A Note on the Frame Semantics of Modal Logic

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Abstract

A new class of frames for modal logic is introduced. They are defined like general frames, but with a different closure condition imposed on the set of allowed valuations. Not closure under modal operators but closure under (arbitrary) unions is demanded. To compare these so-called complete frames with general frames, a common generalization—Boolean frames—is introduced. Then three main results on complete frames are presented. First, complete frames are semantically complete in the sense that every normal modal logic is characterized by a complete frame. Second, frames that are both general and complete are bisimilar to Kripke frames. Third, complete frames can be regarded as Kripke frames where the set of worlds is replaced by a multiset of worlds. We give an example of how this concept may be applied to the theory of concurrent processes.

1 Introduction

A new class of frames is introduced. They are defined like general frames (cf. van Benthem[1]), but with a different closure condition imposed on the set of allowed valuations. Not closure under modal operators but closure under (arbitrary) unions is demanded. To be more precise: A *complete frame* $(W, R, \mathcal{V})$ consists of a set of worlds W, a relation $R \subset W \times W$ and a subset $\mathcal{V}$ of the powerset of W that is closed under Boolean operators and unions, i.e., $(\mathcal{V}, -, \cup)$ has to be a complete Boolean algebra where complements are taken w.r.t. W. A general frame is defined similar but, instead of being complete, $\mathcal{V}$ has to be closed under the modal operator m (see the next section for a definition).

In this paper we will take a closer look at complete frames. Our first concern is a comparison with general frames. As the latter, complete frames are semantically complete for normal modal logics, that is, every modal logic characterized by a general frame is also characterized by some complete frame. To show this we give a construction from general frames to modally equivalent complete frames in section 3. On the other hand, not every theory of a given complete frame is characterized by a general frame. This is because there are theories of complete frames that are not closed under modal substitutions.[1] Thus, from the point of view of modal theories, complete frames may be regarded as a generalization of general frames. But from a model theoretic point of view we get a different picture: We will show in section 4 that a frame that is general and complete is bisimilar to a Kripke frame.

Next, we show that complete frames can be regarded as Kripke frames where the set of worlds is replaced by a multiset (or sequence) of worlds. This allows for frames explicitly mentioning repetition of identical worlds (states). We sketch an application to the theory of concurrent processes and temporal logic.

Before starting on these topics, a common generalization of complete and general frames—Boolean frames—is introduced and the notion of bisimulation is adapted.

2 Basic Definitions

Let us first fix the notation. We consider a language $\mathcal{L}$ of propositional modal logic built in the usual way from propositional variables $p, q, \ldots$, parentheses, the unary operator $\neg$, the binary operator $\vee$ and one unary modality $\Diamond$. The connectives $\Box, \wedge, \rightarrow$ are defined as usual. The restriction to one modality is not essential and only simplifies the notation. A *model* $M = (W, R, V)$ for this language consists of a set of worlds W, a relation R over W and a function V from proposition letters to subsets of W. $V(p)$ denotes the set of worlds where p is true. We write $M, w \models \alpha$ if the model M satisfies the modal formula α in w.[2] We say that $M \models \alpha$ iff $M, w \models \alpha$ for all $w \in W$. A *Kripke frame* $K = (W, R)$ abstracts from a concrete valuation, therefore $K \models \alpha$ iff $(W, R, V) \models \alpha$ for all valuations V. A *general frame* $G = (W, R, \mathcal{V})$ is a Kripke frame plus a restriction $\mathcal{V}$ (that we will call the *valuation range* of G) on the allowed valuations: $\mathcal{V}$ is a subset of the powerset of W and $G \models \alpha$ iff $(W, R, V) \models \alpha$ for all valuations V such that $V(p) \in \mathcal{V}$ for all proposition letters p. Such a model (W, R, V) is said to be *based on* G. Furthermore $(\mathcal{V}, -, \cup, m)$ has to form a modal algebra, i.e., $\mathcal{V}$ is closed under Boolean operations as well as under the modal operator m that is defined by $m(X) = \{y \in W : \exists x \in X : yRx\}$.[3] The (modal) theory of a frame F is the set $\{\alpha : F \models \alpha\}$. When we call two frames *equivalent*, we always mean that they have the same theory. Note that a Kripke frame (W, R) is equivalent to the general frame $(W, R, 2^W)$. In the following we will not distinguish between the general frame $(W, R, 2^W)$ and the Kripke frame (W, R).

