Coalgebraic representations of distributive lattices with operators

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Abstract

We present a framework for extending Stone's representation theorem for distributive lattices to representation theorems for distributive lattices with operators. We proceed by introducing the definition of algebraic theory of operators over distributive lattices. Each such theory induces a functor on the category of distributive lattices such that its algebras are exactly the distributive lattices with operators in the original theory. We characterize the topological counterpart of these algebras in terms of suitable coalgebras on spectral spaces. We work out some of these coalgebraic representations, including a new representation theorem for distributive lattices with monotone operators.

1 Introduction

Boolean algebras with operators were first introduced and investigated in 1951 by Jónsson and Tarski [18, 19] as a common framework for the study of several algebras and logics, including relation algebras, cylindric algebras and modal logics. Using Stone's topological representation of Boolean algebras [37], they showed that every Boolean algebra with operators can be represented as a relational structure. Their result played an important role in the study of many extensions of classical logics, such as normal modal logics [11] and monotone modal logics [14].

Although Jónsson and Tarski considered only Boolean algebras with operators, their result suggested that similar methods can be applied to more general algebraic structures for which a representation theorem is known. For example, an extensive theory of representation of distributive lattices with operators has been developed in the past years [7, 36, 8, 9], either using Stone's representation of distributive lattices in terms of spectral spaces [38], or Priestley's duality [30], an alternative to Stone's original duality.

Building on the original Stone representation theorem, in this paper we present a framework aiming at a general representation theory of distributive lattices with operators in terms of relational structures. Following Rutten [34], we see a relational structure as a coalgebra, allowing the treatment of a large variety of different relational structures in a uniform way.

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Stone duality between distributive lattices and spectral spaces can be lifted to a duality between suitable coalgebras over spectral spaces and algebras over distributive lattices induced by algebraic theories of operators over distributive lattices.

Informally speaking, coalgebras encode operational meaning of systems in terms of the possible next-steps relation a system may engage in; whereas algebras over distributive lattices describes how to construct, up to logical equivalence, formulae for this next-step relations from an intuitionistic (i.e. negation free) logic. The usefulness of our framework stems from the fact that a lifting of Stone duality to a duality between coalgebras and algebras automatically gives rise to non-classical logics that are sound, complete, and expressive with respect to a suitable relational semantics [4].

Our approach greatly differs from most of the work on distributive lattices with operators we are aware of (e.g. [7, 36, 8, 22, 27, 9]), all based on canonical extensions of distributive lattices. Canonical extensions are a very useful technique for enriching topological spaces with relations, obtaining structures departing from classical constructors as studied in topology. Our framework can be casted in terms of classical works on topology, universal algebra and locale theory. For example we consider coalgebras arising from classical hyperspace constructors [24, 26], and theories of operators on distributive lattices inducing functors on distributive lattices that have been greatly studied in the context of domain theory and locales. This goes back to Johnstone [17] where a dual of the Vietoris hyperspace, called Vietoris locale, is described. Winskel [41] used additive and multiplicative operators to describe the Plotkin powerdomain [29], and Robinson [32] established the connection between the work in domain theory and that of Johnstone. Abramsky [1] extended these ideas to give logical descriptions of domains for a large number of other functors. Bonsangue [3] and Brink and Rewitzky [31] applied these ideas to the semantics of programming languages.

We argue that the category of topological spaces in general, and spectral spaces in particular, forms an interesting base category for coalgebras. In fact, most of the work with the aim of giving semantics to coalgebraic logics (as e.g.[33, 15, 25]) can be casted in terms of so-called predicate liftings, as in Pattinson [28].[1] In [21] it was shown that any modal logic given by predicate liftings in the sense of [28] can be described by a functor on Boolean algebras, or, equivalently, by a category of Boolean algebras with operators. In [5] this was generalised to other algebraic categories including distributive lattices.

We proceed as follows. In Section 2 we recall Stone's representation theorem for distributive lattices, and, in Section 3, some basic notions from universal algebra and coalgebra. In Section 4 we introduce a definition of algebraic theory of operators over distributive lattices in a such way that it induces a canonical functor on distributive lattices. As in [5], algebras over the induced functors are exactly algebras of the original theories of distributive lattices with operators. Using Stone duality, we relate these algebras with coalgebras generated by a suitable functor over spectral spaces. This abstract framework is then applied to theories of additive and multiplicative operators. In Section 5 we extend the approach of [5] from sets to posets, that is, we move to ordered algebras to consider theories of monotone operators over distributive lattices, and give a new representation of distributive lattices with monotone operators in terms of the double hyperspace, obtained by composing (in either order [20]) the upper and lower hyperspace constructions, as extensively studied in [40].

