\varphi \vee \tilde{\exists} p.\psi$
- $\tilde{\exists} p.\nabla\Phi \equiv \nabla\tilde{\exists} p.\Phi$

Uniform interpolation of ML

Theorem Modal logic has uniform interpolation.

Proof sketch

$\tilde{\exists}$ is definable in CML^{-}, and hence in ML:

- $\tilde{\exists}p(\varphi \vee \psi) \equiv \tilde{\exists}p.\varphi \vee \tilde{\exists}p.\psi$
- $\tilde{\exists}p.\nabla\Phi \equiv \nabla\tilde{\exists}p.\Phi$
- $\tilde{\exists}p.\odot P \equiv \odot(P \setminus \{p\}) \vee \odot(P \cup \{p\})$

Uniform interpolation of ML

Theorem Modal logic has uniform interpolation.

Proof sketch

$\tilde{\exists}$ is definable in CML^{-}, and hence in ML:

- $\tilde{\exists}p(\varphi \vee \psi) \equiv \tilde{\exists}p.\varphi \vee \tilde{\exists}p.\psi$
- $\tilde{\exists}p.\nabla\Phi \equiv \nabla\tilde{\exists}p.\Phi$
- $\tilde{\exists}p.\odot P \equiv \odot(P \setminus \{p\}) \vee \odot(P \cup \{p\})$
- $\tilde{\exists}p.(P \bullet \Phi) \;\equiv\; P \bullet \tilde{\exists}p.\Phi \,\vee\, (P \cup \{p\}) \bullet \tilde{\exists}p.\Phi$

Overview

- Introduction: reorganizing modal logic
- A modal distributive law
- A game-theoretical perspective
- Uniform interpolation
- Automata
- Axiomatizing ∇
- A coalgebraic generalization
- Concluding remarks

Automata Theory

- automata: finite devices classifying potentially infinite objects
- strong connections with (fixpoint/second order) logic
  Slogan: formulas are automata
- rich history: Büchi, Rabin, Walukiewicz, . . .
- applications in model checking

Automata Theory

- automata: finite devices classifying potentially infinite objects
- strong connections with (fixpoint/second order) logic
  Slogan: formulas are automata
- rich history: Büchi, Rabin, Walukiewicz, . . .
- applications in model checking

Automata can be classified according to

Automata Theory

- automata: finite devices classifying potentially infinite objects
- strong connections with (fixpoint/second order) logic
  Slogan: formulas are automata
- rich history: Büchi, Rabin, Walukiewicz, . . .
- applications in model checking

Automata can be classified according to

- objects on which they operate (words/trees/graphs, . . .)
- transition structure: deterministic/nondeterministic/alternating
- acceptance condition: Büchi/Muller/parity/. . .

A Fundamental Result

- Key result in Rabin's decidability proof for SnS:
  - not the Complementation Lemma, but . . .

A Fundamental Result

- Key result in Rabin's decidability proof for SnS:
  - not the Complementation Lemma, but . . .
  - the simulation of alternating tree automata by nondeterministic ones

A Fundamental Result

- Key result in Rabin's decidability proof for SnS:
  - not the Complementation Lemma, but . . .
  - the simulation of alternating tree automata by nondeterministic ones
- Logically, this corresponds to the elimination of conjunctions

A Fundamental Result

- Key result in Rabin's decidability proof for SnS:
  - not the Complementation Lemma, but . . .
  - the simulation of alternating tree automata by nondeterministic ones
- Logically, this corresponds to the elimination of conjunctions

For the modal μ-calculus,

A Fundamental Result

- Key result in Rabin's decidability proof for SnS:
  - not the Complementation Lemma, but . . .
  - the simulation of alternating tree automata by nondeterministic ones
- Logically, this corresponds to the elimination of conjunctions

For the modal μ-calculus,

- Janin & Walukiewicz introduced modal μ-automata . . .

