A Co-Variety-Theorem for Modal Logic

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ABSTRACT. Coalgebra is dual to algebra. But what—in this sense—is dual to equational logic? Here, modal logic is proposed. A Birkhoff-like covariety theorem for certain kinds of modal logics is proved. Then it is shown that this result specialises to Moss' coalgebraic logic (with propositional variables added) and to infinitary modal logic: the modally definable classes of Kripke frames are those closed under images, subframes and disjoint unions.

1 Introduction

Coalgebras as a generalisation of transition systems have proved to be useful in a variety of application areas, especially in computer science. (For an overview and further references see Jacobs et al. 1998.) In Rutten 1996 the approach is taken to dualise a considerable part of universal algebra. At the end of the paper covarieties are defined and it is shown how they are linked to cofree coalgebras. But the question what a coalgebraic analogue to equational logic could be is left open. Also, Jacobs 1995 raises the question of a Birkhoff-like covariety theorem.

The approach taken here may be summarised by the following slogan: Modal logic is the dual of equational logic. Why that? Trying to dualise Birkhoff's variety theorem, it turns out that a "co-equational" logic should have the following properties: formulas have an interpretation as subsets (of carriers) of coalgebras and formulas have to be invariant under bisimulations. Logics sharing these properties will be called modal logics for coalgebras. This notion of a modal logic is rather unspecific but sufficient to give a proof of a covariety theorem. We then specialise to more "concrete" logics like Moss' coalgebraic logic and infinitary modal logic.

The structure of this paper is as follows. We first recall the basic definitions of coalgebras and cofree coalgebras (section 2). Then modal logics for coalgebras are defined (section 3) and a dualised version of Birkhoff's variety theorem is proved (section 4). Then it is shown that this result carries over to a version of Moss' coalgebraic logic (section 5). Last, it is shown by an example how this can be applied to more familiar modal logics in order to characterise definable classes of Kripke frames (section 6).

I want to add a few historical notes. The idea to link coalgebras and modal logic goes at least back to Barwise and Moss 1996. That this idea is promising also in computer science applications is shown in Kurz 1998. The search for a covariety theorem has been rather popular lately, but the only work I am aware of using modal logic is Gumm and Schröder 1998. They prove a covariety theorem for what they call complete covarieties, that is, for the case $C = 1$ (i.e., no variables) in our terminology .

2 Coalgebras

Coalgebras are given w.r.t. a category $\mathcal{C}$ and an endofunctor $\Omega : \mathcal{C} \to \mathcal{C}$, called the signature or the type of the coalgebra. In what follows, $\mathcal{C}$ will be the category **Set** with sets as objects and functions as morphisms. An Ω-coalgebra $M = (UM, f_M)$ is then given by a set UM and a function $f_M : UM \to \Omega(UM)$ between sets. The Ω-coalgebras form a category $\mathbf{Set}_\Omega$ where a morphism $\alpha : (UM, f) \to (UN, g)$ is a function $\alpha : UM \to UN$ such that $\Omega\alpha \circ f = g \circ \alpha$.

As an example consider the functor Ω given by $\Omega X = \mathcal{P}P \times \mathcal{P}X$ where P is a set of propositional variables and $\mathcal{P}$ denotes the powerset.[1] Then the Ω-coalgebras are the usual *Kripke models*: given a coalgebra M and a world $x \in UM$, we have $f_M(x) = (Q, X)$ where $Q \subset P$ and $X \subset UM$. This determines a Kripke model as follows: Q is the set of propositional variables true in the world x and X is the set of worlds accessible from x. Therefore, in the following we also use the notion Ω-*model* for Ω-coalgebra. The case $P = \{\}$ yields as Ω-coalgebras the *Kripke frames.* To justify that it is useful to consider Ω to be a functor (and not only an operator on sets) we make the important observation that *the notion of a morphism between two Ω-coalgebras coincides with the usual notion of a functional bisimulation or p-morphism.*

Last, a technical requirement: all the functors considered in this paper are supposed to preserve weak pullbacks (this is needed in the proof of

[1]This only defines Ω on sets. On functions Ω is defined in the standard way, $\mathcal{P}$ being the covariant powerset functor: Given $f : X \to Y$, $\Omega f \equiv \mathrm{id}_{\mathcal{P}P} \times \lambda A \in \mathcal{P}X.\{f(a) : a \in A\}$.

proposition 3.3).

2.1 Cofree Coalgebras

We first give the definitions (taken from Rutten 1996) and then show how they are interpreted in the context of modal logic. To define the notion of a *cofree coalgebra* consider the lefthand diagram:

$$
\begin{tikzcd}
TC \arrow[d, "\epsilon_C"'] & \\
C & M \arrow[l, "\alpha"] \arrow[ul, "\alpha^{\#}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
TC \arrow[d, "\epsilon_C"'] & F_K C \arrow[l, "\tau_K^C"'] \\
C & M \arrow[l, "\alpha"] \arrow[ul, "\alpha^{\#}"'] \arrow[u, "\alpha^*"']
\end{tikzcd}
$$

Let Ω be a functor. An Ω-coalgebra TC (and a mapping $\epsilon_C : UTC \to C$) is called *cofree on* C iff for all Ω-coalgebras M and all mappings $\alpha : UM \to C$ there is a unique morphism $\alpha^{\#} : M \to TC$ such that the diagram commutes.

