LOGICS FOR SOCIAL BEHAVIOR

The Lorentz Center

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Tutorial Lecture 1.1

Arithmetic Probability Pooling

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DEDICATED TO

PROFESSOR JANOS ACZEL

26 December 1924 –

in the year of his 90th birthday

1. Probability Mass Functions

● If Ω is a finite or denumerably infinite set of possible states of the world, a function $p: \Omega \to [0,1]$ is a *probability mass function* (*pmf*) if

$$\Sigma_{\omega \in \Omega}\, p(\omega) = 1 .$$

● When Ω is finite, such functions can also represent (*normalized*) *allocations* of a fixed sum of money, or other quantifiable resource, among various enterprises, for example, a cost or benefit sharing arrangement among participants in a cooperative game.

Every *pmf* p on Ω induces a (discrete) probability measure (*pm*) (which, abusing notation, we also denote by p) on 2^{Ω}, defined for every subset A of Ω, by

$$p(A) := \Sigma_{\omega \in A}\, p(\omega).$$

2. Pooling Operators

The Probability Pooling Problem:

What is a reasonable way to aggregate the possibly differing *pmf*s (resp., *pm*s) $p_1,\ldots,p_n$ of n individuals into a single *pmf* (resp, *pm*) p ? We'll treat the case of *pmf*s.

- If $\mathscr{P}$ = the set of all *pmf*s on Ω and n >1,

a *pooling operator* is any function

$$T : \mathscr{P}^n \rightarrow \mathscr{P} .$$

In specifying $\mathscr{P}^n$ as the domain of T , we are implicitly adopting as a pooling axiom the *universal domain condition* (UD). Each n-tuple $(p_1,\ldots,p_n) \in \mathscr{P}^n$ is called a *profile* (of individual probability assessments). UD demands that T produce a *pmf* $p := T(p_1,\ldots,p_n)$ for *every logically possible profile* $(p_1,\ldots,p_n)$.

● The justification of additional axiomatic restrictions on T depends on the intended interpretation of $p := T(p_1,\ldots,p_n)$. Is p conceived of as ….

(i) a rough summary of the current pmfs $p_1,\ldots,p_n$ of n individuals ?

(ii) a compromise adopted by these individuals in order to complete an exercise in group decision making ?

(iii) a “rational” consensus to which all individuals have freely revised their initial pmfs after extensive discussion ?

(iv) the pmf of a decision maker external to the group (who may or may not have assessed his own prior before consulting the group) upon being apprised of the pmfs $p_1,\ldots,p_n$ of n “experts” ?

(v) the revision of the pmf p_i of a particular individual i in the group, upon being apprised of the pmfs of the remaining $n - 1$ individuals in the group, each of whom he considers to be his *epistemic peer* (much discussed recently for $n = 2$) ?

Put the issue of interpretation aside, and consider a natural, and familiar, pooling operator, the *weighted arithmetic mean*

(2.1) $p := T(p_1,\ldots,p_n) = w_1p_1+\cdots+w_np_n$,

where the weights w_i are nonnegative, and sum to 1. Formula (2.1) is interpreted *pointwise*, as asserting for all $\omega \in \Omega$ that

(2.2) $p(\omega) = w_1p_1(\omega)+\cdots+w_np_n(\omega)$.

3. The Charms of Arithmetic Pooling

Note that

(3.1) $\sum_{\omega \in \Omega} [w_1 p_1(\omega)+\cdots+w_n p_n(\omega)] = 1,$

with no need to normalize the summands.

● Other pooling functions based, say on the geometric mean, the harmonic mean, the root-mean-square, etc. lack this property.

● Are weighted arithmetic means unique in this respect? Yes. [J. Aczél & C.Wagner (1980), A characterization of weighted arithmetic means, *SIAM J. on Algebraic and Discrete Methods* **1**, 259-260].

● The solution of Cauchy's functional equation $f(x + y) = f(x) + f(y)$ on the restricted domain $0 \leq x, y, x+y \leq 1$ lies at the core of the proof of this result, and the proofs of many subsequent theorems on probability pooling.

