LOGICS FOR SOCIAL BEHAVIOR

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Tutorial Lecture 1.2

A Critique of Arithmetic Pooling, or Good-bye to UD and SP

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7. Arithmetic Pooling Does Not Commute with Conditioning

Suppose that individuals have assessed priors $p_1,\ldots,p_n$ over Ω. They then jointly come to know that the true state of the world belongs to the subset E of Ω. Should they (i) first update their priors by conditioning on E, and then pool the posteriors $p_1(\cdot\,|E),\ldots,p_n(\cdot\,|E)$, or (ii) first pool the priors to $p = T(p_1,\ldots,p_n)$, and then update p to $p(\cdot\,|E)$?

Let us say that T commutes with conditioning (CC) if, for all subsets E of Ω and all $(p_1,\ldots p_n) \in \mathscr{P}^n$ such that $p_i(E) > 0$, $i = 1,\ldots,n$, we have $T(p_1,\ldots p_n)(E) > 0$, and

(7.1) $T(p_1(\cdot\,|E),\ldots p_n(\cdot\,|E) = T(p_1,\ldots p_n)(\cdot\,|E)$.

- Pooling by weighted arithmetic averaging fails to satisfy CC.

8. Geometric (a.k.a. logarithmic) Pooling

- Recall: $\mathscr{P}$ = { p: p is a pmf on Ω} , and

$\mathscr{P}^n = \{(p_1,\ldots,p_n)$: each $p_i \in \mathscr{P}$. Let

(8.1) $\mathscr{P}^{n+} := \{(p_1,\ldots,p_n) \in \mathscr{P}^n$: there exists $\omega \in \Omega$ such that $p_i(\omega) > 0$, $i = 1,\ldots,n\}$.

- Here, pooling operators $T: \mathscr{P}^{n+} \rightarrow \mathscr{P}$.

- Let $w(1),\ldots,w(n)$ be a sequence of nonnegative weights that sum to 1. If $\{(p_1,\ldots,p_n) \in \mathscr{P}^{n+}$, let

(8.2) $G(p_1,\ldots,p_n)(\omega) := \prod_i p_i(\omega)^{w(i)}$ $(0^0 := 1)$

By the generalized *arithmetic-geometric mean inequality*

(8.3) $\prod_i p_i(\omega)^{w(i)} \leq$ (usually, $<$) $\sum_i w(i)p_i(\omega)$,

and since, as previously noted,

(8.4) $\sum_{\omega \in \Omega} [\sum_i w(i)p_i(\omega)] = 1$, we have

(8.5) $\sum_{\omega \in \Omega} [\prod_i p_i(\omega)^{w(i)}] \leq$ (usually, $<$) 1

Also, since $(p_1,\ldots,p_n) \in \mathscr{P}^{n+}$,

(8.6) $$\Sigma_{\omega \in \Omega} \left[\Pi_i \, p_i(\omega)^{w(i)} \right] > 0.$$

So

(8.7) $$T(p_1,\ldots,p_n)(\omega) := \Pi_i \, p_i(\omega)^{w(i)} \,/\, \Sigma_{\omega \in \Omega} \left[\Pi_i \, p_i(\omega)^{w(i)} \right]$$

defines a pooling operator $T: \mathscr{P}^{n+} \rightarrow \mathscr{P}$.

Theorem 8.1. T, as defined by (8.7), commutes with conditioning. Moreover, T is externally Bayesian (commutes with Jeffrey conditioning, parameterized in terms of Bayes factors.)

See: C. Wagner, Jeffrey conditioning and external Bayesianity, *Logic Journal of the IGPAL* (2009).

Of course, T, as defined by (8.7), does *not* satisfy SP. But it does satisfy…

Normalized State-wise Pooling (NSP):
For each $\omega \in \Omega$, there exists a map $g_\omega: [0,1]^n \to [0,1]$ such that, for all $(p_1,\ldots,p_n) \in \mathscr{P}^{n+}$,

(8.8) $0 < \sum_{\omega \in \Omega} g_\omega(p_1(\omega),\ldots,p_n(\omega)) < \infty$,

and

(8.9) $T(p_1,\ldots,p_n)(\omega) =$

$g_\omega(p_1(\omega),\ldots,p_n(\omega) / \sum_{\omega \in \Omega} g_\omega(p_1(\omega),\ldots,p_n(\omega)$

- NSP is flexible enough to accommodate externally Bayesian pooling, but it is not a panacea.