For the definition of a coalgebra *cofree on* C *for a class* K of coalgebras consider the righthand diagram. The Ω-coalgebra $F_K C$ (and the morphism $\tau_K^C : F_K C \to TC$) is cofree for K iff for any Ω-coalgebra $M \in K$ and any $\alpha : UM \to C$ the unique morphism $\alpha^{\#} : M \to TC$ factors through τ_K^C, i.e., there is a unique $\alpha^* : M \to F_K C$ such that $\alpha^{\#} = \tau_K^C \circ \alpha^*$.[2]

An important observation about the diagrams above is the following: Let $M = (UM, f)$ be an Ω-coalgebra and $\alpha : UM \to C$ a colouring. Then α and f determine a function $\langle \alpha, f \rangle : UM \to C \times \Omega(UM)$ that makes $(UM, \langle \alpha, f \rangle)$ into a $C \times \Omega$-coalgebra. It will be denoted by (M, α). Note that a morphism between $C \times \Omega$-coalgebras (M, α) and (N, β) is an Ω-morphism $\gamma : M \to N$ that additionally satisfies $\beta \circ \gamma = \alpha$. Vice versa, every $C \times \Omega$-coalgebra $M = (UM, f)$ can be considered as an Ω-coalgebra $(UM, \pi_2 \circ f)$ together with a colouring $\pi_1 \circ f : UM \to C$, where π_1, π_2 denote the projections from the product $C \times \Omega$ to its components. A consequence of this correspondence of Ω-coalgebras plus colourings and $C \times \Omega$-coalgebras is for example that the cofree coalgebra on C plus its colouring, (TC, ϵ_C), is the final $C \times \Omega$-coalgebra.

As an example consider $\Omega = \mathcal{P}_\omega$, $\mathcal{P}_\omega X$ being the set of finite subsets of X, and $C = \mathcal{P}P$ as above and let M be an Ω-coalgebra. M is an image-finite Kripke frame. The functions $\alpha : UM \to C$ are valuations:

[2] Categorically speaking, the lefthand diagram tells that T, conceived as a functor $\mathbf{Set} \to \mathbf{Set}_\Omega$, is right adjoint to the forgetful functor $U : \mathbf{Set}_\Omega \to \mathbf{Set}$. In the case of K being a covariety, the righthand diagram says that the forgetful functor restricted to K, $U_K : K \to \mathbf{Set}$, has a right adjoint F_K.

every world in M is assigned the set of propositional variables holding in it. This matches the above observation that (M, α) is a $C \times \Omega$-coalgebra, i.e., a Kripke model. The lefthand diagram then shows that any Kripke model (M, α) has a unique p-morphic image in the model (TC, ϵ_C). We can think of (TC, ϵ_C) as the disjoint union of all models based on Ω-frames with the additional feature that any two bisimilar worlds are identified.Ω-coalgebras (frames). Then $(F_K C, \epsilon_C \circ \tau_K^C)$ similarly can be thought of as (up to bisimulation) the set of all models based on frames in K.

A natural question is what closure properties a class K has to have such that it is determined by its cofree coalgebras. The answer is dual to universal algebra: it has to be a covariety (Rutten 1996).

Definition 2.1 (Covariety). Let $\mathrm{S}, \mathrm{H}, \Sigma$ be the following operators on classes of Ω-coalgebras: closure under subcoalgebras, images of morphisms, and disjoint unions, respectively. V denotes closure under all of these three operators. A class K is called a *covariety* iff $\mathrm{V}K = K$.

Continuing our example, note that the closure V corresponds to closure under generated subframes, p-morphic images and disjoint unions.

The next propositions prove the existence of cofree coalgebras for classes K of coalgebras and that covarieties are generated by their cofree coalgebras. In universal algebra, in order to obtain the free algebra $F_K X$ over a set of variables X for a class of algebras K, one takes the quotient of the term algebra TX with respect to $\bigcap\{\ker(f) \mid f : TX \to A, A \in K\}$ where $\ker(f)$ denotes the kernel of f. The role played by the kernels of morphisms in algebra is dual to the role played by images of morphisms in coalgebra.[3] The image in **Set** of a morphism f is denoted by $\operatorname{Im} f$.

The next result is a special case of Rutten 1996, theorem 14.3, where the reader may find a more complete proof. I only sketch the ideas that will be most important in the sequel.

Proposition 2.2 (Existence of cofree coalgebras).
Let Ω be a functor and C a set such that the functor $C \times \Omega$ has a final coalgebra. Then for any class K of Ω-coalgebras there is a coalgebra $F_K C$ that is cofree on C for K.

Proof. Let TC together with $\epsilon_C : UTC \to C$ be the final $C \times \Omega$-coalgebra. Define $F_K C$ to be the coalgebra with carrier $\bigcup\{\operatorname{Im} f \mid f : M \to TC, M \in K\}$ and τ_K^C to be the inclusion.[4] To see that $F_K C$ is cofree for K, suppose $M \in K$ and consider the righthand diagram above: For every $\alpha : UM \to C$

[3]The duality of relations and predicates or congruences and "mongruences" also plays a central role in Jacobs 1995.