In contrast, a **complete frame** $C = (W, R, \mathcal{V})$ may have a valuation range $\mathcal{V}$ that is not closed under the modal operator. Instead $\mathcal{V}$ has to be a complete Boolean algebra. For comparison of general and complete frames we define a **Boolean frame** as a frame $B = (W, R, \mathcal{V})$ where $\mathcal{V}$ just has to form a Boolean algebra.

One aspect of these different structures is what kinds of logics they allow to describe. In contrast to models, the theory of a general frame is closed under substitutions of

[1] In this respect (and some others to be explained later) complete frames are more similar to models than to Kripke frames.

[2] $M, w \models p$ iff $w \in V(p)$, Boolean connectives as usual, $M, w \models \Diamond\alpha$ iff $\exists v : wRv \;\&\; M, v \models \alpha$.

[3] X is the set of worlds where α is true iff $m(X)$ is the set of worlds where $\Diamond\alpha$ holds.

formulas for propositional variables. The closure under substitutions follows from the fact that the valuation range of a general frame is closed under Boolean and modal operators. It is therefore not surprising that the theory of a Boolean frame is closed only under substitutions of Boolean formulas. Moreover we have the following: The theories of Boolean frames are exactly those theories obtained from the standard formal systems for the modal logic **K** when we replace the rule of substitution by a rule allowing only substitution of Boolean formulas. More precisely, define $\models_b, \vdash_b$ as follows: $\Sigma \models_b \alpha$ iff for all Boolean frames B it holds that $B \models \Sigma$ implies $B \models \alpha$. And $\Sigma \vdash_b \alpha$ iff α may be derived from Σ using as axioms only propositional tautologies and formulas of the form $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$ and as rules only modus ponens, necessitation[4] and the substitution of Boolean formulas for propositional variables. Then

$$\Sigma \vdash_b \alpha \iff \Sigma \models_b \alpha.$$

The proof uses the canonical model of Σ and is the same as for $\Sigma \vdash \alpha \Leftrightarrow \Sigma \models \alpha$ where $\vdash$ denotes derivability with arbitrary substitutions and $\models$ is the consequence relation w.r.t. general frames (cf. van Benthem[1]).

Bisimulation between models is an important notion. Between Kripke frames it is less interesting because it degenerates to isomorphism. But in the case of Boolean (general, complete) frames bisimulation gets interesting again. An example for a general and complete frame that is bisimilar to a Kripke frame but looks rather differently is given in section 5. In order to shorten the definition we introduce the following notation: a relation $Z \subset W \times W'$ induces two functions $\vec{z}: 2^W \to 2^{W'}$ and $\overleftarrow{z}: 2^{W'} \to 2^W$ defined by $\vec{z}\,(X) = \{y \in W' : \exists x \in X : x\,Z\,y\}$ and $\overleftarrow{z}\,(Y) = \{x \in W : \exists y \in Y : x\,Z\,y\}$.

Def 2.1 *Let $B = (W, R, \mathcal{V})$, $B' = (W', R', \mathcal{V}')$ be two Boolean frames. A relation Z is called a* **bisimulation** *between B and B' if Z respects the usual conditions on R and $\vec{z} \circ \overleftarrow{z} = \mathrm{id}_{\mathcal{V}'}$ and $\overleftarrow{z} \circ \vec{z} = \mathrm{id}_{\mathcal{V}}$. An* **isomorphism** *is a bisimulation where Z is a bijective function.*

A consequence of this definition is that if two frames bisimulate then for any model based on one of them we can find a bisimilar one based on the other.[5] This is made possible by the condition that $\mathcal{V}$ and $\mathcal{V}'$ are isomorphic via $\vec{z}$. Therefore, it follows from the corresponding lemma on bisimulation between models that if two frames are bisimilar then they are equivalent.

