Modal Rules are Co-Implications

Alexander Kurz
Institut für Informatik
Ludwig-Maximilians-Universität München
Oettingenstr. 67, 80538 München, Germany
e-mail: kurz@informatik.uni-muenchen.de

May 15, 2000

Abstract

In [9], it was shown that modal logic for coalgebras dualises—concerning definability—equational logic for algebras. This paper establishes that, similarly, modal rules dualise implications: It is shown that a class of coalgebras is definable by modal rules iff it is closed under H (images) and Σ (disjoint unions). As a corollary the expressive power of rules of infinitary modal logic on Kripke frames is characterised.

1 Introduction

The investigation of the relationship of modal logic and coalgebras is motivated by coalgebras being a generalisation of transition systems. A first major achievement was Moss' paper [12] on 'coalgebraic logic' where it was shown how to formulate a modal logic for Ω-coalgebras depending in a canonical way on the functor $\Omega : \mathbf{Set} \to \mathbf{Set}$. Since then, modal logics as a specification language for coalgebras have been investigated in a number of papers, e.g. [7, 13, 11]. On the other hand, it is also interesting to apply categorical and (co)algebraic tools in order to obtain new insights in modal logic. For example, it was shown in [9] that one can characterise the expressive power of infinitary modal logics on Kripke frames by dualising the proof of Birkhoff's variety theorem (which, in turn, characterises the expressive power of equational logic on algebras). Here, we continue this line of research.

We start from the correspondence between implications $\bigwedge_{i\in I} t_i = t'_i \to s = s'$, I a set or class, and algebras. The classical result on implicationally definable classes is (Banaschewski and Herrlich [3]): A class of algebras is implicationally definable iff it is closed under subalgebras and products (and isomorphisms).

Similarly to [9], our aim here is to use the duality of algebras and coalgebras to prove a dual of this theorem for coalgebras. As it turns out, the concept dual to implication is that of a *modal rule*. Theorem 4.1 establishes that a class of coalgebras is rule-definable iff it is closed under images of morphisms and disjoint unions. Theorem 5.2 applies this result to Kripke frames: A class of Kripke frames is definable by rules of infinitary modal logic iff it is closed under images of p-morphisms and disjoint unions.

An algebraically similar but logically different approach is followed by Gumm [6]. There, also the results on equationally and implicationally definable classes of algebras are dualised. But the logic used for coalgebras is different: A formula φ is an element of the carrier of a cofree coalgebra TC and φ holds in a coalgebra M iff for all valuations $\alpha : UM \to C$ the formula φ is not in the image of the induced morphism $\alpha^{\#} : M \to TC$.

A previous version of this draft has been electronically available. The main improvement over this draft is that theorem 4.1 does not depend any more on the existence of cofree coalgebras in $\mathbf{Set}_\Omega$. As a consequence, the application to Kripke frames in theorem 5.2 does not require a bound on the degree of branching of the frames. This result is quite surprising and can not be transferred in an obvious way to the case of the covariety theorem in [9].

2 Coalgebras

We introduce notation and briefly review coalgebras as models for modal logic (for more information see [9]). The classical paper on coalgebras is Rutten [14].

Coalgebras are given w.r.t. a category $\mathcal{C}$ and an endofunctor $\Omega : \mathcal{C} \to \mathcal{C}$. An Ω-coalgebra $M = (UM, f_M)$ is then given by an object $UM \in \mathcal{C}$ and an arrow $f_M : UM \to \Omega(UM)$ in $\mathcal{C}$. Ω-coalgebras form a category $\mathcal{C}_\Omega$ where a *coalgebra morphism* $\alpha : (UM, f_M) \to (UN, f_N)$ is an arrow $\alpha : UM \to UN$ in $\mathcal{C}$ such that $\Omega\alpha \circ f_M = f_N \circ \alpha$.

As an example consider the functor $\Omega : \mathbf{Set} \to \mathbf{Set}$ given by $\Omega X = \mathcal{P}X$ where $\mathcal{P}$ denotes powerset.[1] Then Ω-coalgebras are *Kripke frames*: given a coalgebra M and a world $x \in UM$, $f_M(x)$ is the set of successors of x. Coalgebra morphisms in $\mathbf{Set}_\mathcal{P}$ are functional bisimulations, i.e. p-morphisms.

A remark on epis and monos in $\mathbf{Set}_\Omega$: Since a morphism $\alpha : (UM, f_M) \to (UN, f_N)$ is also a function $\alpha : UM \to UN$, it is immediate that if α is epi (mono) in $\mathbf{Set}$, α is also epi (mono) in $\mathbf{Set}_\Omega$. The converse is also true for epis (see Rutten [14], 4.7), that is, a morphism in $\mathbf{Set}_\Omega$ is epi iff it is surjective. Concerning monos it holds: $\alpha \in \mathbf{Set}_\Omega$ is mono in $\mathbf{Set}$ iff it is strong mono in $\mathbf{Set}_\Omega$ (see the appendix), that is, a morphism in $\mathbf{Set}_\Omega$ is strong mono iff it is injective.