[4]That this defines indeed a coalgebra follows from Rutten 1996, theorems 6.3 and 6.4.

there is $\alpha^{\#} : M \to TC$. By definition of $F_K C$, $\operatorname{Im}\alpha^{\#} \subset \operatorname{Im}\tau_K^C$. Since τ_K^C is mono there is a unique morphism $\alpha^* : M \to F_K C$ with $\tau_K^C \circ \alpha^* = \alpha^{\#}$. $\square$

A functor Ω is *bounded* if there is a set D such that for every Ω-coalgebra M and every $x \in UM$ there is a subcoalgebra N of M such that $x \in UN$ and the cardinality of UN is not greater than the cardinality of D.

The next proposition is quite useful.

Proposition 2.3. *Let Ω be a functor bounded by D and C a set. Then $C \times \Omega$ is bounded by D.*

Proof. Let $M = (UM, f)$ be a $C \times \Omega$-coalgebra and $x \in UM$. Recall that $(UM, \pi_2 \circ f)$ is an Ω-coalgebra. Hence it has an Ω-subcoalgebra N such that $x \in UN$ and $|UN| \leq |D|$. Let α be the restriction of $\pi_1 \circ f$ to UN. It follows that (N, α) is a $C \times \Omega$-subcoalgebra of M containing x and bounded by D. $\square$

Now we can show the important fact that covarieties of bounded functors are generated by a single cofree coalgebra. This is a variation of Rutten 1996, theorem 15.3, where again a complete proof may be found.

Proposition 2.4. *Let Ω be a functor bounded by D and K a covariety of Ω-coalgebras. Then $K = \mathrm{V}\{F_K D\}$.*

Proof. Every bounded functor has a final coalgebra (Rutten 1996, theorem 10.4). It follows from the above propositions that Ω has cofree coalgebras for any class of Ω-coalgebras and for any set C.
"$\supset$": From the proof of proposition 2.2, it follows that $F_K D \in \mathrm{H}\Sigma\mathrm{H}(K)$. Since K a covariety, also $\mathrm{V}\{F_K D\} \subset K$.
"$\subset$": Let $M \in K$ and $x \in UM$. Every one-generated subcoalgebra $\langle x \rangle$ of M is bounded by D. Hence there is a mono $\alpha : \langle x \rangle \to D$, giving rise to a mono $\alpha^* : \langle x \rangle \to F_K D$. It follows that $\langle x \rangle$ is isomorphic to a subcoalgebra of $F_K D$. Now, M can be reconstructed as a homomorphic image of the sum of the one-generated subcoalgebras, therefore $M \in \mathrm{H}\Sigma\mathrm{S}\{F_K D\}$, hence $M \in \mathrm{V}\{F_K D\}$. $\square$

3 Modal Logics for Coalgebras

There are many different kinds of modal logics but most of them share the following features that are essential for a logic for coalgebras: formulas are evaluated in points (worlds, states) and they are invariant under bisimulations. This will be turned into a definition.

Definition 3.1. Let Ω be a functor. A **modal logic for coalgebras** is given by the following:

- a set C, called the set of colours, and a class of formulas $\mathcal{L}_C$,
- for all coalgebras $M \in \mathbf{Set}_\Omega$ and all valuations $\alpha : UM \to C$ a relation $\models_{(M,\alpha)} \subset UM \times \mathcal{L}_C$. (Write $M, \alpha, x \models \varphi$ for $(x, \varphi) \in \models_{(M,\alpha)}$).
- For all Ω-models M, N and all valuations $\alpha : UM \to C$, $\beta : UN \to C$, for all $C \times \Omega$-morphisms $f : (M, \alpha) \to (N, \beta)$, all formulas $\varphi \in \mathcal{L}_C$, and all $x \in UM$, it has to hold

$$M, \alpha, x \models \varphi \;\Leftrightarrow\; N, \beta, f(x) \models \varphi.$$

The last condition says that formulas have to be invariant under bisimulations respecting not only the structure of the Ω-coalgebras but also the given valuations. When there is no danger of confusion, we denote a modal logic for coalgebras by its set of formulas. As usual, $M, x \models \varphi$, $(M \models \varphi)$ are defined by quantifying over all valuations (and all elements) of M.

For the proof of the covariety theorem we need that each $\varphi \in \mathcal{L}_C$ defines a subset of the cofree coalgebra TC. We define:

$$[\![\varphi]\!]^{TC} = \{x \in UTC : TC, x \models \varphi\}.$$

From the invariance of the formulas under bisimulations it follows the **fundamental property** allowing to reduce validity in any model to validity in the cofree models:

$$M \models \varphi \iff \forall \alpha : UM \to C : \mathrm{Im}\, \alpha^{\#} \subset [\![\varphi]\!]^{TC}.$$

Note that, equivalently, this can be stated as $M \models \varphi \;\Leftrightarrow\; \forall \alpha : M \to TC :$ $\mathrm{Im}\, \alpha \subset [\![\varphi]\!]^{TC}$.

As usual, the definition of a satisfaction relation extends to classes of models and classes of formulas and gives rise to operators Mod and Th. For convenience the definitions are listed below:

- If K a class of Ω-coalgebras, then
  $\mathrm{Th}_C(K) = \{\varphi \in \mathcal{L}_C : \forall M \in K : M \models \varphi\}$.
- If Φ a subclass of $\mathcal{L}_C$, then
  $\mathrm{Mod}(\Phi) = \{M \in \mathbf{Set}_\Omega : M \models \varphi \text{ for all } \varphi \in \Phi\}$.