3 Completeness

Here we show that complete frames are semantically complete for normal modal logics by proving that every modal logic that is characterized by a general frame is also characterized by some complete frame. The prove of the following theorem gives a construction that transforms any general frame into an equivalent complete frame. Then an example illustrating this construction is presented. (It may be helpful to look at the example before going through the proof.)

[4] Necessitation means the rule $\alpha \vdash \Box\alpha$.

[5] Note that in case of complete frames the definition of isomorphism is equivalent (via theorem 5.1) to the one given in footnote 9.

Theorem 3.1 *For any general frame there is an equivalent complete frame.*

Proof. Let $G = (W, R, \mathcal{V})$ be a general frame. We will define an equivalent complete frame $B^G = (W^G, R^G, \mathcal{V}^G)$.

For every modal formula α let $Var(\alpha)$ denote the set of propositional variables occurring in α. For every valuation V with $V(p) \in \mathcal{V}$, for all p, and every modal formula α define $\mathcal{V}_{V,\alpha}$ as the Boolean algebra generated by $\{V(p) : p \in Var(\alpha)\}$. Since $\mathcal{V}_{V,\alpha}$ is finite it is also complete and each $(W, R, \mathcal{V}_{V,\alpha})$ is a complete frame. Now define B^G as the disjoint union[6] of all the $(W, R, \mathcal{V}_{V,\alpha})$. Since completeness is preserved under disjoint unions this is indeed a complete frame. It remains to show $G \models \alpha \Leftrightarrow B^G \models \alpha$.
" $\Rightarrow$ ": Suppose $B^G \not\models \alpha$. That is, one of the disjoint frames of which B^G is composed, say $B = (W, R, \mathcal{V}_B)$, rejects α. It follows the existence of a valuation V with $V(p) \in \mathcal{V}_B$ and $(W, R, V) \not\models \alpha$. But since the valuation ranges of general frames are closed under Boolean operators we get $V(p) \in \mathcal{V}$. Hence $(W, R, \mathcal{V}) \not\models \alpha$.
" $\Leftarrow$ ": Suppose $G \not\models \alpha$. That is, there is a valuation V with $V(p) \in \mathcal{V}$ and $(W, R, V) \not\models \alpha$. Hence $(W, R, \mathcal{V}_{V,\alpha}) \not\models \alpha$ and therefore $B^G \not\models \alpha$. **QED.**

Corollary 3.2 *For any Boolean frame there is an equivalent complete frame.*

Proof. The same as above. We didn't use that the valuation ranges of general frames are closed under modal operators. **QED.**

We give an example. Let $G = (\mathbf{N}, \leq, \mathcal{C})$ be the general frame consisting of the natural numbers, the usual $\leq$ relation and the set of all finite and cofinite subsets of $\mathbf{N}$. Now we will build an equivalent complete frame from disjoint unions of "restrictions" of G. Let $B_i = (\mathbf{N}, \leq, \mathcal{C}_i)$ where $\mathcal{C}_i$ is the Boolean algebra generated by $\{\{0\}, \dots \{i-1\}\}$ for $i \geq 0$. B is defined as the disjoint union of the B_i.[7] It is not difficult to see that both frames are equivalent. The intuitive reason is the following. $\mathcal{C}$ prevents modal formulas to change their validity infinitely often. A consequence is that $G \models \Box\Diamond\alpha \to \Diamond\Box\alpha$. This formula does not hold in the Kripke frame $(\mathbf{N}, \leq)$ because $(\mathbf{N}, \leq)$ allows counterexamples that are prevented by $\mathcal{C}$. Now, we can express the same constraint on the possible valuations using infinitely many copies of this Kripke frame and for each of them a much simpler definition of the valuation ranges. The fact that modal formulas does not change their value infinitely often is at least as immediate for B than for G. I think that completeness is easier to visualize than closure under modal operators. The reason is that completeness allows the construction of atoms as intersections of ultrafilters of worlds (see the next section).