2.1 Cofree Coalgebras

We first give the definition of cofree coalgebras and then show how they are interpreted in the context of modal logic. To define the notion of a *cofree coalgebra* consider the diagram below:

$$\begin{tikzcd}
UTC \arrow[d, "\epsilon_C"'] & \\
C & UM \arrow[l, "\alpha"] \arrow[ul, "\alpha^{\#}"']
\end{tikzcd}$$

Let Ω be a functor. An Ω-coalgebra TC (and a mapping $\epsilon_C : UTC \to C$) is called *cofree over C* iff for all Ω-coalgebras M and all mappings $\alpha : UM \to C$ there is a unique morphism $\alpha^\# : M \to TC$ such that the diagram commutes.[2] Compared to universal algebra, the *set of colours C* corresponds to the set of variables and a *colouring* α to a valuation of variables.

As in the case of $\Omega = \mathcal{P}$, cofree coalgebras may not exist in $\mathbf{Set}_\Omega$. This problem can be circumvented by extending the functor Ω on $\mathbf{Set}$ to a functor on $\mathbf{SET}$ [3] by defining $\Omega K = \bigcup\{\Omega X : X \subset K, X \text{ a set}\}$ for classes K. It then follows from a theorem by Aczel and Mendler [2] that for every functor Ω on $\mathbf{Set}$ and all $C \in \mathbf{Set}$ the cofree coalgebras TC exist in $\mathbf{SET}_\Omega$. Thus,

[1] This only defines Ω on sets. On functions Ω is defined in the standard way, $\mathcal{P}$ being the covariant powerset functor: Given $f : X \to Y$, $\Omega f = \lambda A \in \mathcal{P}X.\{f(a) : a \in A\}$.

[2] Note that every morphism $\alpha^\# : M \to TC \in \mathbf{Set}_\Omega$ is by definition of morphisms in $\mathbf{Set}_\Omega$ also a mapping $\alpha^\# : UM \to UTC \in \mathbf{Set}$.

[3] $\mathbf{SET}$ is the category of classes and class maps as in Aczel [1], chapter 7, and Aczel and Mendler [2].

in the following, we will allow, without further mentioning, that cofree coalgebras exist in $\mathbf{SET}_\Omega$ instead of $\mathbf{Set}_\Omega$.

As an example consider $\Omega = \mathcal{P}$ and $C = \mathcal{P}P$ where P is a set of propositional variables. Let M be an Ω-coalgebra, i.e. a Kripke frame. The functions $\alpha : UM \to C$ are valuations: every world x in M is assigned the set of propositional variables holding in x, that is, (M, α) is a *Kripke model.* (Note that an Ω-coalgebra M plus a valuation $\alpha : UM \to C$ is a $C \times \Omega$-coalgebra; and a morphism between $C \times \Omega$-coalgebras $f : (M, \alpha) \to (N, \beta)$ is an Ω-morphism $f : M \to N$ such that $\beta \circ f = \alpha$, see [9].) The diagram above then shows that any Kripke model (M, α) has a unique p-morphic image in the model (TC, ϵ_C). We can think of (TC, ϵ_C) as the disjoint union of all models based on Ω-frames with the additional feature that any two bisimilar worlds are identified.

2.2 Strong-mono-Coreflective Classes of Coalgebras

The concept of a strong-mono-coreflection is a generalisation of the concept of cofreeness and dualises the concept of a strong epireflection (see Borceux [4], I.3.6). Strong monos do appear here because they are the categorical way of describing subcoalgebras (see the appendix). Coreflective classes are used because, on the one hand, they are precisely those classes closed under the operators H (closure under images of coalgebra morphisms) and Σ (closure under disjoint unions (coproducts, sums) of coalgebras), and because, on the other hand, the 'coreflection morphisms' will allow us to see what the defining modal rules will be (section 4).

Let $\Omega : \mathbf{Set} \to \mathbf{Set}$ be a functor. A **strong-mono-coreflection** for a class K of Ω-coalgebras is given by coalgebras $R_K M$ and *strong mono*morphisms $\epsilon^K_M : R_K M \to M$ for all $M \in \mathbf{Set}_\Omega$ such that for all $N \in K$ every morphism $\alpha : N \to M$ factors uniquely through ϵ^K_M:

$$\begin{tikzcd} M & R_K M \arrow[l, "\epsilon^K_M"'] \\ & N \arrow[ul, "\alpha"] \arrow[u, "\alpha^*"'] \end{tikzcd}$$

$R_K M$ is called the *coreflection* of M and ϵ^K_M the *coreflection morphism* of M.

