Some Linear Logic of von Neumann Games

Part 1: Simple strategies

1. A category of games.

A (finite) game consists of (finite) sets U, X of plays for I, II and a pay-off matrix

$$A(u,x) \in \mathbb{R} \qquad u \in U \quad x \in X$$

(Alternatives to $\mathbb{R}$: values V an ordered abelian group (of compact closed category) but later we'll want some convexity structure.)

The $U \times X$-matrix A gives $\mathbb{R}^{\oplus X} \longrightarrow \mathbb{R}^{\oplus U}$ so write this $U \xleftarrow{A} X$.

A map $U \xleftarrow{A} X$ to $V \xleftarrow{B} Y$ is a (crude) reason to prefer B to A, or a proof $A \vdash B$: given by map

$$\begin{tikzcd} U \arrow[d, "f"'] & X \arrow[l, "A"'] \\ V & Y \arrow[l, "B"] \arrow[u, "F"'] \end{tikzcd}$$

such that for all $u \in U$ $y \in Y$

$$A(u, F(y)) \leq B(f(u), y).$$

Whatever we = I might play in A we have something to ~~play~~ in B $u \longmapsto f(u)$ with the property that whatever they = II might play in B we have something for II to play in A where II does better i.e. we do worse.

<u>Theorem</u> Category of games.

[N.B. We could have maps with $\mathbb{R}$-information but skip for now.]

A map $I = (1 \xleftarrow{0} 1)$ to A is an element $u \in U$ such that $A(u,x) \geq 0$ all $x \in X$.

2 Duality

Given $A = (U \xleftarrow{A} X)$ we have its dual

$$A^{\perp} = (X \xleftarrow{-A^t} U)$$

<u>Fact</u>

$$\frac{A \longrightarrow B}{B^{\perp} \longrightarrow A^{\perp}}$$

A map $I \longrightarrow A^{\perp}$ is $A \longrightarrow \perp$ is $x \in X$ such that $A(u,x) \leq 0$ all $u \in U$.

3. Multiplicative structure

The (linear) function space $A \multimap B$ should satisfy

$$\frac{I \longrightarrow A \multimap B}{A \longrightarrow B}$$

So the I plays are $\quad U \Rightarrow V \times Y \Rightarrow X$

II plays are $\quad U \times Y$

and $\quad A \multimap B\,\big((\phi, \Phi), (u,y)\big) = B(\phi(u), y) - A(u, \Phi(y))$.

Interpretation of this game.

This gives a closed structure with corresponding multiplicative conjunction $A \otimes B$ given by

I $\quad U \times V$

II $\quad V \Rightarrow X \times U \Rightarrow Y$

$$A \otimes B\,\big((u,v), (\phi,\psi)\big) = A(u, \phi(v)) + B(v, \psi(u)).$$

Interpretation of this game.

The adjunction $\dfrac{A \otimes B \longrightarrow C}{A \longrightarrow B \multimap C}$

$=$

$A \otimes B \longrightarrow C$ $\qquad$ $U \times V \qquad V \Rightarrow X \times U \Rightarrow Y$

$$\begin{array}{ccc} U \times V & & V \Rightarrow X \times U \Rightarrow Y \\ \downarrow & & \uparrow \\ W & & Z \end{array}$$

$A + B \le C$

$=$

$A \longrightarrow (B \multimap C)$

$$\begin{array}{ccc} U & & X \\ \downarrow & & \uparrow \\ V \Rightarrow W \times Z \Rightarrow Y & & V \times Z \end{array}$$

$A \le C - B$

$=$

3. Additive structure

Product $\quad$ I $\ U \times V$ $\qquad (u, v) \qquad x \longmapsto \alpha(u, x)$

$\quad$ $\underline{\mathrm{II}}$ $\ X + Y$ $\qquad\qquad\qquad y \longmapsto \beta(v, y)$

$\underline{\mathrm{II}}$ gets to choose which game to play; I = we have to be prepared for either.

Sum $\qquad$ evident dual : we = I get to choose.

4. Exponential structure

$!A$ is $\quad \mathrm{I} \quad U$

$\qquad\quad \mathrm{II} \quad U \Rightarrow M(X)$ $\qquad A(u,m) = \sum_{x \in m(u)} A(u,x)$

We have well behaved maps

$$!A \longrightarrow I$$
$$!A \longrightarrow !A \otimes !A$$

and comonad structure

$$!A \longrightarrow A$$
$$!A \longrightarrow !!A .$$

Then we have
FACT Good $\quad !(A \times B) \cong !A \otimes !B \quad$ (more or less automatic)

and so
<u>Thm</u> The Kleisli category for ! is cartesian closed

Map in the Kleisli $A \longrightarrow B$ is

$$\begin{array}{ccccc} A & U & X & M(X) \\ & \downarrow & & \\ B & V & Y & \end{array} \qquad U \xrightarrow{f} V \qquad U \times Y \xrightarrow{m} M(X)$$

such that for all u, y $\quad \sum_{x \in m(u,y)} A(u,x) \leq B(f(u), y)$.