9. Independence Preservation

- As usual, subsets E and F of Ω are independent with respect to p (p-*independent*) if $p(E \cap F) = p(E)p(F)$.

- In what follows, $\Delta := \{p: p$ is a *positive* pmf, i.e., $p(\omega) > 0$ for *all* $\omega \in \Omega\}$, and a probability pooling operator is a map $T: \Delta^n \rightarrow \Delta$.

A number of individuals have asserted that any acceptable probability pooling method T should satisfy

Universal Independence Preservation (UIP): For all $(p_1,\ldots,p_n) \in \Delta^n$, and for all events E and F in Ω, if E and F are p_i-independent for $i = 1,\ldots,n$, then E and F are $T(p_1,\ldots,p_n)$- independent.

● If $|\Omega| \leq 3$, *every* pooling operator satisfies UIP trivially, since subsets E and F of Ω are not independent for $p \in \Delta$ unless one of E or F is equal to the empty set, or to Ω.

● If $|\Omega| = 4$, NSP admits of a rich variety of independence-preserving pooling operators.

Theorem 9.1. Suppose that $|\Omega| = 4$, and $T:\Delta^n \to \Delta$ is of the form

$$T(p_1,\ldots,p_n)(\omega) =$$

$$g_\omega(p_1(\omega),\ldots,p_n(\omega) \,/\, \Sigma_{\omega \in \Omega}\, g_\omega(p_1(\omega),\ldots,p_n(\omega)$$

with at least one of the functions g_ω being Lebesgue measurable. Then T satisfies UIP if and only if there exist real constants $a(1),\ldots,a(n)$ and $b(1),\ldots,b(n)$ such that , for all $(p_1,\ldots,p_n) \in \Delta^n$ and all $\omega \in \Omega$, each $g_\omega = g$, where $g(p_1(\omega),\ldots,p_n(\omega) =$

$$\Pi_{1 \leq i \leq n}\, [p_i(\omega)]^{b(i)} \exp\{a(i)p_i(\omega)[1- p_i(\omega)]\}.$$

The formula $g(p_1(\omega),\ldots,p_n(\omega)) =$

$$\prod_{1\leq i\leq n} [p_i(\omega)]^{b(i)} \exp\{a(i)p_i(\omega)[1- p_i(\omega)]\}$$

yields

(i) a dictatorship of individual d when all $a(i) = 0$, and $b(d) = 1$ and $b(i)= 0$ otherwise.

(ii) geometric pooling when all $a(i) = 0$, and $b(i) = w_i$, as above.

(iii) imposed pooling when all $a(i) = 0$ and all $b(i) = 0$, whence $T(p_1,\ldots,p_n)$ = the uniform distribution on Ω for all $(p_1,\ldots,p_n) \in \Delta^n$.

1. S. Abou-Zaid, Functional equations and related measurements, M.Phil. thesis, U. of Waterloo 1984.

2. C. Sundberg & C. Wagner, A functional equation arising in multi-agent statistical decision theory, *Aeq.Math.* 32(1987), 32-37

Theorem 9.2. When $|\Omega| \geq 5$, a pooling operator $T:\Delta^n \rightarrow \Delta$ satisfies NSP and UIP if and only if it is dictatorial.

See: C. Genest & C. Wagner, Further evidence against independence preservation in expert judgment synthesis, *Aeq. Math.* 32 (1987), 74-86.

● Far from causing despair, this theorem should be taken as a *reductio ad absurdum* of condition UIP, which demands preservation of *every single case of common independence*, whether epistemically significant or not !

But there are lots of cases of purely fortuitous independence….

Example: I consider a die to be fair, and so, for me, E = {2,4,6} = “die comes up even” and F = {3,6} = “die comes up a multiple of three” are independent. The same is true for you if your p(1) = p(5) = p(6) = 1/6, your p(2)=p(4)=1/12, and your p(3) = 1/3. This common independence is completely fortuitous. (It can’t involve considerations of physical independence, since the two events depend on the *same toss* of a die.) So why should it be preserved by pooling ?