Proposition 2.1 (Existence of strong-mono-coreflections). *Let Ω be a functor on* $\mathbf{Set}$ *and K a class of Ω-coalgebras. Then for all $M \in \mathbf{Set}_\Omega$ there is $R_K M \in \mathbf{Set}_\Omega$ and a strong mono $\epsilon^K_M : R_K M \to M \in \mathbf{Set}_\Omega$ such that for all $N \in K$ and all $\alpha : N \to M$ in $\mathbf{Set}_\Omega$ there is a unique $\alpha^* : N \to R_K M$ in $\mathbf{Set}_\Omega$ such that $\epsilon^K_M \circ \alpha^* = \alpha$. Moreover, $R_K M \in \mathrm{H}\Sigma\mathrm{H}K$.*

Proof. Let $M \in \mathbf{Set}_\Omega$. Let $A = \{\alpha : N \to M \mid N \in K\}$ be the collection of all coalgebra morphisms $N \to M$ with $N \in K$. Now, the union of the images of all $\alpha \in A$ defines a subcoalgebra $R_K M$ of M with inclusion ϵ^K_M.[4] Clearly, for each $N \in K$ the above factorisation property holds. Moreover, the fact that union is a quotient of a disjoint union shows that $R_K M \in \mathrm{H}\Sigma\mathrm{H}K$. $\square$

Definition 2.2 (Strong-mono-coreflective-classes, smc). Let Ω be a functor on $\mathbf{Set}$ and K a class of Ω-coalgebras. K is called strong-mono-coreflective, or **smc** for short, iff it is closed under isomorphisms and contains all coreflections $R_K M$.[5]

[4] Every coalgebra morphism $\alpha \in \mathbf{Set}_\Omega$ factors through its image $\mathrm{Im}\,\alpha$, see Rutten [14], theorem 7.1; and the union of images of coalgebra morphisms always exists, see [14], theorem 6.4.

[5] Categorically: K is smc iff it is closed under isomorphisms and the inclusion functor $K \to \mathbf{Set}_\Omega$ has a right adjoint R_K with the counit (i.e. ϵ^K_M) being a strong mono.

Strong-epi-reflective classes of algebras are characterised as being exactly those classes of algebras that are closed under subalgebras and products. Dually, smc-classes of coalgebras are characterised by closure under homomorphic images (denoted by H) and closure under disjoint unions, i.e. coproducts (denoted by Σ).

Proposition 2.3. *Let Ω be a functor on* **Set** *and K a class of Ω-coalgebras. K is smc iff it closed under* H *and* Σ.

Proof. "only if": Closure under H is (the dual) of [4], I.3.6.4. For closure under Σ let $M_i \in K$ (for all $i \in I$) and M the sum of all M_i. Note that $R_K M_i$ isomorphic to M_i. We have to show $R_K M$ isomorphic to M. ϵ_M^K being a strong mono, it suffices to show that ϵ_M^K is epi in **Set** (and hence epi in $\mathbf{Set}_\Omega$). This follows from the observation that every $x \in UM$ is in the image of an inclusion $\mathrm{in}_i : M_i \to M$ and every inclusion factors through ϵ_M^K.
"if": Follows from $R_K M \in \mathrm{H}\Sigma\mathrm{H}K$. □

Closure under H and Σ is equivalent to closure under the operator HΣ. This is dual to the fact that, in universal algebra, SP is closure under subalgebras and products (see Gumm and Schröder [5] for details on closure operators on coalgebras). We can therefore phrase the proposition above as K smc iff $K = \mathrm{H}\Sigma(K)$.

2.3 An Example

To illustrate the notions above and their connection to modal logic we give an example. Let $\Omega = \mathbb{B} \times \mathcal{P}$, where $\mathbb{B}$ is the set of Booleans. That is, every state is assigned (b, Y), where b is a Boolean and Y a set. We interpret b as the truth value of a fixed proposition and Y as the set of successors. A modal language for this functor is build from the usual connectives, modal operators and propositional variables from a set P, plus a propositional constant denoted by *start*. A $\mathbf{Set}_\Omega$-coalgebra $M = (UM, f)$ is a Kripke frame together with a predicate interpreting *start*. To be more precise, let $\alpha : UM \to \mathcal{P}P, x \in UM, p \in P$. Then (boolean cases as usual and π_1, π_2 denoting the projections from the product $\mathbb{B} \times \mathcal{P}$ to its components):

$$\begin{array}{lcl} M, \alpha, x \models \mathit{start} & \Leftrightarrow & \pi_1 \circ f(x) = \mathit{true} \\ M, \alpha, x \models p & \Leftrightarrow & p \in \alpha(x) \\ M, \alpha, x \models \Box\varphi & \Leftrightarrow & \forall y \in \pi_2 \circ f(x) : M, \alpha, y \models \varphi \end{array}$$