A Fundamental Result

- Key result in Rabin's decidability proof for SnS:
  - not the Complementation Lemma, but . . .
  - the simulation of alternating tree automata by nondeterministic ones
- Logically, this corresponds to the elimination of conjunctions

For the modal μ-calculus,

- Janin & Walukiewicz introduced modal μ-automata . . .
- . . . and proved a corresponding simulation result . . .

A Fundamental Result

- Key result in Rabin's decidability proof for SnS:
  - not the Complementation Lemma, but . . .
  - the simulation of alternating tree automata by nondeterministic ones
- Logically, this corresponds to the elimination of conjunctions

For the modal μ-calculus,

- Janin & Walukiewicz introduced modal μ-automata . . .
- . . . and proved a corresponding simulation result . . .
- . . . which lies as the heart of all results on the modal μ-calculus.

Automata & Fixpoint Logics

Theorem (Arnold & Niwiński)

Automata & Fixpoint Logics

Theorem (Arnold & Niwiński)

Elimination of conjunction is preserved under adding fixpoint operators!

Automata & Fixpoint Logics

Theorem (Arnold & Niwiński)

Elimination of conjunction is preserved under adding fixpoint operators!

Hence, by the modal distributive law, conjunctions can be eliminated from the modal μ-calculus.

Corollary (Janin & Walukiewicz)
μML and μCML^{-} (based on $\bigvee, \bullet$) are effectively equi-expressive.

Axiomatizing Fixpoint Logics

(joint work with Luigi Santocanale)

- A connective $\sharp(p_1, \ldots, p_n)$ is a flat fixpoint connective if its semantics is given by the least fixpoint of a modal formula $\gamma(x, p_1, \ldots, p_n)$:

$$\sharp(p_1, \ldots, p_n) \equiv \mu x.\gamma(x, p_1, \ldots, p_n)$$

- Examples: $\langle * \rangle p \equiv \mu x.p \vee \Diamond x$, $pUq \equiv \mu x.q \vee (p \wedge \Diamond x)$.

Axiomatizing Fixpoint Logics

(joint work with Luigi Santocanale)

- A connective $\sharp(p_1, \ldots, p_n)$ is a flat fixpoint connective if its semantics is given by the least fixpoint of a modal formula $\gamma(x, p_1, \ldots, p_n)$:

$$\sharp(p_1, \ldots, p_n) \equiv \mu x.\gamma(x, p_1, \ldots, p_n)$$

- Examples: $\langle * \rangle p \equiv \mu x.p \vee \Diamond x$, $pUq \equiv \mu x.q \vee (p \wedge \Diamond x)$.
- Given set Γ of modal formulas, ML_Γ is extension of ML with $\{\sharp_\gamma \mid \gamma \in \Gamma\}$.
- Example: CTL.

Axiomatizing Fixpoint Logics

(joint work with Luigi Santocanale)

- A connective $\sharp(p_1, \ldots, p_n)$ is a flat fixpoint connective if its semantics is given by the least fixpoint of a modal formula $\gamma(x, p_1, \ldots, p_n)$:

$$\sharp(p_1, \ldots, p_n) \equiv \mu x.\gamma(x, p_1, \ldots, p_n)$$

- Examples: $\langle * \rangle p \equiv \mu x.p \vee \Diamond x$, $pUq \equiv \mu x.q \vee (p \wedge \Diamond x)$.
- Given set Γ of modal formulas, ML_Γ is extension of ML with $\{\sharp_\gamma \mid \gamma \in \Gamma\}$.
- Example: CTL.

Theorem
Sound and complete axiom systems for ML_Γ, uniform and effective in Γ.