4. A Characterization of Weighted Arithmetic Pooling

(K. Lehrer & C. Wagner (1981), *Rational Consensus in Science and Society*, Reidel)

Consider the following axiomatic restrictions on a pooling operator $T : \mathscr{P}^n \rightarrow \mathscr{P}$:

State-wise Pooling (SP). For each $\omega \in \Omega$, there exists a function $f_\omega : [0,1]^n \rightarrow [0,1]$ such that, for all $\omega \in \Omega$,

$$p(\omega) := T(p_1,\ldots,p_n)(\omega) = f_\omega(p_1(\omega),\ldots,p_n(\omega)).$$

(Since $p \in \mathscr{P}$, we must have $\sum_{\omega \in \Omega} p(\omega) = 1$.)

Zero Preservation (ZP). For all $\omega \in \Omega$, if

$p_1(\omega) = \cdots = p_n(\omega) = 0$, then $T(p_1,\ldots,p_n)(\omega) = 0$.

Remark 1. In the presence of SP, ZP reduces to the assertion that $f_\omega(0,\ldots,0) = 0$ for all $\omega \in \Omega$.

Remark 2. SP alone allows for different pooling methods for different states of the world. But, as we'll see, ZP forces such pooling to be uniform across all $\omega \in \Omega$.

Theorem 4.1. If $|\Omega| \geq 3$, a pooling operator $T : \mathscr{P}^n \to \mathscr{P}$ satisfies SP and ZP if and only if there exists a sequence $w_1,\ldots,w_n$ of nonnegative numbers, summing to 1, such that, for all $\omega \in \Omega$,

$$T(p_1,\ldots,p_n)(\omega) = w_1 p_1(\omega)+\cdots+w_n\, p_n(\omega).$$

Proof. *Sufficiency* : obvious.

Necessity : Denote by X, Y, etc. elements of $[0,1]^n$, and by **c** the n-dimensional vector $(c,\ldots,c)$.

Let ω_1, ω_2, and ω_3 denote arbitrary *distinct* elements of Ω.

Suppose that $\mathbf{0} \le X, Y, X+Y \le \mathbf{1}$.

Define $P = (p_1,\ldots,p_n) \in \mathscr{P}^n$ by

(4.1) $(p_1(\omega_1), \ldots, p_n(\omega_1)) = X$,

(4.2) $(p_1(\omega_2), \ldots, p_n(\omega_2)) = Y$,

(4.3) $(p_1(\omega_3), \ldots, p_n(\omega_3)) = \mathbf{1} - X - Y$, and

(4.4) $(p_1(\omega), \ldots, p_n(\omega)) = \mathbf{0}$ otherwise.

By SP and ZP, (4.1) – (4.4) imply

(4.5) $f_{\omega 1}(X) + f_{\omega 2}(Y) + f_{\omega 3}(\mathbf{1} - X - Y) = 1$.

Now define $Q = (q_1,\ldots,q_n) \epsilon \mathscr{P}^n$ by

(4.6) $(q_1(\omega_1), \ldots, q_n(\omega_1)) = \mathbf{0}$,

(4.7) $(q_1(\omega_2), \ldots, q_n(\omega_2)) = X + Y$,

(4.8) $(q_1(\omega_3), \ldots, q_n(\omega_3)) = \mathbf{1} - X - Y$, and

(4.9) $(q_1(\omega), \ldots, q_n(\omega)) = \mathbf{0}$ otherwise.

By SP and ZP, (4.6) – (4.9) yield

(4.10) $f_{\omega 2}(X+Y) + f_{\omega 3}(\mathbf{1} - X - Y) = 1.$

Recalling (4.5) that

$$f_{\omega 1}(X) + f_{\omega 2}(Y) + f_{\omega 3}(\mathbf{1} - X - Y) = 1,$$

we get

(4.11) $f_{\omega 2}(X+Y) = f_{\omega 1}(X) + f_{\omega 2}(Y)$

Setting $Y = \mathbf{0}$ in (4.11) yields $f_{\omega 1}(X) = f_{\omega 2}(X)$ for all $X \epsilon [0,1]^n$. But since ω_1 and ω_2 were arbitrary, the functions f_ω are *identically equal* to some function $f: [0,1]^n \rightarrow [0,1]$.