The states x satisfying the first clause are called states marked by *start*. Next, we want to axiomatise a subclass of these Kripke frames by modal rules. A modal rule φ/ψ (where φ, ψ are modal formulas) is interpreted via

$$M \models \varphi/\psi \iff \forall \alpha : UM \to \mathcal{P}P : M, \alpha \models \varphi \;\Rightarrow\; M, \alpha \models \psi$$

Modal axioms are rules with true premise. Now, consider the following rules:

$$\begin{array}{ll} \text{(refl)} & \Box p \to p \\ \text{(trans)} & \Box p \to \Box\Box p \\ \text{(start)} & \mathit{start} \to \Box p \,/\, p \end{array}$$

The first two are the well-known axioms defining reflexivity and transitivity on Kripke frames. The third one is the start rule from Kröger [8]. In the presence of reflexivity and transitivity it expresses that every state has to be reachable from a state marked by *start*.

Call Φ the set of the three rules above and let K be the class of Kripke frames defined by Φ. We show that K is smc. Define $R_K M$ as the largest subcoalgebra of M satisfying Φ (that is, to find $R_K M$, take the largest subcoalgebra of M that is reflexive and transitive and then cut off all states that are not reachable by a state marked by *start*). $\epsilon_M^K : R_K M \to M$ is the canonical embedding and it is a strong mono since it is injective. Recalling the definition of a coreflective subcategory, it remains to show that for all $N \in \mathbf{Set}_\Omega$ satisfying Φ it holds that for all $f : N \to M$ there is a unique $g : N \to R_K M$ such that $f = \epsilon_M^K \circ g$:

$$\begin{tikzcd}
M & R_K M \arrow[l, "\epsilon_M^K"'] \\
& N \arrow[ul, "f"] \arrow[u, dotted, "g"']
\end{tikzcd}$$

Consider a factorisation $N \xrightarrow{e} \operatorname{Im} f \xrightarrow{m} M$ of f. Since rules are invariant under taking images (see proposition 3.5) it follows that $\operatorname{Im} f \models \Phi$. Moreover $m : \operatorname{Im} f \to M$ is a subcoalgebra of M and since $R_K M$ is the largest subcoalgebra of M satisfying Φ, m factors through ϵ_M^K as $m = \epsilon_M^K \circ g'$ for some g'. Now, $g = g' \circ e$ is the required morphism and g is uniquely determined since ϵ_M^K is injective.

Finally, let us note that K is closed under images and disjoint unions (coproducts) but not under subcoalgebras. Hence K is an example of a coquasivariety that is not a covariety.

3 Modal Logics for Coalgebras

There are many different kinds of modal logics but most of them share the following features that are essential for a logic for coalgebras: formulas are evaluated in points (worlds, states) and they are invariant under bisimulations. Compared to the paper on covarieties [9] the definition below changed a little: Since we have no requirement that the functor Ω is bounded, a logic has to have formulas for arbitrary large sets of colours.

Definition 3.1. Let Ω be a functor. A **modal logic for coalgebras** $\mathcal{L}$ is given by the following:

- a class *Col* of sets (the sets in *Col* are called sets of colours of $\mathcal{L}$), where *Col* contains for each cardinal κ a set with cardinality $\geq \kappa$, and for each $C \in Col$ a class of formulas $\mathcal{L}_C$,
- for all $C \in Col$, for all $M \in \mathbf{Set}_\Omega$, and for all valuations $\alpha : UM \to C$ a relation $\models^C_{(M,\alpha)} \subset UM \times \mathcal{L}_C$. (Write $M, \alpha, x \models \varphi$ for $(x, \varphi) \in \models^C_{(M,\alpha)}$.)
- for all $C \in Col$, $M, N \in \mathbf{Set}_\Omega$, $\alpha : UM \to C$, $\beta : UN \to C$, $\varphi \in \mathcal{L}_C$, $x \in UM$, and all $C \times \Omega$-morphisms[6] $f : (M, \alpha) \to (N, \beta)$, it has to hold

$$M, \alpha, x \models \varphi \;\Leftrightarrow\; N, \beta, f(x) \models \varphi.$$

The last condition says that formulas have to be invariant under bisimulations respecting not only the structure of the Ω-coalgebras but also the given valuations. As usual, $M, x \models \varphi$, ($M \models \varphi$) are defined by quantifying over all valuations (and all elements) of M.

Formulas $\varphi \in \mathcal{L}_C$ define subsets of the cofree coalgebra (TC, ϵ_C). It is useful to introduce the following notation:

$$[\![\varphi]\!]^{TC,\epsilon_C} = \{x \in UTC : TC, \epsilon_C, x \models \varphi\}.$$

[6] Recall that $f : (M, \alpha) \to (N, \beta)$ is a $C \times \Omega$-morphisms iff f is an Ω-morphism such that $\beta \circ f = \alpha$.