Overview

- Introduction: reorganizing modal logic
- A modal distributive law
- A game-theoretical perspective
- Uniform interpolation
- Automata
- Axiomatizing ∇
- A coalgebraic generalization
- Concluding remarks

Axiomatizing ∇

(joint work with Alessandra Palmigiano)

Axiomatizing ∇

(joint work with Alessandra Palmigiano)

- (Equi-expressiveness with ML trivially provides axiomatization)

Axiomatizing ∇

(joint work with Alessandra Palmigiano)

- (Equi-expressiveness with ML trivially provides axiomatization)
- Aim: Axiomatize ∇ ‘in its own terms’

Axiomatizing ∇

(joint work with Alessandra Palmigiano)

- (Equi-expressiveness with ML trivially provides axiomatization)
- Aim: Axiomatize ∇ 'in its own terms'
- Observation: axiomatization of ∇ is independent to that of negation
- Change setting to positive modal logic: ($=$ $\neg$-free residu of classical ML)

Axiomatizing ∇

(joint work with Alessandra Palmigiano)

- (Equi-expressiveness with ML trivially provides axiomatization)
- Aim: Axiomatize ∇ 'in its own terms'
- Observation: axiomatization of ∇ is independent to that of negation
- Change setting to positive modal logic: ($= \neg$-free residu of classical ML)
- Our approach is algebraic.

Algebraic approach

- Positive modal algebra: structure $\mathbb{A} = \langle A, \wedge, \vee, \top, \bot, \Diamond, \Box \rangle$ with
  - $A := \langle A, \wedge, \vee, \top, \bot \rangle$ a distributive lattice, and
  - $\Box, \Diamond$ unary operations on A satisfying:

$$\begin{array}{rclcc} \Diamond(a \vee b) & = & \Diamond a \vee \Diamond b & & \Diamond\bot = \bot \\ \Box(a \wedge b) & = & \Box a \wedge \Box b & & \Box\top = \top \\ \Box a \wedge \Diamond b & \leq & \Diamond(a \wedge b) & & \\ \Box(a \vee b) & \leq & \Box a \vee \Diamond b & & \end{array}$$

- Modal algebra: $\mathbb{A} = \langle A, \wedge, \vee, \top, \bot, \neg, \Diamond, \Box \rangle$ with
  - $\langle A, \wedge, \vee, \top, \bot, \neg \rangle$ a Boolean algebra
  - $\Box$ and $\Diamond$ satisfy, in addition to the axioms above: $\neg\Diamond a = \Box\neg a$.

Axioms for ∇

Positive modal ∇-algebra: $\mathbb{A} = \langle A, \wedge, \vee, \top, \bot, \nabla \rangle$ with

- $\langle A, \wedge, \vee, \top, \bot \rangle$ a distributive lattice, and ∇ satisfying
- $\nabla 1.$ If $\leq$ is full on α and β, then $\nabla\alpha \leq \nabla\beta$,
  $\nabla 2a.$ $\nabla\alpha \wedge \nabla\beta \leq \bigvee\{\nabla\{a \wedge b \mid (a,b) \in Z\} \mid Z \in \alpha \bowtie \beta\}$,
  $\nabla 2b.$ $\top \leq \nabla\varnothing \vee \nabla\{\top\}$,
  $\nabla 3a.$ If $\bot \in \alpha$, then $\nabla\alpha \leq \bot$,
  $\nabla 3b.$ $\nabla\alpha \cup \{a \vee b\} \;\leq\; \nabla(\alpha \cup \{a\}) \vee \nabla(\alpha \cup \{b\}) \vee \nabla(\alpha \cup \{a,b\})$.

Modal ∇-algebra: $\mathbb{A} = \langle A, \wedge, \vee, \top, \bot, \neg, \nabla \rangle$ with

- $\langle A, \wedge, \vee, \top, \bot, \neg \rangle$ a Boolean algebra, and ∇ satisfying $\nabla 1$ – $\nabla 3$ and:
- $\nabla 4.$ $\neg\nabla\alpha = \nabla\{\bigwedge \neg\alpha, \top\} \vee \nabla\varnothing \vee \bigvee\{\nabla\{\neg a\} \mid a \in \alpha\}$.