5. Dialectica Modality

Gödel's Dialectica Interpretation   Princeton 1942
Dialectica 1958

$A \longmapsto GA$ I U $(u, \phi) \longmapsto A(u, \phi(u))$.
II $U \Rightarrow X$
Comonad

Interpretation of game I plays but II plays knowing what I plays

So traditional style value is evident viz

$$\max_u \min_x A(u, x)$$

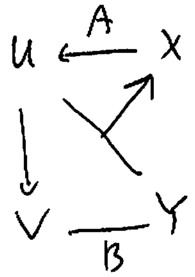
Dually a monad FA $X \Rightarrow U$, X $\psi, x \mapsto A(\psi(x), x)$
Interpretation of game II plays but I plays knowing II's play
value $\min_x \max_u A(u, x)$

The trivial fact $\max_u \min_x A(u,x) \leq \min_x \max_u A(u,x)$
follows from $GA \xrightarrow{\varepsilon} A \xrightarrow{\eta} FA$ or
$(\text{or } \Box A \longrightarrow A \longrightarrow \Diamond A)$

The proof
$$\begin{array}{ccc} U & - & U \Rightarrow X \\ \downarrow & & \uparrow \\ X \Rightarrow U & - & X \end{array}$$
is via constant functions

[Cf $\exists u \, \forall x \, \phi \vdash \forall x \, \exists u \, \phi$].

Interpretation of Kleisli map

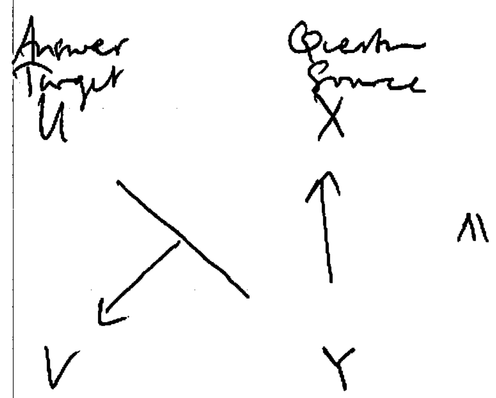


For each u we might play in A we get $f(u)$ to play in B and (dependent on u) a map from possible plays y of II to plays $F(u,y)$ in A, showing we do better in B.

Analogous to problem reduction Blass

[interpretation of Milner's tactics]

computer reductions (Bundy, Galenson, Plotkin)



Part 2 Mixed strategies

6. Change of base

Let $D(-)$ be the collection of probability distributions on $-$.

Apply D to each Hom-set.

Get a category enriched in convex spaces

Multiplicative structure: fine

Additive structure: small problem eg

$$\begin{tikzcd} W \arrow[d] & Z \\ U \times V & X+Y \arrow[u] \end{tikzcd}$$

Prob distn on

$W \Rightarrow U \times W \Rightarrow V \times X \Rightarrow Z \times Y \Rightarrow Z$

vs

Prob distn's on

$(W \Rightarrow U \times X \Rightarrow Z)$ & on $(W \Rightarrow V \times Y \Rightarrow Z)$

not the same though there is a retraction.

Exponentials: more complicated (but perhaps we don't care).

Instinct: its boring;
something else: we are not exploiting the idea of the value of a mixed strategy !!!!

7. Bi-Kleisli category

We have (restricted) $\begin{cases}\text{comonads}\\ \text{monads}\end{cases}$:

comonad

$$(U \xrightarrow{A} X) \longmapsto U \xrightarrow{A_e} D(X)$$

$$(u, \mu) \mapsto \sum_x A(u,x)\mu(x)$$

"$uA\mu$"

monad

$$U \xrightarrow{A} X \longmapsto D(U) \xrightarrow{eA} X$$

$$(\lambda, x) \mapsto \sum_u \lambda(u) A(u,x)$$

"$\lambda A x$"

These commute so ~~there is~~ a trivial distribution law + we get the caty

$$\begin{tikzcd} U \arrow[r, "A"] \arrow[d, "\phi"'] & D(X) \\ D(V) \arrow[r, "B"'] & Y \arrow[u, "\Phi"'] \end{tikzcd}$$

s.t. for all u, y

$$uA\Phi \leq \phi B y.$$

Good as now we see the mixed strategies we couldn't see before.