From the invariance of the formulas under bisimulations it follows the **fundamental property** allowing to reduce validity w.r.t. a valuation in any model to validity in the cofree models:

$$M, \alpha \models \varphi \iff \operatorname{Im} \alpha^{\#} \subset [\![\varphi]\!]^{TC,\epsilon_C}.$$

Next, we show that dualising the concept of an implication in algebra we obtain the notion of a modal rule. The reader might want to recall the semantics of implications in universal algebra. First two basic facts: Let X be a set of variables and TX be the term algebra over variables X. Then every valuation $\alpha : X \to A$ has a unique lifting to an algebra morphism $\alpha^\flat : TX \to A$. And every algebra morphism $\alpha^\flat : TX \to A$ determines a congruence relation on TX that we denote by $\ker \alpha^\flat$. Next, consider an implication $\bigwedge_{i \in I} t_i = t_i' \to s = s'$. It determines two congruence relations P, Q on the carrier of TX, P standing for the relation induced by $\bigwedge_{i \in I} t_i = t_i'$ and Q for the relation induced by $s = s'$. Now, it is not difficult to see that the implication holds in an algebra A iff $P \subset \ker \alpha^\flat \Rightarrow Q \subset \ker \alpha^\flat$ for all $\alpha : X \to A$. This characterisation of implications is dualised by the following definition of modal rules.

Definition 3.2 (Rules). Given two formulas $\varphi, \psi \in \mathcal{L}_C$ we call the expression φ/ψ a rule. The class of all rules built from formulas in $\mathcal{L}_C$ is denoted by Ru_C. Define $\models$

$$M \models \varphi/\psi \text{ iff } \forall \alpha : UM \to C : \operatorname{Im} \alpha^{\#} \subset [\![\varphi]\!]^{TC,\epsilon_C} \Rightarrow \operatorname{Im} \alpha^{\#} \subset [\![\psi]\!]^{TC,\epsilon_C},$$

Definition 3.3 (rule-definable). Let $\Omega : \mathbf{Set} \to \mathbf{Set}$ be a functor and $\mathcal{L}$ be a modal logic for Ω-coalgebras. $K \subset \mathbf{Set}_\Omega$ is rule-definable iff there are classes $\Phi_C \subset \mathrm{Ru}_C$, $C \in \mathit{Col}$, such that $M \in K \Leftrightarrow \forall C \in \mathit{Col} : M \models \Phi_C$.

Up to now, we have only required that formulas of modal logic are evaluated in points and are invariant under bisimulations. We need an additional property that guarantees enough expressive power.

Definition 3.4. A modal logic for coalgebras $\mathcal{L}$ is called **expressive** if for all $C \in \mathit{Col}$ and every $C \times \Omega$-subcoalgebra S of (TC, ϵ_C) there is a formula φ such that $US = \{x \in TC : TC, \epsilon_C, x \models \varphi\}$.

An important consequence of our definition of a modal logic for coalgebras is that rules are preserved under images and disjoint unions.

Proposition 3.5. *Let $\mathcal{L}$ be a modal logic for Ω-coalgebras and let K be a rule-definable class of Ω-coalgebras. Then K is closed under the operators* H *and* Σ.

Proof. Let $C \in \mathit{Col}$ and $\varphi, \psi \in \mathcal{L}_C$.

"H": Suppose $M \in K$ and $f : M \to N$ epi in $\mathbf{Set}_\Omega$. We have to show $M \models \varphi/\psi \implies N \models \varphi/\psi$. For a contradiction assume that $N \not\models \varphi/\psi$, i.e. there exist $\beta : UN \to C$ and $y \in UN$ s.t. $N, \beta, y \models \varphi$ and $N, \beta, y \not\models \psi$. Define $\alpha : UM \to C$ by $\alpha = \beta \circ f$, i.e. f is a $C \times \Omega$-bisimulation between (M, α) and (N, β). Now, since f epi in $\mathbf{Set}_\Omega$ implies f epi in $\mathbf{Set}$ and since $\models$ is compatible with $C \times \Omega$-bisimulations, there is $x \in UM$ such that $M, \alpha, x \models \varphi$ and $M, \alpha, x \not\models \psi$, which is the desired contradiction.

"Σ": Similar to the above. Let $(M_i)_{i \in I}$ be a family of models in K and $M = \Sigma_{i \in I} M_i$. Suppose $M \not\models \varphi/\psi$. That is, there exist $\alpha : UM \to C$ and $x \in UM$ such that $M, \alpha, x \models \varphi$ and $M, \alpha, x \not\models \psi$. Since sums in $\mathbf{Set}_\Omega$ are constructed as sums in $\mathbf{Set}$[7] there is a $j \in I$ such that $x \in M_j$. Now, using that the inclusion in_j of M_j into M is a bisimulation shows $M_j, i_j, x \models \varphi$ and $M_j, i_j, x \not\models \psi$. □

[7] Categorically: $U : \mathbf{Set}_\Omega \to \mathbf{Set}$ creates all colimits, see [14], theorem 4.5.