Results

- Given a PMA $A = \langle A, \wedge, \vee, \top, \bot, \Diamond, \Box \rangle$, define $\nabla \alpha := \Box \bigvee \alpha \wedge \bigwedge \Diamond \alpha$, and put $A^{\nabla} := \langle A, \wedge, \vee, \top, \bot, \nabla \rangle$.
- Conversely, given a PMA$_{\nabla}$ $\langle B, \wedge, \vee, \top, \bot, \nabla)\rangle$, define $\Diamond a := \nabla\{a, \top\}$ and $\Box a := \nabla\varnothing \vee \nabla\{a\}$, and put $B^{\Diamond} := \langle B, \wedge, \vee, \top, \bot, \Diamond, \Box \rangle$.
- Extend to maps: $f^{\nabla} := f$ and $f^{\Diamond} := f$ whenever applicable.

Theorem The functors $(\cdot)^{\nabla}$ and $(\cdot)^{\Diamond}$
- give a categorical isomorphism between the categories PMA and PMA$_{\nabla}$,
- and similarly for the categories MA and MA$_{\nabla}$.

Results

- Given a PMA $A = \langle A, \wedge, \vee, \top, \bot, \Diamond, \Box \rangle$, define $\nabla\alpha := \Box \bigvee \alpha \wedge \bigwedge \Diamond\alpha$, and put $A^{\nabla} := \langle A, \wedge, \vee, \top, \bot, \nabla \rangle$.
- Conversely, given a PMA$_{\nabla}$ $\langle B, \wedge, \vee, \top, \bot, \nabla) \rangle$, define $\Diamond a := \nabla\{a, \top\}$ and $\Box a := \nabla\varnothing \vee \nabla\{a\}$, and put $B^{\Diamond} := \langle B, \wedge, \vee, \top, \bot, \Diamond, \Box \rangle$.
- Extend to maps: $f^{\nabla} := f$ and $f^{\Diamond} := f$ whenever applicable.

Theorem The functors $(\cdot)^{\nabla}$ and $(\cdot)^{\Diamond}$

- give a categorical isomorphism between the categories PMA and PMA$_{\nabla}$,
- and similarly for the categories MA and MA$_{\nabla}$.

Corollary $\nabla 1$ – $\nabla 4$ form a complete axiomatization of ∇.

Results

- Given a PMA $A = \langle A, \wedge, \vee, \top, \bot, \Diamond, \Box \rangle$, define $\nabla \alpha := \Box \bigvee \alpha \wedge \bigwedge \Diamond \alpha$, and put $A^{\nabla} := \langle A, \wedge, \vee, \top, \bot, \nabla \rangle$.
- Conversely, given a PMA$_{\nabla}$ $\langle B, \wedge, \vee, \top, \bot, \nabla) \rangle$, define $\Diamond a := \nabla \{a, \top\}$ and $\Box a := \nabla \varnothing \vee \nabla \{a\}$, and put $B^{\Diamond} := \langle B, \wedge, \vee, \top, \bot, \Diamond, \Box \rangle$.
- Extend to maps: $f^{\nabla} := f$ and $f^{\Diamond} := f$ whenever applicable.

Theorem The functors $(\cdot)^{\nabla}$ and $(\cdot)^{\Diamond}$

- give a categorical isomorphism between the categories PMA and PMA$_{\nabla}$,
- and similarly for the categories MA and MA$_{\nabla}$.

Corollary $\nabla 1$ – $\nabla 4$ form a complete axiomatization of ∇.

Corollary Description of topological Vietoris construction in terms of ∇.

Carioca Axioms for ∇

(joint work with Marta Bilkova & Alessandra Palmigiano)

A set $B \in \wp\wp(S)$ is a full redistribution of a set $A \in \wp\wp(S)$ if

- $\bigcup B = \bigcup A$
- $\beta \cap \alpha \neq \varnothing$ for all $\beta \in B$ and all $\alpha \in A$

The set of redistributions of A is denoted as $FRDB(A)$.