The reader may have noted that in our example the valuation ranges of the frames B_i are closed under the modal operator m. That is, B is not only a complete frame but also a general frame. But this is an exception. Generally this disjoint-union-construction will destroy modal closure. This follows from theorem 4.2 and the incompleteness of Kripke frames. As an example consider the general frame $G = (\mathbf{N}, <, >, \mathcal{C})$, $\mathcal{C}$ as above. The appropriate language should now contain two modalities $\Diamond_<$, $\Diamond_>$. Note that $\mathcal{C}$ is closed

[6] In the disjoint union $(W, R, \mathcal{V}) = \biguplus_I (W_i, R_i, \mathcal{V}_i)$ W, R are as usual. $\mathcal{V}$ contains all unions of sets of the form $\{(i, w) : w \in X\}$ for some X in $\mathcal{V}_i$.

[7] Comparing the definition of B with the one of B^G in the proof, it is conspicuous that here the disjoint union is formed of fewer frames. The difference is not essential and is only to simplify both the example and the proof.

under the operators $m_<$ and $m_>$ and that we still have $G \models \Box_< \Diamond_< \alpha \to \Diamond_< \Box_< \alpha$. Now, make the same disjoint union construction as above, that is, let B be the disjoint union of the frames $B_i = (\mathbf{N}, <, >, \mathcal{C}_\mathbf{i})$, $\mathcal{C}_i$ as above. Again B and G are equivalent. But the $\mathcal{C}_i$ are not closed under $m_>$. Hence B is not a general frame.

We hope to have shown that completeness is one reason why complete frames may be useful: they allow (at least in some examples) a quite natural semantics in cases where the usual Kripke frames don't suffice.

4 A Comparison of General and Complete Frames

We first characterize complete frames in terms of Kripke frames plus an equivalence relation over the set of worlds. Then we show that a frame that is general and complete is bisimilar to a Kripke frame.

Any Kripke frame (W, R) together with an equivalence relation $\sim$ over W determines uniquely a complete frame $(W, R, \mathcal{V}^\sim)$ if we define $\mathcal{V}^\sim$ to be the set of all unions of equivalence classes of $\sim$, i.e., $\mathcal{V}^\sim = \{X \subset W : \forall x, y \in W : x \sim y \ \&\ x \in X \ \Rightarrow \ y \in X\}$. Conversely, every complete frame $C = (W, R, \mathcal{V})$ determines uniquely an equivalence relation $\sim_\mathcal{V}$, where $\sim_\mathcal{V}$ is defined by the partition of W containing the atoms[8] of $\mathcal{V}$. That the atoms of the complete Boolean algebra $\mathcal{V}$ form indeed a partition of W follows from the fact that a complete field of sets is atomic (cf. Sikorski[5], chapter 25).

In a complete field of sets the atoms are exactly the intersections of the ultrafilters. Therefore they are minimal classes of worlds being forced to fulfill the same propositions. In the next section we will pursue this idea and consider the atoms as states (of a program execution, for example). There (cf. theorem 5.1) it will be important to note that the above constructions of an equivalence relation out of a valuation range and that of a valuation range out of an equivalence relation are inverse to each other. That is $\sim_{(\mathcal{V}^\sim)} = \sim$ and $\mathcal{V}^{(\sim_\mathcal{V})} = \mathcal{V}$.

Now we want to show that a frame that is general and complete is bisimilar to a Kripke frame. Given a complete frame $C = (W, R, \mathcal{V})$ we define a Kripke frame $C^\sim = (W^\sim, R^\sim, 2^{W^\sim})$: $W^\sim$ is the partition of W induced by $\sim$ (writing $\sim$ for $\sim_\mathcal{V}$) and $R^\sim$ is defined by $[w]R^\sim[v] \ \Leftrightarrow \ \exists w_1 \sim w : \exists v_1 \sim v : w_1 R v_1$, where $[w]$ denotes the equivalence class of w w.r.t. $\sim$.

