Sequence Frames

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Abstract

In this contribution we will give a short introduction to a new semantics for modal and temporal logic called sequence frames. The interest in sequence frames arises from their use as models for concurrent processes. In contrast to Kripke frames (W, R_r) consisting of sets of worlds W, we consider sequences (or multisets) of worlds, i.e., functions $w : I \to W$. We will show that sequence frames can be employed to give a stutterinvariant semantics for a linear temporal logic with conventional next-time operator. The last section contains some interesting results about the relationship of sequence frames, general frames and Kripke frames.

1 Introduction

Sequences of program states are the standard models used for linear temporal logic like for example TLA by Lamport [5] and TLR by Manna and Pnueli [2]. These logics are essentially modal logics provided with a Kripke style possible world semantics. There are two levels of Kripke semantics, namely models and frames. Both are lacking the possibility to express the idea of a sequence of states. Models because they are too specific in the sense that every world has a fixed valuation, frames because they totally abstract away from the possible valuations. We will define a new class of structures lying in between, the sequence frames. They will allow us to take account of state repetitions in the semantic interpretation of the temporal operators. We will give an application to the problem of stutter invariance in the next section. The last section considers sequence frames from an abstract model theoretic viewpoint. In fact they are similar to the well-known general frames, the difference lying in distinct closure conditions that have to be imposed on the set of the allowed valuations.

This section presents the basic definitions that are almost direct consequences of the corresponding definitions for Kripke frames. For an excellent introduction to modal logics and Kripke frames see Hughes and Cresswell [1]. In the following we keep all definitions as abstract as possible and specialize to linear temporal logic only in the next section.

A *sequence frame* $S = (w : I \to W, R_r)$, or (w, R_r) for short, resembles a Kripke frame (W, R_r) but the set of worlds W is replaced by a sequence of worlds $w : I \to W$. The elements of I are called points (of time, for example), the elements of W worlds or states. The relations R_r are on I. For example, as frames for Manna-Pnueli-like linear temporal logic (see [3] for example) we use $(w : \mathbf{N} \to W, \leq, \sigma)$, where w is a sequence over the natural numbers, $\leq$ interpretes as usual the always operator $\Box$ and the succesor relation σ the next operator $\bigcirc$.

A sequence model M is based on $S = (w : I \to W, R_r)$ iff $M = (w : I \to W, R_r, V)$ and V a function from the set of propositional variables into the powerset of W. $V(p)$ denotes the set of worlds where p is true. The definition of validity w.r.t. a point i in a sequence model M goes as follows:

$$\begin{array}{lll} M, i \models p & \text{iff} & w(i) \in V(p) \\ M, i \models \neg\alpha & \text{iff} & M, i \not\models \alpha \\ M, i \models \alpha \to \beta & \text{iff} & M, i \not\models \alpha \text{ or } M, i \models \beta \\ M, i \models \Box_r\alpha & \text{iff} & \text{for all } j \text{ with } iR_rj : M, j \models \alpha \end{array}$$

Please note that only the first clause of the definition has been changed as compared to the validity definition of Kripke models (cf [1]). As usual, $M \models \alpha$ iff $M, i \models \alpha$ for all $i \in I$ and $S \models \alpha$ iff $M \models \alpha$ for all sequence models based on S. Note that sequence frames $(w : I \to W, R_r, V)$ with w one-to-one are equivalent to Kripke frames (I, R_r).

Of course, every sequence model $(w : I \to W, R_r, V)$ is equivalent to the Kripke model (I, R_r, V') defined by $i \in V'(p) :\Leftrightarrow w(i) \in V(p)$ and, conversely, every Kripke model (I, R_r, V) is equivalent to the sequence model $(id : I \to I, R_r, V)$. Thus, taking models as the semantic structure to work with, no differences appear when temporal (or modal) formulas are interpreted over sequences of states. Probably this is one reason why sequences did not receive much model theoretic attention. Nevertheless, on the level of frames this semantic structure shows some interesting new features. Central to their understanding is the notion of a p-morphism.

A *strong p-morphism* f between two sequence frames $S = (w, R_r)$ and $S' = (w', R'_r)$ is a mapping f from I onto I' satisfying

$$\begin{array}{lll} iR_rj & \text{implies} & f(i)R'_rf(j), \\ w(i) = w(j) & \text{implies} & w'(f(i)) = w'(f(j)) \\ f(i)R'_rj' & \text{implies} & \exists j : iR_rj \;\&\; f(j) = j' \\ w'(f(i)) = w'(f(j)) & \text{implies} & w(i) = w(j) \end{array}$$

for all corresponding relations R_r, R'_r. The first and second condition state that every path in S corresponds via f to a path in S'. The third condition, the so-called p-morphism condition, says that every step $i'R'j'$ in the image S' corresponds to a step iRj in S if $f(i) = i'$. These three conditions define a *p-morphism*. The last condition assures that if two points in S' have identical states then so do their pre-images. Please note that strong p-morphisms are essentially bisimulations (one only had to allow f to be a relation instead of a function).