∇-Axioms:

$$\text{If } \leq \text{ is full on } \alpha \text{ and } \beta\text{, then } \nabla\alpha \leq \nabla\beta. \tag{∇1}$$

$$\bigwedge \left\{\nabla\alpha \mid \alpha \in A\right\} \leq \bigvee \left\{\nabla\{\bigwedge\beta \mid \beta \in B\} \;\mid\; B \in FRDB(A)\right\} \tag{∇2}$$

$$\nabla\{\bigvee\alpha \mid \alpha \in A\} \leq \bigvee\{\nabla\beta \mid \in \text{ is full on } \beta \text{ and } A\}. \tag{∇3}$$

Overview

- Introduction: reorganizing modal logic
- A modal distributive law
- A game-theoretical perspective
- Uniform interpolation
- Automata
- Axiomatizing ∇
- A coalgebraic generalization
- Concluding remarks

Almost all of this has been generalized to the level of coalgebras

(for weak pullback-preserving set functors)

Almost all of this has been generalized to the level of coalgebras

(for weak pullback-preserving set functors)

(partly joint work with Clemens Kupke & Alexander Kurz)

Kripke Structures as Coalgebras

Kripke Structures as Coalgebras

- Represent $R \subseteq S \times S$ as map $\sigma_R : S \to \wp(S)$:

$$\sigma_R(s) := \{t \in S \mid Rst\}.$$

- Kripke frame $\langle S, R \rangle \sim$ coalgebra $\langle S, \sigma_R \rangle$

Kripke Structures as Coalgebras

- Represent $R \subseteq S \times S$ as map $\sigma_R : S \to \wp(S)$:

$$\sigma_R(s) := \{t \in S \mid Rst\}.$$

- Kripke frame $\langle S, R\rangle \sim$ coalgebra $\langle S, \sigma_R\rangle$
- Kripke model = Kripke frame + assignment (valuation)
- A valuation is a map $V : \mathsf{X} \to \wp(S)$

Kripke Structures as Coalgebras

- Represent $R \subseteq S \times S$ as map $\sigma_R : S \to \wp(S)$:

$$\sigma_R(s) := \{t \in S \mid Rst\}.$$

- Kripke frame $\langle S, R \rangle \sim$ coalgebra $\langle S, \sigma_R \rangle$
- Kripke model $=$ Kripke frame $+$ assignment (valuation)
- A valuation is a map $V : \mathsf{X} \to \wp(S)$
- represent this as a map $\sigma_V : S \to \wp(\mathsf{X})$:

$$\sigma_V(s) := \{p \in \mathsf{X} \mid s \in V(p)\}.$$

Kripke Structures as Coalgebras

- Represent $R \subseteq S \times S$ as map $\sigma_R : S \to \wp(S)$:

$$\sigma_R(s) := \{t \in S \mid Rst\}.$$

- Kripke frame $\langle S, R\rangle \sim$ coalgebra $\langle S, \sigma_R\rangle$
- Kripke model $=$ Kripke frame $+$ assignment (valuation)
- A valuation is a map $V : \mathsf{X} \to \wp(S)$
- represent this as a map $\sigma_V : S \to \wp(\mathsf{X})$:

$$\sigma_V(s) := \{p \in \mathsf{X} \mid s \in V(p)\}.$$

- Combine σ_V and σ_R into map $\sigma_{V,R} : S \to \wp(\mathsf{X}) \times \wp(S)$:
- Kripke model $\langle S, R, V\rangle \sim$ coalgebra $\langle S, \sigma_{V,R}\rangle$

Coalgebra

- Coalgebra is

  a general mathematical theory for evolving state-based systems

Coalgebra

- Coalgebra is
  a general mathematical theory for evolving state-based systems
- It provides a natural framework for notions like
  - behavior

Coalgebra

- Coalgebra is
  a general mathematical theory for evolving state-based systems
- It provides a natural framework for notions like
  - behavior
  - bisimulation/behavioral equivalence

Coalgebra

- Coalgebra is
  a general mathematical theory for evolving state-based systems
- It provides a natural framework for notions like
  - behavior
  - bisimulation/behavioral equivalence
  - invariants

Coalgebra

- Coalgebra is
  a general mathematical theory for evolving state-based systems
- It provides a natural framework for notions like
  - behavior
  - bisimulation/behavioral equivalence
  - invariants
- A coalgebra is a structure $\mathbb{S} = \langle S, \sigma : S \to \mathsf{F}S \rangle$,
  where F is the type of the coalgebra.