Up to now, we have only required that formulas of the modal logic are evaluated in points and are invariant under bisimulations. To prove the covariety theorem, we need an additional property that guarantees some expressive power.

Definition 3.2. A modal logic for coalgebras $\mathcal{L}_C$ is called **expressive** if for every $C \times \Omega$-subcoalgebra S of (TC, ϵ_C) there is a formula φ such that $US = \{x \in UTC : TC, \epsilon_C, x \models \varphi\}$.[5]

An important consequence of our definition of a modal logic for coalgebras is that formulas are preserved under taking subframes, images or disjoint unions.

Proposition 3.3. *Let $\mathcal{L}_C$ be a modal logic for coalgebras and $\mathcal{O}$ any of the operators* $\mathrm{S}, \mathrm{H}, \Sigma, \mathrm{V}$ *and K any class of coalgebras. Then* $\mathrm{Th}_C(\mathcal{O}(K)) = \mathrm{Th}_C(K)$.

Proof. The proof is essentially the same as the well-known proof that modally definable classes of frames are closed under generated subframes, p-morphic images and disjoint unions, but it also contains some features specific to the coalgebraic setting that might be interesting.

"S": Let $N \in K, N \models \varphi$ and $f : M \to N$ be a mono. We have to show $M \models \varphi$. For a contradiction assume $M \not\models \varphi$. That is, there exists $\alpha : UM \to C$ and $x \in UM$ such that $M, \alpha, x \not\models \varphi$. Now, $f : M \to N$ mono implies that f is injective as a function on sets.[6] Therefore we find a function $\beta : UN \to C$ such that $\beta \circ f = \alpha$. Hence f is a $C \times \Omega$-morphism from (M, α) to (N, β). Now, by invariance of φ under $C \times \Omega$-bisimulations, we get a contradiction to $N \models \varphi$.

"H": Similar, but use that epis in $\mathbf{Set}_\Omega$ are surjective as mappings on sets.

"Σ": Let $(M_i)_{i \in I}$ be a family of models in K and $M = \Sigma_{i \in I} M_i$. Suppose $M \not\models \varphi$. That is, there exists $\alpha : UM \to C$ and $x \in UM$ such that $M, \alpha, x \not\models \varphi$. Since sums in $\mathbf{Set}_\Omega$ are constructed as sums in $\mathbf{Set}$ there is a $j \in I$ such that $x \in M_j$.[7] Now, using that the inclusion of M_j into M is a bisimulation shows $M_j \not\models \varphi$. □

4 A Covariety Theorem

In this section we prove the covariety theorem using essentially the three properties discussed in the last section, i.e., formulas are invariant under

[5]It would suffice to require that for every invariant Ω-subcoalgebra S of TC there is a formula φ such that $US = [\![\varphi]\!]^{TC}$. But is there any reasonable modal logic for coalgebras where this requirement is really weaker?

[6]Categorically speaking, this is due to the fact that the functor $U : \mathbf{Set}_\Omega \to \mathbf{Set}$ that maps a coalgebra to its carrier preserves monos (Ω preserving weak pullbacks and the domain of the mono being non-empty), see Rutten 1996, Prop 4.7.

[7]Categorically: $U : \mathbf{Set}_\Omega \to \mathbf{Set}$ creates (hence preserves) all colimits (and therefore sums), see Rutten 1996, theorem 4.5.

bisimulations, formulas define subsets of the cofree coalgebras and expressiveness.

For the remainder of the paper $\mathcal{L}_C$ will denote a modal logic for coalgebras as defined in the previous section.

Proposition 4.1. *Let Φ be a subclass of $\mathcal{L}_C$. Then* $\mathrm{Mod}(\Phi)$ *is a covariety.*

Proof. We have to show that $\mathrm{VMod}(\Phi) \subset \mathrm{Mod}(\Phi)$ (the other direction is obvious). By proposition 3.3 it holds $\mathrm{Th}_C\mathrm{VMod}(\Phi) = \mathrm{Th}_C\mathrm{Mod}(\Phi)$. Hence $\mathrm{ModTh}_C\mathrm{VMod}(\Phi) = \mathrm{ModTh}_C\mathrm{Mod}(\Phi)$. Now, using $\mathrm{VMod}(\Phi) \subset \mathrm{ModTh}_C\mathrm{VMod}(\Phi)$ and $\mathrm{ModTh}_C\mathrm{Mod}(\Phi) = \mathrm{Mod}(\Phi)$ shows our claim. □

The next proposition is the key to the covariety theorem. Though not essential for the proof it might be interesting to note that K below is not required to be a covariety.