As for standard frames we have the *p-morphism lemma* that states that $S \models \alpha \Rightarrow S' \models \alpha$ for all formulas α, i.e., p-morphisms are truth-preserving. For strong p-morphisms it holds furthermore $S' \models \alpha \Rightarrow S \models \alpha$. It is interesting to note that the strong p-moprphism definition applied to Kripke frames yields isomorphisms.

2 Two examples

The following two examples are related to the notion of *stuttering invariance* which plays an important role in composing concurrent systems and refining specifications. In the first example two sequences are said to be *stuttering equivalent* iff they can be transformed into the same sequence by eliminating subsequences of identical states. More formally (we consider sequence frames $S = (w : \mathbf{N} \to W, \leq)$ where $\mathbf{N}$ are the natural numbers and $\leq$ is interpreted as usual) we first define a mapping $\natural : \mathbf{N} \to \mathbf{N}$ by

$$\natural_w(i) := \begin{cases} 0 & \text{if } i = 0 \\ \natural_w(i-1) & \text{if } w(i) = w(i-1) \,\&\, i \geq 1 \\ \natural_w(i-1)+1 & \text{otherwise} \end{cases}$$

Next, given a frame S we define the frame $S^\natural := (w^\natural : \natural_w(\mathbf{N}) \to W, \leq)$, where $w^\natural(\natural_w(i)) := w(i)$ and we say that two frames S, S' are stuttering equivalent iff $S^\natural = S'^\natural$.

Now the point is that we can use the concept of sequence frames to define a semantic for a temporal logic *with next-time operator* that is stuttering invariant, i.e., $S^\natural = S'^\natural \Rightarrow (S \models \alpha \Leftrightarrow S' \models \alpha)$ for all formulas α. We define *linear stuttering frames* to be of the form $(w : \mathbf{N} \to W, \leq, \sigma_s)$ where $\mathbf{N}$, $\leq$ as above and σ_s the next-step relation defined by

$$i\sigma_s j :\Leftrightarrow i \leq j \,\&\, w(i) \neq w(j) \,\&\, (i \leq k < j \Rightarrow w(k) = w(i)).$$

Equipped with this next-step relation jumping over subsequent identical states it is not difficult to see that $S^\natural$ is a strong p-morphic image of S. Thus, invariance under stuttering is a consequence of the p-morphism lemma.

In the second example, we will see that sequence frames are also the natural underlying structure for the temporal logic TLR recently presented by Kesten, Manna and Pnueli [2]. Assuming what they call left-continuity we can write a frame for TLR in our formalism as $(w : \mathbf{N} \to W, \leq, \sigma)$ where $w(2j+1) = w(2j)$ and $i\sigma j :\Leftrightarrow (i = j \,\&\, i$ even or $j = i+1 \,\&\, i$ odd). Here, two sequences are said to be stuttering equivalent iff they can be transformed into the same sequence by eliminating states (s_{2j-1}, s_{2j}) whenever $s_{2j-1} = s_{2j}$. As above, invariance of propositional formulas under stuttering follows from the p-morphism lemma.

3 Boolean Frames

This section presents a model theoretic comparison of sequence, Kripke and general frames. A common generalization of these, a so called boolean frame, is introduced and their relationships are analyzed. For more details and results consult [4].

First let us recall the definition of a *general frame* $G = (W, R_r, \mathcal{V})$. It consists of a Kripke frame (W, R_r) and a subset $\mathcal{V}$ of the powerset of W restricting the possible valuations: a model (W, R_r, V) is said to be based on G iff $V(p) \in \mathcal{V}$ for all propositional variables p. Furthermore $\mathcal{V}$ has to be closed under boolean operators (i.e., finite unions and complements) and the modal operators

$$m_r(X) := \{w \in W : \exists v(wR_rv \ \& \ v \in X)\}.$$

These closure conditions ensure that a theory of general frames is closed under substitutions. Please note that every Kripke frame (W, R_r) can be regarded as the general frame $(W, R_r, \mathcal{P}(W))$, $\mathcal{P}(W)$ denoting the powerset of W.