Coalgebra

- Coalgebra is

  a general mathematical theory for evolving state-based systems

- It provides a natural framework for notions like

  - behavior
  - bisimulation/behavioral equivalence
  - invariants

- A coalgebra is a structure $\mathbb{S} = \langle S, \sigma : S \to \mathsf{F}S \rangle$,
  where F is the type of the coalgebra.

- Sufficiently general to model notions like:
  input, output, non-determinism, interaction, probability, ...

Coalgebra

- Coalgebra is

  a general mathematical theory for evolving state-based systems

- It provides a natural framework for notions like

  - behavior
  - bisimulation/behavioral equivalence
  - invariants

- A coalgebra is a structure $\mathbb{S} = \langle S, \sigma : S \to \mathsf{F}S \rangle$, where F is the type of the coalgebra.

- Sufficiently general to model notions like: input, output, non-determinism, interaction, probability, . . .

- Type of Kripke models is K_{X}, with $\mathsf{K}_{\mathsf{X}} S = \wp(\mathsf{X}) \times \wp(S)$
  Type of Kripke frames is K, with $\mathsf{K}S = \wp(S)$

Examples

- C-streams: $\mathsf{F}S = C \times S$
- finite words: $\mathsf{F}S = C \times (S \uplus \{\downarrow\})$
- finite trees: $\mathsf{F}S = C \times ((S \times S) \uplus \{\downarrow\})$
- deterministic automata: $\mathsf{F}S = \{0,1\} \times S^C$
- labeled transition systems: $\mathsf{F}S = (\wp S)^A$
- (non-wellfounded) sets: $\mathsf{F}S = \wp S$
- topologies: $\mathsf{F}S = \wp\wp(S)$

Coalgebra and Modal Logic

Coalgebra and Modal Logic

- Coalgebras are a natural generalization of Kripke structures

Coalgebra and Modal Logic

- Coalgebras are a natural generalization of Kripke structures
-

$$\frac{\text{Modal Logic}^{*}}{\text{Coalgebra}} = \frac{\text{Equational Logic}}{\text{Algebra}}$$

Coalgebra and Modal Logic

- Coalgebras are a natural generalization of Kripke structures
-

$$\frac{\text{Modal Logic}^{*}}{\text{Coalgebra}} = \frac{\text{Equational Logic}}{\text{Algebra}}$$

* with fixpoint operators

Relation Lifting

- $\mathsf{K}S := \wp(S)$
- Kripke frame is pair $\langle S, \sigma : S \to \mathsf{K}S \rangle$
- Lift $Z \subseteq S \times S'$ to $\overline{\mathsf{K}}(Z) \subseteq \mathsf{K}S \times \mathsf{K}S'$:

$$\overline{\mathsf{K}}(Z) \ := \ \{(T, T') \mid \forall t \in T \exists t' \in T'.Ztt' \text{ and } \forall t' \in T' \exists t \in T.Ztt'\}$$

- Z is full on T and T' iff $(T, T') \in \overline{\mathsf{K}}(Z)$.

Relation Lifting

- $\mathsf{K}S := \wp(S)$
- Kripke frame is pair $\langle S, \sigma : S \to \mathsf{K}S \rangle$
- Lift $Z \subseteq S \times S'$ to $\overline{\mathsf{K}}(Z) \subseteq \mathsf{K}S \times \mathsf{K}S'$:

$$\overline{\mathsf{K}}(Z) \;:=\; \{(T, T') \mid \forall t \in T \exists t' \in T'.Ztt' \text{ and } \forall t' \in T' \exists t \in T.Ztt'\}$$

- Z is full on T and T' iff $(T, T') \in \overline{\mathsf{K}}(Z)$.