Proposition 4.2. *Let K be a class of Ω-coalgebras and $\varphi \in \mathcal{L}_C$. Then the following are equivalent:*

(1) $K \models \varphi$

(2) $F_K C \models \varphi$

(3) $UF_K C \subset [\![\varphi]\!]^{TC}$

Proof. (1) $\Rightarrow$ (2): The proof of proposition 2.2 shows $F_K C \in \mathrm{H\Sigma H}(K)$. Now apply proposition 3.3.
(2) $\Rightarrow$ (3): the fundamental property implies in particular that for the inclusion $\tau_K^C : F_K C \to TC$ it holds $\mathrm{Im}(\tau_K^C) \subset [\![\varphi]\!]^{TC}$. (And $\mathrm{Im}(\tau_K^C) = UF_K C$ by definition of $F_K C$.)
(3) $\Rightarrow$ (1): Suppose $M \in K$ and $\alpha : M \to TC$. Since $F_K C$ cofree $\mathrm{Im}(\alpha^\#) \subset \mathrm{Im}(\tau_K^C)$. Now, $\mathrm{Im}(\tau_K^C) = UF_K C \subset [\![\varphi]\!]^{TC}$ and the fundamental property yields $M \models \varphi$. □

Proposition 4.3. *Let $\mathcal{L}_C$ be expressive. Then* $\mathrm{Th}_C(K) = \mathrm{Th}_C(K') \implies F_K C = F_{K'} C$.

Proof. We first show that there is $\varphi \in \mathcal{L}_C$ such that $UF_K C = [\![\varphi]\!]^{TC}$. By expressiveness there is φ such that $UF_K C = \{x \in UTC : TC, \epsilon_C, x \models \varphi\}$. Clearly, $[\![\varphi]\!]^{TC} \subset UF_K C$. For the converse show $F_K C \models \varphi$. Choose any $\alpha : UF_K C \to C$ and any $x \in UF_K C$. Since $\alpha^\#$ factors through τ_K^C we have $TC, \epsilon_C, \alpha^\#(x) \models \varphi$, and therefore (by bisimulation) $F_K C, \alpha, x \models \varphi$. We have shown $F_K C \models \varphi$, hence $UF_K C = [\![\varphi]\!]^{TC}$. The proposition now follows from (1) $\Leftrightarrow$ (3) of proposition 4.2. □

Theorem 4.4 (Covariety Theorem). *Let Ω be a functor bounded by C and let $\mathcal{L}_C$ be an expressive modal logic for coalgebras. Then a class K is a covariety iff* $\mathrm{ModTh}_C(K) = K$.

Proof. "if" is proposition 4.1. For the converse, let K' denote $\mathrm{ModTh}_C(K)$. $K' \supset K$ follows from ModTh_C being a closure operator. It remains to show that K is a covariety only if $K' \subset K$. Suppose $M \in K'$. By proposition 2.4 it follows $M \in \mathrm{V}\{F_{K'}C\}$. By proposition 4.3 one gets $M \in \mathrm{V}\{F_K C\}$ and again by proposition 2.4 $M \in K$. □

Corollary 4.5. *Under the assumptions of the theorem it follows for all classes K that* $\mathrm{ModTh}_C(K) = \mathrm{V}(K)$.

Proof. "$\subset$": $K \subset \mathrm{V}(K)$ implies $\mathrm{ModTh}_C(K) \subset \mathrm{ModTh}_C\mathrm{V}(K)$. Hence $\mathrm{ModTh}_C(K) \subset \mathrm{V}(K)$ by the theorem.

"$\supset$": follows from proposition 4.1 and from $\mathrm{V}, \mathrm{ModTh}_C$ being closure operators. □

5 Coalgebraic Logic

We collected in the notion of a modal logic for coalgebras the properties needed to prove the covariety theorem. But we had no requirements on the existence of logical operators and other features usually associated with a logic. Here, we show that the coalgebraic logic of Moss 1997 can be adapted to meet our requirements: essentially we only have to add propositional variables.

5.1 A short introduction to coalgebraic logic

First the definition of coalgebraic logic from Moss 1997 is given. For a detailed discussion and results, the reader is referred to the original paper. We need the category **SET** of classes and set-continuous functions. The functors Ω are on **SET** and have to be set-based, standard and to preserve weak pullbacks. (A functor Ω on **Set** is extended to a functor on **SET** by defining $\Omega K = \bigcup\{\Omega X : X \subset K, X \text{ a set}\}$ for classes K.) The Ω-language $\mathcal{CL}_\Omega$ is defined to be the least class satisfying:

$$\begin{array}{lcl} \Phi \subset \mathcal{CL}_\Omega,\ \Phi \text{ a set} & \Longrightarrow & \bigwedge \Phi \in \mathcal{CL}_\Omega \\ \varphi \in \mathcal{CL}_\Omega & \Longrightarrow & \neg\varphi \in \mathcal{CL}_\Omega \\ \varphi \in \Omega(\mathcal{CL}_\Omega) & \Longrightarrow & \varphi \in \mathcal{CL}_\Omega \end{array}$$

Due to the first clause $\bigwedge\{\}$, denoted by *true*, is in $\mathcal{CL}_\Omega$and $\mathcal{CL}_\Omega$ is a proper class. The last clause uses the fact that Ω is a functor on **SET** and as such can also be applied to classes of formulas. The second clause is not a proper part of $\mathcal{CL}_\Omega$ as defined in Moss 1997 but it is shown there that negation may be added.