4 Rule-Definable Classes of Coalgebras

We have already seen that rule-definable classes are closed under H and Σ. To show the converse, one uses that every class K closed under H and Σ is strong-mono-coreflective (smc) and then shows that K is 'defined' by its coreflection morphisms.

Theorem 4.1 (Characterisation of rule-definable classes). *Let $\Omega : \mathbf{Set} \to \mathbf{Set}$ be a functor and $\mathcal{L}$ an expressive modal logic for Ω-coalgebras. Then a class K is definable by rules of $\mathcal{L}$ iff K is closed under* H *and* Σ.

Proof. "only if" is proposition 3.5. For "if" note that K is smc by proposition 2.3. The defining rules are now determined by the coreflection morphisms $\epsilon_M^K : rkm \to M$. Define Φ_C for $C \in Col$ as follows. For $M \in \mathbf{Set}_\Omega$, $|UM| \leq |C|$, choose an injective mapping $i : UM \to C$. By expressiveness there are formulas $\varphi_M^C, \psi_M^C \in \mathcal{L}_C$ such that $[\![\varphi_M^C]\!]^{TC,\epsilon_C} = \operatorname{Im} i^\#$ and $[\![\psi_M^C]\!]^{TC,\epsilon_C} = \operatorname{Im}(i^\# \circ \epsilon_M^K)$. Let $\Phi_C = \{\varphi_M^C/\psi_M^C : M \in \mathbf{Set}_\Omega, |UM| \leq |C|\}$. We have to show $N \in K \Leftrightarrow \forall C \in Col : N \models \Phi_C$.
" $\Rightarrow$ ": Let $\varphi_M^C/\psi_M^C \in \Phi_C$ and suppose $N, \beta \models \varphi_M^C$, i.e. $\operatorname{Im} \beta^\# \subset [\![\varphi_M^C]\!]^{TC,\epsilon_C}$. By definition of φ_M^C there is $i : UM \to C$ with $[\![\varphi_M^C]\!]^{TC,\epsilon_C} = \operatorname{Im} i^\#$. Hence, $\beta^\#$ factors through $i^\#$ as $\beta^\# = i^\# \circ f$ for some $f : N \to M$. Since $N \in K$ and K is smc, f factors through ϵ_M^K as $f = \epsilon_M^K \circ g$ for some $g : N \to R_K M$. It follows $\operatorname{Im} \beta^\# = \operatorname{Im}(i^\# \circ \epsilon_M^K \circ g) \subset \operatorname{Im}(i^\# \circ \epsilon_M^K) = [\![\psi_M^C]\!]^{TC,\epsilon_C}$, i.e. $N, \beta \models \psi_M^C$.
" $\Leftarrow$ ": Let $N \in \mathbf{Set}_\Omega$. Choose $C \in Col$, $|C| \geq |UN|$, and $i : UN \to C$ such that $\operatorname{Im} i^\# = [\![\varphi_N^C]\!]^{TC,\epsilon_C}$. We show $\operatorname{Im} i^\# = \operatorname{Im}(i^\# \circ \epsilon_N^K)$ (which implies, since $i^\#$ and ϵ_N^K injective, $N \simeq R_K N$ and hence $N \in K$). "$\supset$" is obvious and $\operatorname{Im} i^\# \subset [\![\psi_N^C]\!]^{TC,\epsilon_C} = \operatorname{Im}(i^\# \circ \epsilon_N^K)$ holds due to $N, i \models \varphi_N^C/\psi_N^C$. ☐

Remark. In the case of $\Omega = \mathcal{P}$ the cofree coalgebras TC do not exist in $\mathbf{Set}_\Omega$ but in $\mathbf{SET}_\Omega$. This has no effect on the proof since for all $\alpha^\# : M \to TC$, $M \in \mathbf{Set}_\Omega$, also $\operatorname{Im} \alpha^\# \in \mathbf{Set}_\Omega$. Note that this reasoning cannot be transferred to the proof of the covariety theorem in [9] since there one needs to consider coreflections $R_K TC$ (called $F_K C$ in [9]) of the cofree coalgebras which usually are only in $\mathbf{Set}_\Omega$ if $TC \in \mathbf{Set}_\Omega$ (which is not the case for $\Omega = \mathcal{P}$ and $C \neq \{\}$).

5 Rule Definable Classes of Kripke Frames

The generality of theorem 4.1 allows for many applications. For example it is possible to give a version of this theorem for coalgebraic logic (Moss [12]). For covarieties instead of smc-classes this has been carried out in [9]. Coalgebraic logic has the advantage that it gives a definition of a modal logic for coalgebras for all functors Ω (preserving weak pullbacks). But here we only want to give one example of a (concrete) modal logic. We choose $\Omega = \mathcal{P}$ and show that our theorem becomes a statement about rule-definable classes of Kripke frames.