Proposition

- Z is a bisimulation iff $(\sigma(s), \sigma'(s')) \in \overline{\mathsf{K}}(Z)$ for all $(s, s') \in Z$.

Relation Lifting

- $\mathsf{K}S := \wp(S)$
- Kripke frame is pair $\langle S, \sigma : S \to \mathsf{K}S \rangle$
- Lift $Z \subseteq S \times S'$ to $\overline{\mathsf{K}}(Z) \subseteq \mathsf{K}S \times \mathsf{K}S'$:

$$\overline{\mathsf{K}}(Z) \;:=\; \{(T,T') \mid \forall t \in T \exists t' \in T'.Ztt' \text{ and } \forall t' \in T' \exists t \in T.Ztt'\}$$

- Z is full on T and T' iff $(T,T') \in \overline{\mathsf{K}}(Z)$.

Proposition

- Z is a bisimulation iff $(\sigma(s), \sigma'(s')) \in \overline{\mathsf{K}}(Z)$ for all $(s,s') \in Z$.
- $\mathbb{S}, s \Vdash \nabla\Phi$ iff $(\sigma(s), \Phi) \in \overline{\mathsf{K}}(\Vdash)$.

Moss' Coalgebraic Logic

- Moss: generalize this to (almost) arbitrary functor

Moss' Coalgebraic Logic

- Moss: generalize this to (almost) arbitrary functor
- Define the language $\mathsf{CML}_{\mathsf{F}}$ by

$$\varphi ::= \bot \mid \top \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \nabla_{\mathsf{F}} \alpha$$

where $\alpha \in \mathsf{F}(Fma)$

Moss' Coalgebraic Logic

- Moss: generalize this to (almost) arbitrary functor
- Define the language $\mathsf{CML}_{\mathsf{F}}$ by

$$\varphi \ ::= \ \bot \mid \top \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \nabla_{\mathsf{F}}\alpha$$

where $\alpha \in \mathsf{F}(Fma)$

- Semantics: $\mathbb{S}, s \Vdash \nabla_{\mathsf{F}}\alpha$ iff $(\sigma(s), \alpha) \in \overline{\mathsf{F}}(\Vdash)$.

Moss' Coalgebraic Logic

- Moss: generalize this to (almost) arbitrary functor
- Define the language $\mathsf{CML}_{\mathsf{F}}$ by

$$\varphi \ ::= \ \bot \mid \top \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \nabla_{\mathsf{F}} \alpha$$

  where $\alpha \in \mathsf{F}(Fma)$

- Semantics: $\mathbb{S}, s \Vdash \nabla_{\mathsf{F}} \alpha$ iff $(\sigma(s), \alpha) \in \overline{\mathsf{F}}(\Vdash)$.
- The 'nabla for Kripke **models**' is: •!

The coalgebraic distributive law

- Consider $\bigwedge : \wp(Fma) \to Fma$, then $\mathsf{F}\bigwedge : \mathsf{F}\wp(Fma) \to \mathsf{F}Fma$

The coalgebraic distributive law

- Consider $\bigwedge : \wp(Fma) \to Fma$, then $\mathsf{F}\bigwedge : \mathsf{F}\wp(Fma) \to \mathsf{F}Fma$
- $\Xi \in \mathsf{F}\wp S$ is a redistribution of $A \in \wp\mathsf{F}S$ if $\alpha(\overline{\mathsf{F}}{\in_S})\Xi$, for all $\alpha \in A$.

$$\bigwedge\{\nabla_{\mathsf{F}}\alpha \mid \alpha \in A\} \;\equiv\; \bigvee\{\nabla_{\mathsf{F}}(\mathsf{F}\bigwedge)(\Xi) \mid \Xi \text{ a redistribution of } A\}$$

Axioms for ∇_F

(joint work with Clemens Kupke & Alexander Kurz)

Axioms for ∇_{F}

(joint work with Clemens Kupke & Alexander Kurz)

- Consider $\bigwedge, \bigvee : \wp(Fma) \to Fma$, then $\mathsf{F}\bigwedge : \mathsf{F}\wp(Fma) \to \mathsf{F}Fma$
- $\Xi \in \mathsf{F}\wp S$ is a redistribution of $A \in \wp\mathsf{F}S$ if $\alpha(\overline{\mathsf{F}}\in_S)\Xi$, for all $\alpha \in A$.