Given a coalgebra M the semantics is given by the least relation $\models_\Omega \subset UM \times \mathcal{CL}_\Omega$ such that (let $x \in UM$):

$$\begin{array}{lcl} x \models_\Omega \varphi \text{ for all } \varphi \in \Phi, \Phi \subset \mathcal{CL}_\Omega,\ \Phi \text{ a set} & \Rightarrow & x \models_\Omega \bigwedge \Phi \\ x \not\models_\Omega \varphi & \Rightarrow & x \models_\Omega \neg\varphi \\ \text{there is } w \in \Omega(\models_\Omega) \text{ s.t. } \Omega\pi_1(w) = f(x), \Omega\pi_2(w) = \varphi & \Rightarrow & x \models_\Omega \varphi \end{array}$$

where π_1, π_2 denote the projections from the product $UM \times \mathcal{CL}_\Omega$ to its components. The last clause makes use of $\models_\Omega$ being in **SET** and applies Ω to it.

For an example take again $\Omega = C \times \mathcal{P}$. A typical formula of $\mathcal{CL}_\Omega$ is of the kind (c, Φ) where $c \in C$ a colour and $\Phi \subset \mathcal{CL}_\Omega$ a set of formulas. Satisfiability for such formulas is described by the next proposition.

Proposition 5.1. *Let $\Omega = C \times \mathcal{P}$, $c \in C$, $\Phi \subset \mathcal{CL}_\Omega$, $M = (UM, f)$ a Ω-coalgebra, and $x \in UM$. Then*

$$x \models_\Omega (c, \Phi) \ \Leftrightarrow \ \begin{cases} \forall y \in \pi_2 \circ f(x) : \exists \varphi \in \Phi : y \models_\Omega \varphi & \text{and} \\ \forall \varphi \in \Phi : \exists y \in \pi_2 \circ f(x) : y \models_\Omega \varphi & \text{and} \\ c = \pi_1 \circ f(x) \end{cases}$$

Proof. See Moss 1997. □

Using the modal operators $\Box, \Diamond$, the first two clauses of the righthand side of the above equivalence may be written as: $x \models \Box \bigvee \Phi \wedge \bigwedge \Diamond \Phi$, where $\Diamond\Phi = \{\Diamond\varphi : \varphi \in \Phi\}$.

5.2 Coalgebraic logic with variables (colours)

In order to get a version of coalgebraic logic that allows us to state a covariety theorem for it we need to add colours to the logic. How should we do that? Consider the algebraic case: Given a signature Ω, an equation with variables from X is a pair of terms for the signature $X + \Omega$. Analogously, a formula for Ω-coalgebras with colours from C is a formula for the signature $C \times \Omega$. To interpret $C \times \Omega$-formulas in Ω-coalgebras we use (see section 2.1) that an Ω-coalgebra M and a valuation $\alpha : UM \to C$ determine a $C \times \Omega$-coalgebra (M, α).

This allows for the following definition:

Definition 5.2 (Coalgebraic logic with colours). Let Ω be a functor, and C a set. The coalgebraic logic for Ω with colours from C, denoted by

$\mathcal{CL}_{C,\Omega}$, is given by the set of colours C, the class of formulas $\mathcal{CL}_{C\times\Omega}$ and by the relation $\models_{C,\Omega}$ defined by

$$M, \alpha, x \models_{C,\Omega} \varphi \;\Leftrightarrow\; x \models_{C\times\Omega} \varphi,$$

where M an Ω-coalgebra, $\alpha : UM \to C$, $x \in UM$, $\varphi \in \mathcal{CL}_{C\times\Omega}$, and $\models_{C\times\Omega}$ the semantics of coalgebraic logic defined above w.r.t. the $C \times \Omega$-coalgebra (M, α).

It is not difficult to see that $\mathcal{CL}_{C,\Omega}$ meets the requirements of a modal logic for coalgebras stated in section 3. The only thing we have still to ensure is expressiveness.

In Moss 1997, *uniform* functors are defined and it is proven that for uniform functors Ω the logic $\mathcal{CL}_\Omega$ has characterising formulas: Let T be the final Ω-coalgebra and $t \in UT$. Then there is $\varphi_t \in \mathcal{CL}_\Omega$ such that $s \models_\Omega \varphi_t \;\Leftrightarrow\; s = t$.

Now suppose that $C \times \Omega$ is uniform and has a final coalgebra (TC, ϵ_C). Let S be a subcoalgebra of (TC, ϵ_C). Define $\varphi = \bigvee_{t \in US} \varphi_t$. It follows $TC, \epsilon_C, x \models \varphi \;\Leftrightarrow\; x \in US$. Hence $\mathcal{CL}_{C\times\Omega}$ is expressive.

We have just shown that coalgebraic logic with colours meets the assumptions of our covariety theorem 4.4. Therefore:

Theorem 5.3. *Let Ω be a functor on* **Set** *bounded by C and let $C \times \Omega$ be uniform. Then a class K of Ω-coalgebras is a covariety iff it is definable by a class of formulas of $\mathcal{CL}_{C\times\Omega}$.*

6 Modal Logic

In this section we choose for Ω the powerset functor and show how to translate the coalgebraic logic $\mathcal{CL}_{C,\Omega}$ into infinitary modal logic. Since it turns out that both logics have the same expressive power, the covariety theorem 5.3 becomes a statement about modally definable classes of Kripke frames.