So, for $\mathbf{0} \le X,Y,\ X+Y \le \mathbf{1}$, (4.11) becomes

$$f(X+Y) = f(X) + f(Y), \tag{4.12}$$

which is Cauchy's equation for vectors, but on a restricted domain.

● From f we extract n scalar functions $f^{<i>} : [0,1]\to[0,1]$, $i = 1,\ldots,n$, by

$$f^{<i>}(x) := f(0,\ldots,0,x,0,\ldots,0), \tag{4.13}$$

where x denotes the i^{th} coordinate.

● From the obvious extension of (4.12) to n summands, we get

$$f(x_1,\ldots,x_n) = f^{<1>}(x_1) +\cdots+f^{<n>}(x_n), \tag{4.14}$$

and, if $0 \le x, y, x+y \le 1$, then

$$f^{<i>}(x + y) = f^{<i>}(x) + f^{<i>}(y). \tag{4.15}$$

It is crucial to be able to extend $f^{<i>}$ to a nonnegative function on the domain $[0,\infty)$ so that (4.15) continues to hold for all $x, y \ge 0$….

See (J. Aczél & C. Wagner, *C.R. Math. Rep. Acad. Sci. Canada*, **3** (1981), 139-142) for details of this extension.

● Since the additivity of $f^{<i>}$ also extends to finitely many summands, it follows immediately (with φ denoting $f^{<i>}$ for the sake of simplicity) that

(4.16) $\varphi(mx) = m\,\varphi(x)$, $m = 1,2,\ldots$

● If $x = (k/m)t$, then $mx = kt$, and so

$m\varphi(x) = \varphi(mx) = \varphi(kt) = k\varphi(t)$, whence

(4.17) $\varphi[(k/m)t] = (k/m)\varphi(t)$.

● Suppose that $\varphi(1) = w$. Setting $t = 1$ above shows that

(4.18) $\varphi(r) = wr$ for every rational $r > 0$.

Also, $\varphi(x + y) = \varphi(x) + \varphi(y) \Rightarrow \varphi(0) = 0$, so (4.18) holds for all rational $r \geq 0$.

● Since $\varphi(y) \geq 0$ for all $y \geq 0$,

$$\varphi(x + y) = \varphi(x) + \varphi(y) \geq \varphi(x)$$

i.e., φ is weakly increasing, with, as noted above, $\varphi(r) = wr$, for all rational $r \geq 0$.

● For each $x \geq 0$, let (r_j) and (R_j) be, respectively, increasing and decreasing rational sequences converging to x. Then

(4.19) $\quad wr_j = \varphi(r_j) \leq \varphi(x) \leq \varphi(R_j) = wR_j$,

and so

(4.20) $\quad \varphi(x) = wx$ for all $x \geq 0$.

● From (4.20), with $\varphi = f^{<i>}$, $i = 1,\ldots,n$, along with $f(x_1,\ldots,x_n) = f^{<1>}(x_1) + \cdots + f^{<n>}(x_n)$, it follows that there exist nonnegative weights $w_1,\ldots,w_n$ such that, for all $X = (x_1,\ldots,x_n) \epsilon [0,1]^n$,

(4.21) $\quad f(x_1,\ldots,x_n) = w_1 x_1 + \cdots + w_n x_n$.

Set $x_i = p_i(\omega)$, where $(p_1,\ldots,p_n) \epsilon \mathscr{P}^n$. Then

(4.22) $p(\omega) := T(p_1,\ldots,p_n)(\omega) =$

$f(p_1(\omega),\ldots,p_n(\omega)) = w_1p_1(\omega) + \cdots + w_np_n(\omega)$, and since $\Sigma_{\omega \in \Omega}\, p(\omega) = 1$,

(4.23) $w_1 + \cdots + w_n = 1.$ □

● See also the beautiful paper of Kevin McConway,

Marginalization and linear opinion pools, *J. Amer. Stat. Assoc.* **76** (1981) 410-414, conceptualized in terms of the pooling of all profiles of probability measures *defined on every possible sigma algebra* **A** *on* Ω. In this context, McConway shows that the commutativity of the family { $T_{\mathbf{A}}$ } with marginalization (restriction to a sub-sigma algebra) is equivalent to postulating “event-wise pooling.”