$$\sum_x A(u,x)\Phi_y(x) \leq \sum_v \phi_u(v) B(v,y)$$

$$\sum_y B(v,y)\Psi_z(y) \leq \sum_w \psi_v(w) C(w,z)$$

+ deduce

$$\sum_x A(u,x)\Big(\sum_y \Phi_y(x)\Psi_z(y)\Big) \leq \sum_{\substack{v\\ y}} \phi_u(v) B(v,y)\Psi_z(y)$$

$$\leq \sum_w \Big(\Big(\sum_v \phi_u(v)\psi_v(w)\Big) C(w,z)\Big).$$

Multiplicative problems

Compare

$$\begin{tikzcd} U\times V \arrow[d] & D(V\Rightarrow X,\ U\Rightarrow Y) \\ D(W) & Z \arrow[u] \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} U \arrow[d] & D(X) \\ D(V\Rightarrow W \times Z\Rightarrow Y) & V\times Z \arrow[u] \end{tikzcd}$$

Problems.

$$\begin{tikzcd} D(M)\times D(N) \\ D(M\times N) \arrow[u, "\Delta"] \end{tikzcd}$$

and

Additive problems

$$\begin{tikzcd} W \arrow[d] & D(Z) \\ D(U\times V) & X+Y \arrow[u] \end{tikzcd}$$

& this problem.

What to do? Feels like we have a result up to retracts: so try splitting them?

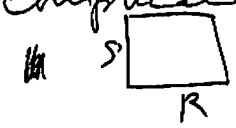
8. Convex spaces.

Monad D such that $D(X)$ consists of formal sums $\sum \lambda_i x_i$ with $\lambda_i \geq 0$ $\sum \lambda_i = 1$, i.e. formal convex sums: distribution monad. ~~(D is commutative)~~. └ generated by $\lambda a + (1-\lambda) b$.

The D-algebras are convex spaces:

D is commutative so ~~there is~~ a monoidal closed structure: concrete but complicated

There are obvious products



There are coproducts:

The free D-algebra on n generators is $\Delta(n-1)$ the $n-1$ dimensional simplicial set: $D(0) = 0$ has dimension $-\infty$

Observe $\quad D(n) + D(m) = D(n+m)$

dim $\quad a = n-1 \quad b = m-1 \quad a+b+1 = n+m-1$

$$\dim(X \times Y) = \dim X + \dim Y \}.$$

$$D(n) \otimes D(m) = D(n \times m)$$

(And $D(1) = 1 = \{*\}$ is the unit for the $\otimes$)

$a = n-1 \quad b = m-1 \quad ab + a + b = nm - 1.$

$R \multimap S$ = convex maps R to S with convex structure

So $D(n) \multimap S = S^n$

$\dim(D(r) \multimap D(s)) = r(s-1).$ (Special case $D(0)$.)

9 New category of games

<u>Notation</u> U, X etc now convex spaces (finite)

<u>Idea</u> Generalised spaces of probabilities/strategies replacing $D(m)$, $D(n)$ etc.

<u>N.B.</u> Rescaling is automatic: there is no absolute size.

Games $\quad U \otimes X \xrightarrow{A} \mathbb{R}$ $\quad$ (A was bilinear in mixed strategies)

Maps
$$\begin{array}{cc} U & X \\ f\downarrow & \uparrow F \\ V & Y \end{array} \quad \text{s.t.} \quad \begin{tikzcd} & U\otimes X \arrow[dr, "A"] \arrow[d, phantom, "\wedge\!\backslash"] & \\ U\otimes Y \arrow[ur, "U\otimes F"] \arrow[dr, "f\otimes Y"'] & {} & \mathbb{R} \\ & V\otimes Y \arrow[ur, "B"'] & \end{tikzcd}$$

Multiplicative structure

Eg $\quad A \multimap B \quad [(U \multimap V) \times (Y \multimap X)] \otimes (U \otimes Y) \xrightarrow{\text{"}B-A\text{"}} \mathbb{R}$

Additive structure

$$A \times B \qquad (U \times V) \otimes (X + Y) \xrightarrow{A \| B} \mathbb{R}$$

(using $I = 1$ in Convex spaces)

Exponential structure

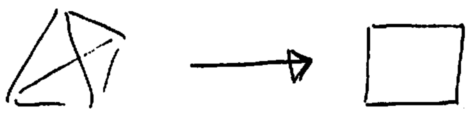
In progress: probably restrict convex spaces to those with a natural commoid structure.

10. Moral

Following the mathematics gives an extended notion of game ~~in which~~ in which

Players can be restricted to using strategies constructed in simple ways
(without any issue of complexity)

Misleading example



Instead of an arbitrary strategy on $4 = 2 \times 2$
we need to play a product strategy
ie. a pair of strategies on 2 and on 2.

(Generally the geometry is more complicated.)