Proposition 4.1 *$wRv \ \&\ w_1 \sim w \ \Rightarrow \ \exists v_1 : v_1 \sim v \ \&\ w_1 R v_1$ iff $\mathcal{V}$ closed under m.*

Theorem 4.2 *Let $C = (W, R, \mathcal{V})$ be a complete frame and $C^\sim$ as above. Then $\mathcal{V}$ is closed under m iff the relation $Z = \{(w, [w]) : w \in W\} \subset W \times W^\sim$ is a bisimulation between C and $C^\sim$.*

We have shown that a frame that is complete and general is bisimilar to a Kripke frame. Now we will show that the converse also holds.

Proposition 4.3 *Let $(W, R, \mathcal{V})$ and $(W', R', \mathcal{V}')$ be two Boolean frames and Z a bisimulation between them. Then it holds for all $X, X_i \in \mathcal{V}$:*

- $\overleftarrow{Z}(m'(\overrightarrow{Z}(X))) = m(X)$,

[8] $X \in \mathcal{V}$ is an atom iff $X \neq \{\}$ and $\forall Y \in \mathcal{V} : \{\} \neq Y \subset X \ \Rightarrow \ Y = X$.

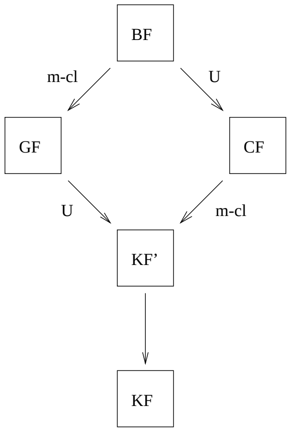
- $\overleftarrow{z}\,(\bigcup_i(\overrightarrow{z}\,(X_i))) = \bigcup_i(X_i)$.

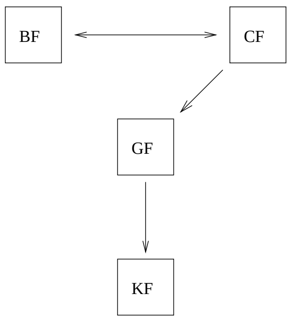
It follows that modal closure and completeness are preserved by bisimulations. We therefore have the following theorem.

Theorem 4.4 *A frame that is bisimilar to a Kripke frame is a general, complete frame.*

The interest in the class of general and complete frames may be explained as follows: As indicated above the atoms are minimal classes of worlds being forced to fulfill the same propositions. If we want to interpret the atoms moreover as states being a set of complete information, it seems sensible that identical states not only fulfill the same propositions, but also agree in all modal formulas. A sufficient condition for this is that these identical states may be identified by some bisimulation. But theorem 4.2 shows that there is a bisimulation identifying all identical states if and only if the complete frame is also a general frame.

We can summarize the results of the last two sections in two pictures drawing hierarchies of classes of frames. The first one gives a semantic interpretation in terms of closure conditions and bisimulation. The second one classifies according to classes of logics that are characterizable.





In the first picture $A \rightarrow B$ signifies $A \supset B$. The class KF$'$ is meant to contain all frames $(W, R, \mathcal{V})$ that are bisimilar to a Kripke frame. Because of the theorems above, we have KF$'$ = GF $\cap$ CF. An example of an interesting frame of this class is presented in section 5. 'm-cl' stands for modal closure and 'U' for completeness. In the second picture $A \rightarrow B$ signifies: for every frame in B there is an equivalent frame in A.

5 Multisets of Worlds - An Application

In this section we will see how complete frames can be understood as Kripke frames with multisets of worlds. An example of a possible application in temporal logics of concurrent programs is given.