Axioms:

$$\text{From } \alpha\overline{\mathsf{F}}(\leq)\beta \text{ derive } \nabla\alpha \leq \nabla\beta. \tag{$\nabla 1$}$$

$$\bigwedge\{\nabla_{\mathsf{F}}\alpha \mid \alpha \in A\} = \bigvee\{\nabla_{\mathsf{F}}(\mathsf{F}\bigwedge)(\Xi) \mid \Xi \text{ a redistribution of } A\} \tag{$\nabla 2$}$$

$$\nabla\{\bigvee\alpha \mid \alpha \in A\} \leq \bigvee\{\nabla\beta \mid \beta\overline{\mathsf{F}}(\in)A\}. \tag{$\nabla 3$}$$

Axioms for ∇_{F}

(joint work with Clemens Kupke & Alexander Kurz)

- Consider $\bigwedge, \bigvee : \wp(Fma) \to Fma$, then $\mathsf{F}\bigwedge : \mathsf{F}\wp(Fma) \to \mathsf{F}Fma$
- $\Xi \in \mathsf{F}\wp S$ is a redistribution of $A \in \wp\mathsf{F}S$ if $\alpha(\overline{\mathsf{F}}\in_S)\Xi$, for all $\alpha \in A$.

Axioms:

$$\text{From } \alpha\overline{\mathsf{F}}(\leq)\beta \text{ derive } \nabla\alpha \leq \nabla\beta. \tag{∇1}$$

$$\bigwedge\{\nabla_{\mathsf{F}}\alpha \mid \alpha \in A\} = \bigvee\{\nabla_{\mathsf{F}}(\mathsf{F}\bigwedge)(\Xi) \mid \Xi \text{ a redistribution of } A\} \tag{∇2}$$

$$\nabla\{\bigvee\alpha \mid \alpha \in A\} \leq \bigvee\{\nabla\beta \mid \beta\overline{\mathsf{F}}(\in)A\}. \tag{∇3}$$

Completeness is on its way

Overview

- Introduction: reorganizing modal logic
- A modal distributive law
- A game-theoretical perspective
- Uniform interpolation
- Automata
- Axiomatizing ∇
- A coalgebraic generalization
- Concluding remarks

Concluding remarks

The modal distributive law is a fundamental principle,

Concluding remarks

The modal distributive law is a fundamental principle,

with many applications/manifestations:

Concluding remarks

The modal distributive law is a fundamental principle,

with many applications/manifestations:

- logic

Concluding remarks

The modal distributive law is a fundamental principle,

with many applications/manifestations:

- logic
- game theory

Concluding remarks

The modal distributive law is a fundamental principle,

with many applications/manifestations:

- logic
- game theory
- automata theory

Concluding remarks

The modal distributive law is a fundamental principle,

with many applications/manifestations:

- logic
- game theory
- automata theory
- coalgebra

Concluding remarks

The modal distributive law is a fundamental principle,

with many applications/manifestations:

- logic
- game theory
- automata theory
- coalgebra
- . . .

Further research

Further research

- proof theory

Further research

- proof theory
- completeness for fixpoint logics

Further research

- proof theory
- completeness for fixpoint logics
- algebraic aspects of ∇

Further research

- proof theory
- completeness for fixpoint logics
- algebraic aspects of ∇
- logics for coalgebra

Further research

- proof theory
- completeness for fixpoint logics
- algebraic aspects of ∇
- logics for coalgebra
- role of negation

Further research

- proof theory
- completeness for fixpoint logics
- algebraic aspects of ∇
- logics for coalgebra
- role of negation
- constructive content