Note that the covariety theorem 5.3 does not hold for $\Omega = \mathcal{P}$ because $\mathcal{P}$ has no final coalgebra in $\mathbf{Set}_\Omega$. But it does hold for any powerset functor $\mathcal{P}_\kappa$ restricting the cardinality of the subsets: $\mathcal{P}_\kappa X = \{Y \subset X : |Y| < \kappa\}$. Note that, for $\kappa \geq 1$, $\mathbf{Set}_{\mathcal{P}_\kappa}$ is the class of Kripke frames with degree of branching smaller than κ. $\mathbf{Set}_{\mathcal{P}_0}$ only contains the empty Kripke frame.

We denote with $\mathcal{ML}$ the infinitary modal logic built from propositional variables, the constant $\bot$, the operators $\neg, \Box$ and conjunctions $\bigwedge$ over any set of formulas. $\bigvee$ and $\Diamond$ are defined as abbreviations. The set of propositional variables is called P and we write $\mathcal{ML}(P)$ for the class of formulas taking only variables from P. In this section symbols without subscripts like

$\models$ and Mod will refer to modal logic. $\mathcal{CL}_{C,\Omega}$ and $\models_{C,\Omega}$ are as in the previous section. For $\Phi \subset \mathcal{CL}_{C\times\Omega}$ define $\mathrm{Mod}_{C,\Omega}(\Phi) = \{M \in \mathbf{Set}_\Omega : M \models_{C,\Omega} \Phi\}$.

In the following let $\Omega = \mathcal{P}_\kappa$ and $C = \mathcal{P}P$ for a fixed set of propositional variables P. We give translations from $\mathcal{CL}_{C,\Omega}$ to $\mathcal{ML}(P)$ and vice versa and show that they have equal expressive power on $\mathbf{Set}_\Omega$.[8]

The translation $(\cdot)^{\mathcal{ML}}$ from $\mathcal{CL}_{C,\Omega}$ to $\mathcal{ML}(P)$ is defined as to be expected for negation and conjunction. For formulas in $C \times \mathcal{P}_\kappa(\mathcal{CL}_{C\times\Omega})$ let $c \in C$ and Φ a subset of $\mathcal{CL}_{C\times\Omega}$ and

$$(c,\Phi)^{\mathcal{ML}} = \bigwedge_{p\in c} p \;\wedge\; \bigwedge_{p\notin c} \neg p \;\wedge\; \Box \bigvee \Phi^{\mathcal{ML}} \;\wedge\; \bigwedge \Diamond\, \Phi^{\mathcal{ML}},$$

with $\Phi^{\mathcal{ML}} = \{\varphi^{\mathcal{ML}} : \varphi \in \Phi\}$ and $\Diamond\Phi^{\mathcal{ML}} = \{\Diamond(\varphi^{\mathcal{ML}}) : \varphi \in \Phi\}$ if $\Phi \neq \{\}$. $\Phi = \{\}$ means that there is no successor, so define $\{\}^{\mathcal{ML}} = \bot$.

Proposition 6.1. *Let Ω be $\mathcal{P}_\kappa$ or $\mathcal{P}$, P a set, $C = \mathcal{P}P$, M a Kripke frame in $\mathbf{Set}_\Omega$ and $\varphi \in \mathcal{CL}_{C\times\Omega}$. Then for all $x \in M$ it holds $M, x \models_{C,\Omega} \varphi \;\Leftrightarrow\; M, x \models \varphi^{\mathcal{ML}}$.*

Proof. By induction on the structure of φ (use proposition 5.1). □

For the translation in the other direction, let $\varphi \in \mathcal{ML}(P)$. Define $(\cdot)^{\mathcal{CL}}$ via

$$p^{\mathcal{CL}} = \bigvee_{c\ni p} (c, true) \;\vee\; \bigvee_{c\ni p} (c, \{\}),$$

$$(\Box\varphi)^{\mathcal{CL}} = \bigvee_{c\in\mathcal{P}P} (c, \{\varphi^{\mathcal{CL}}\}) \;\vee\; \bigvee_{c\in\mathcal{P}P} (c, \{\}).$$

Proposition 6.2. *Let $\kappa \geq 2$ be cardinal, Ω be $\mathcal{P}_\kappa$ or $\mathcal{P}$, P a set, $C = \mathcal{P}P$, M a Kripke frame in $\mathbf{Set}_\Omega$ and $\varphi \in \mathcal{ML}(P)$. Then for all $x \in M$ it holds $M, x \models \varphi \;\Leftrightarrow\; M, x \models_{C,\Omega} \varphi^{\mathcal{CL}}$.*

Proof. Because of $\kappa \geq 2$ the translated formula $\varphi^{\mathcal{CL}}$ is indeed in $\mathcal{CL}_{C\times\Omega}$. Now, $M, x \models \varphi \;\Leftrightarrow\; M, x \models_{C,\Omega} \varphi^{\mathcal{CL}}$ is again proved by induction on the structure of φ. □

The propositions above imply that the expressive power of modal and coalgebraic logic coincide:

[8] For the case $P = \{\}$ the translation is already treated in Moss 1997. Our case is similar.