Let us first define the new concept and only then show how it relates to complete frames. We will speak of sequences of states rather than of multisets of worlds in view of the example at the end of the section. In our terminology a sequence of states is just a mapping $s : W \rightarrow S$ where S is the set of states and W an index set. We will see later that the index set and not the set of states corresponds to the set of possible worlds in a Kripke style semantics, hence the notation. The index set may be ordered by a relation R, that is, $R \subset W \times W$. On the other hand, as the notion of state already suggests, the valuations V will be over S. What do we gain by this construction? Multiple occurrences of the same state become representable on the level of frames. Of course, on the level of models we already had the ability to force two worlds to fulfill the same propositions. But with Kripke frames this was not possible. The merits of this new structures will be illustrated in the example below.

In view of the above discussion we define a **sequence frame** to be a structure $(s : W \to S, R)$ where s is a function and $R \subset W \times W$. A *sequence model* M consists of a frame and an assignment of truth values to propositional variables. M *is based on* $P = (s : W \to S, R)$ iff $M = (s : W \to S, R, V)$ and V a function from the set of propositional variables into the powerset of S. $V(p)$ denotes the set of states where p is true. The definition of validity w.r.t. a point w in a sequence model M goes as follows:

$$\begin{array}{lll} M, w \models p & \text{iff} & s(w) \in V(p) \\ \neg, \vee & & \text{as usual} \\ M, w \models \Diamond\alpha & \text{iff} & \text{there is a } v \text{ with } wRv \;\&\; M, v \models \alpha \end{array}$$

Only the first clause of the definition has been changed as compared to the validity definition of Kripke models. As usual, $M \models \alpha$ iff $M, w \models \alpha$ for all $w \in W$ and $P \models \alpha$ iff $M \models \alpha$ for all sequence models M based on P.

Now, we will show that sequence frames offer just a different notation for complete frames. Recall from section 4 that every complete frame $C = (W, R, \mathcal{V})$ is uniquely determined by the Kripke frame (W, R) and the equivalence relation $\sim_{\mathcal{V}}$. $W^{\sim} = \{[w] : w \in W\}$ has been defined as the respective set of equivalence classes. Given $C = (W, R, \mathcal{V})$, we define the corresponding sequence frame $C^s = (s : W \to W^{\sim}, R)$, where s is given by $s(w) = [w]$. Conversely, given a sequence frame $P = (s : W \to S, R)$ we define the corresponding complete frame $P^c = (W, R, \mathcal{V}^c)$ where $\mathcal{V}^c$ contains all unions of equivalence classes of $w \sim v \;\Leftrightarrow\; s(w) = s(v)$.

Theorem 5.1 *Given a complete frame $C = (W, R, \mathcal{V})$ and a sequence frame $P = (s : W' \to S', R')$ it holds:*

- $C^s \models \alpha$ *iff* $C \models \alpha$ *and* $P^c \models \alpha$ *iff* $P \models \alpha$,
- $(P^c)^s$ *isomorphic to* P *and* $(C^s)^c$ *isomorphic to* C.

The proof uses the characterization of complete frames as Kripke frames plus an equivalence relation (see section 4) and the fact that a sequence frame $P = (s : W \to S, R)$ is determined up to isomorphism[9] by the Kripke frame (W, R) and the equivalence relation $w \sim v \;\Leftrightarrow\; s(w) = s(v)$.