We denote with $\mathcal{ML}$ the infinitary modal logic built from a proper class of propositional variables **Prop**, the constant $\perp$, the operators $\neg, \Box$ and conjunctions over any set of formulas. $\bigvee$ and $\Diamond$ are defined as abbreviations. When $P \subset \mathbf{Prop}$ and P a set we write $\mathcal{ML}(P)$ for the class of formulas taking only variables from P.

In order to apply theorem 4.1 we need the following lemma:

Lemma 5.1. *The collection of all $\mathcal{ML}(P)$ where P ranges over subsets of* **Prop** *is an expressive modal logic for coalgebras w.r.t. the functor $\mathcal{P}$. Furthermore, the classes $\mathrm{Ru}_{\mathcal{P}P}$ (see definition 3.2) are the classes of rules of $\mathcal{ML}(P)$.*

Proof. Instantiating Col of definition 3.1 by $\{\mathcal{P}P : P \subset \textbf{Prop}, P \text{ a set}\}$ and $\models^{C}_{(M,\alpha)}$ by the usual satisfaction relation of modal logic, it is immediate that the conditions of definition 3.1 are met. Expressiveness can be shown as in [9]. Next, let $\varphi/\psi \in \mathrm{Ru}_{\mathcal{P}P}$ and M be a Kripke frame. Then, according to the definition of a rule in modal logic, $M \models \varphi/\psi$ iff $\forall \alpha : UM \to \mathcal{P}P : M, \alpha \models \varphi \;\Rightarrow\; M, \alpha \models \psi$, matching definition 3.2. □

Theorem 5.2. *Let K be a class of Kripke frames. Then K is rule-definable iff K is closed under p-morphic images and disjoint unions.*

Proof. Recall that $Col = \{\mathcal{P}P : P \subset \textbf{Prop}, P \text{ a set}\}$.

"if": By lemma 5.1 and theorem 4.1.

"only if": K is rule-definable, that is, there is a class $\Phi \subset \{\varphi/\psi : \varphi, \psi \in \mathcal{ML}\}$ such that $K = \{M \in \textbf{Set}_\Omega : M \models \Phi\}$. Let $K_{\mathcal{P}P} = \{M \in \textbf{Set}_\Omega : M \models \Phi \cap \mathrm{Ru}_{\mathcal{P}P}\}$. We can then write $K = \bigcap\{K_{\mathcal{P}P} : \mathcal{P}P \in Col\}$ and, by proposition 3.5, $K = \bigcap\{\mathrm{H}\Sigma K_{\mathcal{P}P} : \mathcal{P}P \in Col\}$. Now, it follows from a general fact on closure operators that $K \supset \mathrm{H}\Sigma \bigcap\{K_{\mathcal{P}P} : \mathcal{P}P \in Col\}$ and, therefore, $K \supset \mathrm{H}\Sigma K$. □

6 Conclusion

This paper showed that the duality between quotients in algebra and subcoalgebras in coalgebra does not only allow for a dual of Birkhoff's variety theorem but also for a dual of the result characterising implicationally definable classes of algebras. Moreover, it was shown that the modal concept corresponding to an implication is not that of a formula $\varphi \to \psi$ but that of a rule φ/ψ.

Let us mention that the duality of algebras and coalgebras has been used here as a heuristics. The proof of theorem 4.1 is not the formal (categorical) dual of a corresponding proof for algebras since it depends on the category of sets (and coalgebras over **Set** are dual to algebras over $\textbf{Set}^{op}$). But it is possible to give an account of the duality of modal and equational logic which makes the duality precise in a categorical sense. This will appear in [10].

Acknowledgements

I want to thank Alexander Knapp for comments on a previous draft.

References

[1] Peter Aczel. *Non-Well-Founded Sets.* Center for the Study of Language and Information, Stanford University, 1988.

[2] Peter Aczel and Nax Mendler. A final coalgebra theorem. In D. H. Pitt et al, editor, *Category Theory and Computer Science*, volume 389 of *LNCS*, pages 357–365. Springer, 1989.

[3] B. Banaschewski and H. Herrlich. Subcategories defined by implications. *Houston Journal of Mathematics*, 2(2):149–171, 1976.

[4] Francis Borceux. *Handbook of Categorical Algebra.* Cambridge University Press, 1994.

[5] H. P. Gumm and T. Schröder. Covarieties and complete covarieties. In B. Jacobs, L. Moss, H. Reichel, and J. Rutten, editors, *Coalgebraic Methods in Computer Science (CMCS'98)*, volume 11 of *Electronic Notes in Theoretical Computer Science*, 1998.

[6] H. Peter Gumm. Equational and implicational classes of co-algebras. extended abstract. *RelMiCS'4. The 4th International Seminar on Relational Methods in Logic, Algebra and Computer Science, Warsaw*, 1998.