Lemma 6.3. *Let $\Omega = \mathcal{P}_\kappa$ for a cardinal κ, P a set and $C = \mathcal{P}P$. Then a class of Kripke frames is $\mathcal{ML}(P)$-definable w.r.t. $\mathbf{Set}_\Omega$ iff it is $\mathcal{CL}_{C,\Omega}$-definable.*

Proof. "only if": A class of frames is $\mathcal{ML}(P)$-definable w.r.t. $\mathbf{Set}_\Omega$ iff it is of the form $\mathrm{Mod}(\Phi) \cap \mathbf{Set}_\Omega$ for some class of $\mathcal{ML}(P)$-formulas Φ. We have to show that there is a class of $\mathcal{CL}_{C,\Omega}$-formulas Ψ such that $\mathrm{Mod}(\Phi) \cap \mathbf{Set}_\Omega = \mathrm{Mod}_{C,\Omega}(\Psi)$. In the case $\kappa = 0$ only $\mathbf{Set}_\Omega$ itself is $\mathcal{ML}(P)$-definable and we can choose Ψ to be empty. If $\kappa = 1$ all models are sets (the transition relation is empty) and only $\{\}$ and $\mathbf{Set}_\Omega$ are $\mathcal{ML}(P)$-definable. We can choose Ψ to be either $\{\mathit{true}\}$ or $\{\neg \mathit{true}\}$. For $\kappa \geq 2$ we choose $\Psi = \Phi^{\mathcal{CL}}$ and now $\mathrm{Mod}(\Phi) \cap \mathbf{Set}_\Omega = \mathrm{Mod}_{C,\Omega}(\Psi)$ follows easily from the second proposition above and the respective definitions.
"if": It suffices to show that $\mathrm{Mod}(\Psi^{\mathcal{ML}}) \cap \mathbf{Set}_\Omega = \mathrm{Mod}_{C,\Omega}(\Psi)$ for any $\Psi \subset \mathcal{CL}_{C\times\Omega}$. This follows from the first of the above propositions. □

Theorem 6.4. *Let κ be a cardinal and $\mathcal{K}$ the class of frames with degree of branching $< \kappa$. Then $K \subset \mathcal{K}$ is $\mathcal{ML}$-definable[9] w.r.t. $\mathcal{K}$ iff K is closed under disjoint unions, generated subframes and p-morphic images.*

Proof. "if": Let Ω be $\mathcal{P}_\kappa$. Then $\mathcal{K}$ is $\mathbf{Set}_\Omega$. Ω is bounded, hence bounded by a set $\mathcal{P}P$. And since $\mathcal{P}P \times \Omega$ is uniform and K a covariety by assumption, theorem 5.2 implies that $K \subset \mathcal{K}$ is definable by a class of $\mathcal{CL}_{\mathcal{P}P\times\Omega}$ formulas. By the lemma above $K \subset \mathcal{K}$ is definable w.r.t. $\mathcal{K}$ by a class of $\mathcal{ML}(P)$ formulas.
"only if": K is $\mathcal{ML}$-definable w.r.t. $\mathcal{K}$ means that K is $\mathcal{ML}(P)$-definable w.r.t. $\mathcal{K}$ for some P. It follows by the lemma above that K is $\mathcal{CL}_{\mathcal{P}P\times\Omega}$-definable, hence by proposition 4.1 that K is a covariety, i.e., K is closed under disjoint unions, generated subframes and p-morphic images. □

7 Discussion and Conclusion

The question of this paper was what kind of logic could play the role for coalgebras that equational logic plays for algebras. And the answer was: modal logic. We arrived at that conclusion by dualising the proof of Birkhoff's variety theorem (section 4). The properties a logic has to have to make the proof possible were summarised in the notion of an expressive modal logic for coalgebras (section 3). Then it was shown that the coalgebraic logic of Moss is an example of such a logic (section 5) and the covariety theorem was used to obtain a definability result for infinitary modal logics (theorem 6.4).

[9] $\mathcal{ML}$-definable means $\mathcal{ML}(P)$-definable for some set P.

It should be emphasised that theorem 6.4 does not only hold for classes of Kripke frames with bounded degree of branching but also for classes of frames of all modal signatures that correspond to uniform and bounded functors $\Omega : \mathbf{Set} \to \mathbf{Set}$.

The approach of the paper can be extended in several ways. First, a more careful treatment of coalgebraic logic would show that, due to the boundedness of the functors, a set of formulas instead of a proper class of formulas would suffice to get theorems 5.3. Second, we could replace the requirement that the functor Ω is bounded by the condition that for any set C a final $C \times \Omega$-coalgebras exist.[10] Another way to overcome the restriction to bounded functors might be to allow coalgebras that have a proper class as carrier. (In Aczel and Mendler 1989 it is shown that then every (set-based) functor has a final coalgebra.)

Furthermore it would be interesting to compare our approach with the one of H. Peter Gumm 1998. He also proves a covariety theorem along the lines of universal algebra but uses a different logical approach: a formula φ is an element of the cofree coalgebra and φ holds in a coalgebra M iff for all valuations $\alpha : UM \to C : \varphi \notin \operatorname{Im} \alpha^{\#}$.

Topics of ongoing research are the following questions: How much further can we go in dualising similar results from algebra?[11] What is a categorical formulation of the duality principle we used in section 4? How will this coalgebra/algebra duality relate to the well-known duality of Kripke frames/modal algebras?

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[10] Categorically: that $U : \mathbf{Set}_\Omega \to \mathbf{Set}$ has a right adjoint

[11] For example, it has recently been shown in Kurz 1999 that the modal logic analogue of implications are modal rules. Definability theorems can be proved along the same line as in this paper.

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