But there are cases of mutually agreed upon independence worth preserving under pooling, based, for example, on agreed upon physical independence (often of random variables), or some other prior theoretical commitment. This is best understood by considering

10. Partition Independence

● Recall: If events E and F are independent, so are E and F^c, E^c and F, and E^c and F^c. Indeed, if any of these pairs of events are independent, so are all the others. So the fundamental notion of independence is that of *partition independence*, where countable partitions $\mathcal{E} = \{ E_j \}$ and $\mathcal{F} = \{ F_k \}$ are p-*independent* if $p(E_j \cap F_k) = p(E_j)\, p(F_k)$, for all j and k in the relevant index sets.

The sets E_j (resp.,F_k) are called the *blocks* of $\mathcal{E}$ (resp., $\mathcal{F}$).

● The notion of partition independence can be extended to more than two partitions in the obvious way. (What is commonly called the "total independence" of events $A_1,\ldots,A_n$ is equivalent to the independence of the n 2-block partitions $\{A_1,A_1^c\},\ldots,\{A_n,A_n^c\}$.)

● Independence of finitely many random variables defined on Ω reduces to the independence of the partitions of Ω induced by these random variables.

Example: If $X: \Omega \rightarrow_{onto} U = \{ u_j \}$ and $Y: \Omega \rightarrow_{onto} V = \{ v_k \}$ are discrete random variables, then X and Y are p-independent if and only if $p(X=u_j \ \& \ Y=v_k) = p(X=u_j)p(Y=v_k)$ for all j and k. But this amounts to p-independence of partitions $\{ E_j \}$ and $\{ F_k \}$, where $E_j = \{\omega \in \Omega: X(\omega) = u_j\}$ and $F_k = \{\omega \in \Omega: Y(\omega) = v_k\}$.

11. How to Preserve Partition Independence in a Principled Way

Assumptions (not as general as possible):

- Ω = a countable set of possible states of the world

- 2^{Ω} = the set of all subsets of Ω.

- $p_1,\ldots,p_n$ are *positive* pmfs on Ω (or their induced pms on 2^{Ω}, depending on context).

- Countable partitions $\mathcal{E} = \{ E_j \}$ and $\mathcal{F} = \{ F_k \}$ of Ω are p_i – independent, $i = 1,\ldots,n$.

A recipe for pooling $p_1,\ldots,p_n$ *to a probability measure* q *on* 2^Ω *such that partitions* $\mathcal{E}$ *and* $\mathcal{F}$ *are* q *– independent* :

(i) Let $p := w_1p_1 + \cdots + w_np_n$ (or the result of any other way of pooling of $p_1,\ldots,p_n$).

(ii) The family $\mathcal{E} \times \mathcal{F} := \{ E_j \cap F_k \}$ is a partition of Ω (each $E_j \cap F_k$ is nonempty, since $p_i(E_j \cap F_k) = p_i(E_j)\, p_i(F_k) > 0$)

(iii) For all $A \in 2^\Omega$, set

(11.1) $q(A) = \Sigma_{j,\,k}\, p(E_j)p(F_k)\, p(A|E_j \cap F_k)$,

where $E_j \cap F_k \in \mathcal{E} \times \mathcal{F}$.

Theorem 11.1. The partitions $\mathcal{E}$ and $\mathcal{F}$ are q – independent.

Proof. $q(E_j \cap F_k) = p(E_j)p(F_k)$. But then $q(E_j) = p(E_j)$ and $q(F_k) = p(F_k)$.

Note. q comes from p by *Jeffrey conditioning* on the partition $\mathcal{E} \times \mathcal{F}$.

● Among all r satisfying $r(E_j \cap F_k) = p(E_j)p(F_k)$, and therefore preserving the independence of $\mathcal{E}$ and $\mathcal{F}$, q is closest (uniquely so) to p for both the Kullback-Leibler divergence

(11.2) $KL(r,p) := \Sigma_\omega\, r(\omega)\, \mathit{ln}\, [r(\omega)/ p(\omega)]$

and the Hellinger metric

(11.3) $H(r,p) := \Sigma_\omega\, [\, r(\omega)^{1/2} - p(\omega)^{1/2}]^2$.

Moral: Insisting on Universal Domain more or less forces one to pool in accord with SP or NSP. Both of the latter are radically anti-holistic. Abandoning UD (which frees us from SP and NSP) allows one the flexibility to preserve epistemically significant agreement on independence under pooling in a principled way.