[7] Bart Jacobs. Towards a duality result in coalgebraic modal logic. In Horst Reichel, editor, *Coalgebraic Methods in Computer Science (CMCS'00)*, volume 33 of *Electronic Notes in Theoretical Computer Science*, pages 163–198, 2000.

[8] Fred Kröger. *Temporal Logic of Programs.* Springer, 1987.

[9] Alexander Kurz. A co-variety-theorem for modal logic. In *Proceedings of Advances in Modal Logic 2, Uppsala, 1998.* Center for the Study of Language and Information, Stanford University. To appear.

[10] Alexander Kurz. *Logics for Coalgebras and Applications to Computer Science.* PhD thesis, Ludwig-Maximilians-Universität München. Forthcoming.

[11] Alexander Kurz. Specifying coalgebras with modal logic. *Theoretical Computer Science*, 260(1-2), 2000. To appear.

[12] Lawrence Moss. Coalgebraic logic. *Annals of Pure and Applied Logic*, 96:277–317, 1999.

[13] Martin Rößiger. From modal logic to terminal coalgebras. *Theoretical Computer Science*, 260(1-2), 2000. To appear.

[14] J. J. M. M. Rutten. Universal coalgebra: A theory of systems. Technical Report CS R 9652, CWI, Amsterdam, 1996.

A Strong Monomorphisms

We recall the definition of strong monos according to Borceux [4], I.4.3 and state some properties relating strong monos and coalgebras.

Definition A.1 (Strong mono). A mono $f : M \to N$ is called strong iff for all epis $g : X \to Y$ and all $u : X \to M, v : Y \to N$ such that $f \circ u = v \circ g$ there is a (necessarily unique) $w : Y \to M$ such that $w \circ g = u$ and $f \circ w = v$:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "u"'] & Y \arrow[d, "v"] \arrow[dl, "w"'] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

An immediate consequence of this definition is the following useful proposition.

Proposition A.2 (Extremal monos). *A strong mono m is extremal, that is, if m factors as $m = f \circ e$ with e epi, then e is iso.*

The use of strong monos in our setting is explained by the following proposition:

Proposition A.3.

1. *If $f \in \mathbf{Set}_\Omega$ is mono as a mapping in $\mathbf{Set}$ then f is strong mono in $\mathbf{Set}_\Omega$.*
2. *Every morphism $f \in \mathbf{Set}_\Omega$ factors uniquely as an epi followed by a strong mono. Moreover, this factorisation is obtained as the epi/mono factorisation of f in $\mathbf{Set}$.*
3. *Strong monos in $\mathbf{Set}_\Omega$ are monos in $\mathbf{Set}$.*

Proof.

1. Let $f, g, u, v \in \mathbf{Set}_\Omega$ as in the diagram above and f mono in $\mathbf{Set}$, g epi in $\mathbf{Set}_\Omega$. Since g epi in $\mathbf{Set}$ (Rutten [14], 4.7) the required w exists as a mapping in $\mathbf{Set}$. It remains to show that w is a morphism in $\mathbf{Set}_\Omega$, i.e. $\Omega w \circ f_Y = f_M \circ w$ where f_Y, f_M are the structure maps of the coalgebras Y, M, respectively. Drawing the appropriate diagram, this follows from g, u morphisms in $\mathbf{Set}_\Omega$, g epi in $\mathbf{Set}$ and $w \circ g = u$ in $\mathbf{Set}$.
2. Let $h : X \to Y \in \mathbf{Set}_\Omega$ and $UX \xrightarrow{e} \operatorname{Im} h \xrightarrow{m} UY$ the factorisation of h in $\mathbf{Set}$ through its image $\operatorname{Im} h$. We have to show that $\operatorname{Im} h$ can be equipped in a unique way with a coalgebra structure such that e, m become morphisms in $\mathbf{Set}_\Omega$. This follows from the "diagonal fill in":

$$\begin{tikzcd} UX \arrow[r, "e"] \arrow[d, "\Omega e \circ f_X"'] & \operatorname{Im} h \arrow[d, "f_Y \circ m"] \arrow[dl, "f_{\operatorname{Im} h}"'] \\ \Omega \operatorname{Im} h \arrow[r, "\Omega m"'] & \Omega UY \end{tikzcd}$$

   The diagonal fill in exists because either $\operatorname{Im} h \neq \{\}$ and Ωm mono[8] and e epi or $\operatorname{Im} h = \{\}$ and the empty map makes the diagram commute.

   That m is strong in $\mathbf{Set}_\Omega$ follows from (1). Uniqueness of the factorisation in $\mathbf{Set}_\Omega$ may be found in [4], I.4.4.5.
3. By (2), a strong mono $h : X \to Y$ in $\mathbf{Set}_\Omega$ factors as epi/strong-mono $h = e \circ m$. Since h is also extremal and e is epi, it follows e iso. m is strong, so is h.

□

[8] $\operatorname{Im} h \neq \{\}$ and m mono implies m split mono, hence Ωm mono.