Limits in Categories of Coalgebras

Alexander Kurz
Institut für Informatik
Ludwig-Maximilians-Universität München
Oettingenstr. 67, 80538 München, Germany
e-mail: kurz@informatik.uni-muenchen.de

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Abstract

It is shown how limits can be constructed in category of coalgebras.

1 Introduction

In categories of coalgebras colimits usually exist and are constructed as in the base category. Here we show that under reasonable conditions also limits exist and how they are constructed.

Our proof uses factorisation structures for sinks, a concept dual to the factorisation structures for sources that are used in Adamek, Herrlich and Strecker [1] to analyse "algebraic" categories.

We present two proofs. A first version for the special case of coalgebras over the base category **Set**. In this case we can leave the apparatus of factorisation structures implicit. The second proof for the general case is actually the same, the factorisation structures serving only the purpose to allow for a clear and **Set**-independent way to state the precise assumptions that are necessary to make the proof possible.

2 Limits in Categories of Coalgebras over Set

Before proving the theorem we define *sinks* and give two propositions.

Definition 2.1 (sinks). A sink $(B, (s_i)_{i \in I})$ consists of an object B and a collection of morphisms $s_i : A_i \to B$ with common codomain B.

We frequently write sinks as (s_i) and composition of sinks with a morphism f as $f \circ (s_i) = (f \circ s_i)$.[1] Also, we use the same terminology for sinks as for morphisms. For example, a sink is called epi, or an *epi sink*, iff $f \circ (s_i) = g \circ (s_i)$ implies $f = g$.

The next propositions generalise facts known about morphisms in $\mathbf{Set}_\Omega$ to sinks. The proofs are almost literally the same as for morphisms, see Rutten [2], 4.7 and 2.4.1.

Proposition 2.2. *A sink (e_i) is epi in $\mathbf{Set}_\Omega$ iff it is epi in $\mathbf{Set}$.*

Proposition 2.3. *Epi sinks are final, that is, for any epi sink $(e_i)_{i \in I}$ in $\mathbf{Set}_\Omega$ and any function f in $\mathbf{Set}$, it holds: if $f \circ e_i$ in $\mathbf{Set}_\Omega$ for all $i \in I$, then $f \in \mathbf{Set}_\Omega$.*

Remark. The theorem below is interesting because we prove more than we stated, namely how the limits in $\mathbf{Set}_\Omega$ are obtained: as certain subcoalgebras of the cofree coalgebras. For details we refer to the proof.

Theorem 2.4. *Let Ω be a functor on $\mathbf{Set}$ such that the underlying functor $U : \mathbf{Set}_\Omega \to \mathbf{Set}$ has a right adjoint F. Then $\mathbf{Set}_\Omega$ is complete.*

Proof. Let $D : \mathcal{I} \to \mathbf{Set}_\Omega$ be a diagram in $\mathbf{Set}_\Omega$. Let $c_i : L \to UDi$ be a limit of UD in $\mathbf{Set}$. Consider the cofree coalgebra FL over L and let $\epsilon : UFL \to L$ be the arrow given by the counit of the adjunction.

Let A be a coalgebra and $f_i : A \to Di$ a cone for the diagram D. Since L is a limit of UD, there is a unique $g : UA \to L \in \mathbf{Set}$ such that $Uf_i = c_i \circ g$. Since FL is cofree g lifts to a unique $g^* : A \to FL$ such that $\epsilon \circ Ug^* = g$.

We have seen that every cone $f_i : A \to Di$ gives rise to a $g^* : A \to FL$. Consider the sink (s_j) consisting of all these g^*. We can now define the limit of D: Let C be the subcoalgebra whose carrier consists of all images of all s_j and let $m : C \to FL$ be the corresponding embedding. That C and m are well defined as elements of $\mathbf{Set}_\Omega$ follows from Rutten [2], theorems 6.3 and 6.4.

Note that by definition of m and C, there is a sink (e_j) such that $(s_j) = m \circ (e_j)$, the e_j being the same as the s_j but the codomain restricted to C. Moreover, since every element of UC is in the image of some s_j, the sink (e_j) is epi.

[1] This notation is somewhat dangerous in the case of empty sinks $(B, \{\})$ because then the object B is not implicit anymore in the collection (s_i).

To complete the definition let $l_i = c_i \circ \epsilon \circ Um$ be a cone. By definition of (s_j), for all $i \in \mathcal{I}$ we have that $l_i \circ (Ue_j) = c_i \circ \epsilon \circ (Us_j)$ is a sink in $\mathbf{Set}_\Omega$, hence l_i in $\mathbf{Set}_\Omega$ by the fact that (e_j) is epi and proposition 2.3. Furhtermore, l_i is a cone for D because it is a cone for UD.

To complete the proof we have to show that every cone in $\mathbf{Set}_\Omega$ over D factors uniquely through $l_i : C \to Di$. The existence follows from the definition of C, uniqueness from m being an embedding. □

3 The general theorem

We use some terminology about factorisation structures. For details the reader is referred to Adamek, Herrlich, Strecker [1].

Theorem 3.1. *Let $\mathcal{C}$ be a category that has $(EpiSink, Mono)$-factorisations of sinks.*[2] *Let $U : \mathcal{C} \to \mathcal{X}$ be a faithful functor with right adjoint F and suppose that epi sinks are final.*[3] *Then $\mathcal{C}$ has every type of limit that $\mathcal{X}$ has. In particular, $\mathcal{C}$ is complete if $\mathcal{X}$ is.*

Proof. The proof is almost literally the same as in the previous section. Only that the definition of C via union of images is now replaced by the following. Let (s_j) be as before. By assumption on $\mathcal{C}$, (s_j) factors as $m \circ (e_j)$ with m being mono and (e_j) an epi sink. Define C to be the domain of m. Since (e_j) is final by assumption, the remainder of the proof is as above. □

References

[1] J. Adamek, H. Herrlich, and G. Strecker. *Abstract and Concrete Categories.* John Wiley & Sons, 1990.

[2] Jan Rutten. Universal coalgebra: A theory of systems. Technical Report CS R 9652, CWI, Amsterdam, 1996.

[2] That is, every sink (s_j) factors as $m \circ (e_j)$ where m mono and (e_j) an epi sink.

[3] That is, for $f \in \mathcal{X}$ and (e_j) an epi sink, $f \circ Ue_j$ has a U-preimage in $\mathcal{C}$ for all e_j implies that f has a U-preimage. Note that U-preimages are unique because U is faithful. Definition 2.3 is a special case.