Next, we will give an example of the use of complete frames. We consider propositional linear temporal logic (LTL). A Kripke frame for LTL is $N = (\mathbf{N}, \leq, \sigma)$, where $\mathbf{N}$ is the set of natural numbers, $\leq$ as usual and σ the successor relation. The language contains two

[9] An *isomorphism between sequence frames* $(s : W \to S, R)$, $(s' : W' \to S', R')$ is a bijection $f : W \to W'$ such that f is an isomorphism between (W, R) and (W', R') and furthermore $s(w) = s(v) \;\Leftrightarrow\; s'(f(w)) = s'(f(v))$.

modal operators $\Box$ and $\circ$ that are interpreted by the relations $\leq$ and σ respectively. LTL itself is defined as the set of all modal formulas valid in N (for details see e.g. Goldblatt[2]). Now, in computer science, a well-known problem with this semantics is that it is not stutter invariant (see, e.g., [4] or [3] for a discussion). That is, given a formula α and a model M for α there may be a stutter variant M' of M with $M' \not\models \alpha$. A stutter variant of M is a model that differs from M only because some repetitions of identical states have been inserted or eliminated. A semantics that is not stutter invariant causes problems concerning composition and refinement. Now, well-known again, this defect is not an inherent property of LTL itself but depends strongly on the semantic assumptions. For example, LTL may be rendered stutter invariant by replacing the relation σ by a relation σ_s with the following property: $w\sigma_s v$ if, given w, v is the next point whose state is different and, if no such state exists, $w\sigma_s(w+1)$. To prove that these two semantics are indeed equivalent the easiest way is the following: Let us call LTL$_s$ the logic characterized by the class of sequence frames $(s : \mathbf{N} \to \mathbf{N}, \leq, \sigma_s)$. Now, identity of LTL and LTL$_s$ follows easily from the fact that each frame $P = (s : \mathbf{N} \to \mathbf{N}, \leq, \sigma_s)$ is bisimilar (in the sense of definition 2.1) to the frame $P' = (s' : \mathbf{N} \to \mathbf{N}, \leq, \sigma)$ where s' is defined as the sequence that is obtained from s by eliminating all finite repetitions of identical states.

An objection to this example might be that the well-established bisimulation between models is enough to prove the identity of LTL and LTL$_s$. So why should we work with sequence frames? The relationship of sequence frames to models is similar to that of Kripke frames to models. One reason of their usefulness is that they offer a convenient way to define the appropriate classes of models. Often, one prefers to work with classes of models being based on certain frames with some agreeable properties. On the other hand, as the example suggests, there are cases where Kripke frames do not allow to define the models one is interested in. Sequence frames may repair this defect. For example, here, sequence frames allow a notion of indistinguishability between worlds which is independent of the interpretation of any particular proposition. Another reason why sequence frames are useful is that the notion of a bisimulation is lifted to the level of frames. For example, the class of frames bisimilar to the Kripke frame $(\mathbf{N}, \leq, \sigma)$ contains 'stutter invariant' frames $(s : \mathbf{N} \to \mathbf{N}, \leq, \sigma_s)$.

6 Conclusion

Besides general frames, complete frames are a second generalization of the concept of a Kripke frame. This can also be formulated in the following way: There are classes of Kripke models that are based on a complete frame but on no Kripke frame. One consequence of this fact was that we get semantical completeness (section 3). Another was that the theories may not be closed under (modal) substitutions (section 2). When we want to avoid this, one possibility is to consider frames that are complete and general.[10] We showed that these are exactly those (Boolean) frames that are bisimilar to Kripke frames (section 4). We therefore can transfer all we know about the logics of Kripke frames. On the other hand, we have the possibility to define frames with properties (like stutter invariance) that are not shared by Kripke frames (section 5).

Another interesting point is the way we can look at complete frames. The starting point was a rather abstract variation of the closure condition of the valuation ranges of

[10] Another is to work with classes of complete frames whose theories are closed under substitutions.

Boolean frames. But then we saw that we can conceive complete frames as Kripke frames plus an equivalence relation on the set of possible worlds (section 4), or as Kripke frames with the set of worlds replaced by multisets of worlds (or sequences of states). The reason was that completeness of a field of sets guarantees that the algebra is atomic, that is, we can rebuild any set from atoms. These atoms generally do not exist in general frames. Therefore, one has to work with subsets of the powerset of the set of worlds to restrict the possible valuations. Section 5 showed how atoms may be used to give a more implicit definition of these restrictions. An application was the interpretation of atoms as states of program executions.

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