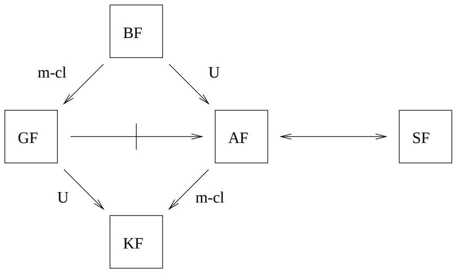
We have seen that both sequence frames and general frames are generalizations of Kripke frames. An obvious question is how do they relate to one another. That they differ follows from the observation that, generally, theories of sequence frames are not closed under substitutions. A simple example is the frame $(\{(1, w), (2, w)\}, \{(1, 2)\})$, verifying $(p \to \Box p)$ but not $(\Diamond p \to \Box\Diamond p)$:

Recalling the definition of a sequence model, we see that a sequence frame $S = (w : I \to W, R_r)$ can be regarded as a Kripke frame (I, R_r) plus a restriction on the possible valuations. How can we express that restriction as a subset $\mathcal{V}$ of the powerset of I? To this end, we define a *boolean frame* to be a frame $(I, R_r, \mathcal{V})$ where $\mathcal{V}$ is closed under boolean operators. That is, a boolean frame is a general frame except that it may not be closed under modal operators. Now, given $S = (w : I \to W, R_r)$, let I^w be the partition of I induced by the equivalence relation $i \sim j \Leftrightarrow w(i) = w(j)$ and $\mathcal{V}^w$ the union-hull of I^w (i.e., $\mathcal{V}^w$ is the smallest set containing I^w and being closed under (finite and infinite) unions). $\mathcal{V}^w$ is called the valuation range of S. It is easy to see that the boolean frame $(I, R_r, \mathcal{V}^w)$ and the sequence frame $S = (w : I \to W, R_r)$ are equivalent (that is, they validate exactly the same formulas).

Having translated every sequence frame into an equivalent boolean frame, the next step is to identify the subclass of boolean frames that corresponds to sequence frames. We say that a boolean frame $(I, R_r, \mathcal{V})$ is *atomic* iff the union-hull of the set $\mathcal{V}^{at} := \{X \in \mathcal{V} : X \neq \{\}$ and $Y \in \mathcal{V} \ \& \ \{\} \neq Y \subset X \Rightarrow Y = X\}$ equals $\mathcal{V}$. The notion *atomic frame* is justified by the fact that $\mathcal{V}^{at}$ is a partition of I. The elements of $\mathcal{V}^{at}$ can therefore be regarded as the atoms of $\mathcal{V}$. Now, we define for every atomic boolean frame $(I, R_r, \mathcal{V})$ the corresponding sequence frame $(w : I \to \mathcal{V}^{at}, R_r)$ where w maps each point of I to the element of $\mathcal{V}^{at}$ containing it. Again it is not difficult to see that these two frames are equivalent.

To extend our picture of the relationship of the four classes of frames (standard, general, atomic, boolean) let us present two further results from [4]. The first states that any atomic frame $(I, R_r, \mathcal{V})$ with $\mathcal{V}$ closed under modal operators is a strong p-morphic preimage of a Kripke frame. That is, an atomic frame being also a general frame is 'almost' a Kripke frame. The second gives a nice characterization of atomic frames: a boolean frame $(I, R_r, \mathcal{V})$ is atomic iff $\mathcal{V}$ is closed under infinite unions. The 'only-if' direction is evident from the defintion of atomic frames. To see the 'if' direction observe that $\mathcal{V}$ is closed under complements and hence also under infinite intersections. The atoms are then obtained as the nonempty intersections of the ultrafilters over $\mathcal{V}$.

We are now able to summarize these results in the following picture:



The set KF is defined to be the set of all strong p-morphic pre-images of Kripke frames. GF, BF, AF, SF represent the general, boolean, atomic and sequence frames respectively. The intended meaning of the arrow '$\rightarrow$' is: ($A \rightarrow B$ iff for every frame in B there is a frame in A having the same theory), but as a relation between boolean frames it can also be read as $A \supset B$.

The picture tells us that we come from BF to GF by demanding the valuation ranges of the frames to be closed under modal operators and from BF to AF by demanding them to be closed under infinite unions. Further, demanding closure under infinite unions from a general frame or closure under modal operators from an atomic frame yields a frame in KF.

Speaking algebraically, we can express this as follows: a boolean frame corresponds to a boolean algebra $(\mathcal{V}, \cup, \cap, -, \{\}, I)$, a general frame is a modal algebra $(\mathcal{V}, \cup, \cap, -, \{\}, I, m_r)$, which is defined to be boolean algebra closed under modal operators and sequence frames correspond to boolean algebras $(\mathcal{V}, \cup, \cap, -, \{\}, I)$ with $(\mathcal{V}, \subset, \cup, \cap)$ being a complete lattice.

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