5. Pooling Under SP Alone

Theorem 5.1. (Aczél, Ng, & Wagner, 1984) Suppose that $|\Omega| \geq 3$ and T: $\mathscr{P}^n \to \mathscr{P}$ satisfies SP, so that for each $\omega \in \Omega$, there exists a function $f_\omega : [0,1]^n \to [0,1]$ such that $p(\omega) := T(p_1,\ldots,p_n)(\omega) = f_\omega(p_1(\omega),\ldots,p_n(\omega))$.

- If Ω is finite, then, for all $\omega \in \Omega$, either

(i) $p(\omega) = \Sigma_i w_i p_i(\omega)$

with all $w_i \geq 0$ and $w_1 + \cdots + w_n = 1$, or

(ii) $p(\omega) = \Sigma_i w_i p_i(\omega) + [1 - \Sigma_i w_i]q(\omega)$,

where all $w_i \in [-1,1]$, $\Sigma_i w_i < 1$, and q is a pmf on Ω.

- If Ω is denumerably infinite, then either (i) above holds, or (ii) holds, but with all $w_i \geq 0$.

Details of proof in …..

J. Aczél, C.T. Ng, and C. Wagner, Aggregation theorems for allocation problems, *SIAM J. on Algebraic and Discrete Methods* **5** (1984), 1-8.

- *Example with negative weights* ($|\Omega| = m$)

$w_1 = \cdots = w_{n-1} = 0$, $w_n = -1/(m-1)$, and $q(\omega) = 1/m$ for all $\omega \in \Omega$, which yields

$$p(\omega) = [-1/(m-1)]\, p_n(\omega) + 1/(m-1)$$

Exercise. Show that all weights w_i are nonnegative (though they need not sum to 1) if and only if, for all $\omega \in \Omega$, and all X and Y in $[0,1]^n$,

$$X \geq Y \quad \Rightarrow \quad f_\omega(X) \geq f_\omega(Y)$$

(*Weak Dominance)*

6. SP is stringent

SP looks innocuous at first glance, but it is really quite stringent, restricting pooling to weighted arithmetic averaging, with the same weights for each state, or to an affine version thereof. (More on the problems with such pooling in tutorial 1.2)

More strikingly, if probability values are restricted (as they always are in practice) to a *finite* subset of [0,1], SP *alone rules out any reasonable pooling methods.*

See: Shattuck & Wagner, An impossibility theorem for allocation aggregation,
J. Philos. Logic, published online
24 June 2014)

Details…..

● Let V denote the of set of values that may be assigned as probabilities. Suppose that V satisfies the closure conditions

(6.1) $0 \in V$,

(6.2) $x \in V \Rightarrow 1 - x \in V$, and

(6.3) $x, y \in V \;\&\; x + y \leq 1 \Rightarrow x + y \in V$.

● The subsets V of [0,1] satisfying the preceding conditions fall into just two radically distinct categories:

Theorem 6.1

A subset V of [0,1] satisfying (6.1) – (6.3) is either *dense in* [0,1], or *finite.* In particular, every *discrete* subset of [0,1] satisfying these closure conditions must be finite.

● A pooling operator $T: \mathscr{P}_V{}^n \to \mathscr{P}_V$ is *dictatorial* if there exists $d \in \{1,\ldots,n\}$ such that, for all $(p_1,\ldots,p_n) \in \mathscr{P}_V{}^n$, $T(p_1,\ldots,p_n) = p_d$. T is *imposed* if there exists $q \in \mathscr{P}_V$ such that, for all $(p_1,\ldots,p_n) \in \mathscr{P}_V{}^n$, $T(p_1,\ldots,p_n) = q$.

Theorem 6.2

Suppose that Ω is finite and $|\Omega| \geq 3$. Let $\mathscr{P}_V := \{$ *pmf*s p on Ω : $p(\omega) \in V$ for all $\omega \in \Omega\}$. A pooling operator $T: \mathscr{P}_V{}^n \to \mathscr{P}_V$ satisfies SP if and only if it is dictatorial, or imposed. If T satisfies ZP as well, T must be dictatorial.

● When V = {0,1}, this yields a corollary of Franz Dietrich on *judgment aggregation*.

(Judgment aggregation: (im)possibility theorems, *J. Economic Theory* **126** (2006), 286-298.