[1] Given $T{:}Set \to Set$, a predicate lifting is a natural transformation $\lambda_X{:}2^X \to 2^{TX}$, which lifts predicates over X to predicates over TX (2 is here the contravariant powerset functor). The import of predicate liftings λ stems from the fact that they give semantics to modal operators $\Box_\lambda$ in a canonical way: Given a T-coalgebra (X, ξ) and a predicate $A \subseteq X$, $\Box_\lambda A$ is defined as $\Box_\lambda A = \xi^{-1}(\lambda_X(A))$.

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2 Stone duality for distributive lattices

A *distributive lattice* D is a partial order that has join and meets for arbitrary finite subsets, and it satisfies the distributive law

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

We denote by $\top$ the empty meet, and by $\perp$ the empty join. For example, the two-element partial order 2 with $\perp \leq \top$ is a distributive lattice. Distributive lattices with functions preserving both finite joins and finite meets form a category called DLat.

In what follows we will often refer to Stone's representation theorem for distributive lattice [38]. We mention below the main ingredients of Stone's result. The *points* of a distributive lattice D are defined by the set $\mathcal{S}(D)$ of morphisms $f{:}D \to 2$ in DLat. This set of points can be equipped with a topology with basic opens defined, for every $x \in D$, by

$$\triangle(x) = \{f{:}D \to 2 \mid f(x) = \top\}$$

In particular, the distributive lattice 2 has one single point, the identity morphism $id_2{:}2 \to 2$, and two basic opens, namely $\triangle(\perp) = \emptyset$ and $\triangle(\top) = \{id_2\}$.

Definition 2.1 *A topological space X is* spectral *if it is sober and its compact opens form a basis for X and are closed under finite intersections. If, moreover, the compact opens are closed under complement, then X is said to be a* Stone space.

For every distributive lattice D, the topological space $\mathcal{S}(D)$ is spectral. Spectral spaces can be organized into a category Spec by taking as morphisms all continuous maps with inverse preserving compact opens. Examples of spectral spaces are Scott domains taken with the Scott topology [29].

Our interest in spectral spaces is justified by the following observation. Since finite unions of compact opens are again compact, it follows that if X is a spectral space then the set $\mathcal{K}(X)$ of its compact opens is a distributive lattice. Moreover, if $f{:}X \to Y$ is a morphism between the spectral spaces X and Y then

$$\mathcal{K}(f) = f^{-1}{:}\mathcal{K}(Y) \to \mathcal{K}(X)$$

is well-defined and preserves finite meets and finite joins of the distributive lattice $\mathcal{K}(Y)$. Thus we have a functor $\mathcal{K}{:}\,\mathsf{Spec} \to \mathsf{DLat}^{op}$.

Lemma 2.2 *The assignment $D \mapsto \mathcal{S}(D)$ for each distributive lattice D can be extended to a functor from DLat^{op} to Spec which is the right adjoint of $\mathcal{K}$.*

For a spectral space X, the unit $\eta_X{:}X \to \mathcal{S}(\mathcal{K}(X))$ of the above adjunction, given by the assignments

$$x \mapsto p_x{:}\mathcal{K}(X) \to 2 \;\text{ where }\; p_x(o) = \top \iff x \in o \;\text{ for } o \in \mathcal{K}(X),$$

is an isomorphism. The same holds also for the co-unit $\triangle(-){:}D \to \mathcal{K}(\mathcal{S}(D))$. This result, originally due to M. Stone [38], can be viewed as a generalization of his famous representation theorem for Boolean algebras [37].

Theorem 2.3 *The functors $\mathcal{S}$ and $\mathcal{K}$ establish a dual equivalence between the categories* DLat *and* Spec*:*

$$\begin{tikzcd} \mathsf{Spec} \arrow[r, bend left, "\mathcal{K}"] & \mathsf{DLat} \arrow[l, bend left, "\mathcal{S}"] \end{tikzcd} \tag{1}$$

In other words, every distributive lattice D is isomorphic in DLat to $\mathcal{K}(\mathcal{S}(D))$ via $\triangle$, and every spectral space X is isomorphic in Spec to $\mathcal{S}(\mathcal{K}(X))$ via η_X. By definition of $\triangle(D)$ we have the following corollary.

Corollary 2.4 *For each distributive lattice D and $d_1, d_2 \in D$, if $d_1 \not\leq d_2$ then there exists a morphism $f{:}D \to 2$ such $f(d_1) = \top$ but $f(d_2) = \bot$.*

3 Algebras and coalgebras

An *algebraic theory* $\mathbb{T} = (\Sigma, E)$ consists of a set Σ of function symbols σ, each with an associated arity n_σ, and a set E of equations consisting of pairs (e_l, e_r), where e_l and e_r are expressions formed from a set of variables V by applying the given function symbols.

A $\mathbb{T}$-*algebra* is a set A together with a corresponding function $\sigma_A{:}A^{n_\sigma} \to A$ for each function symbol $\sigma \in \Sigma$, such that, independently of the way we substitute elements of A for the variables, each equation in E holds in A as an identity. A homomorphism between two $\mathbb{T}$-algebras A and B is a function $f{:}A \to B$ such that $\sigma_B \circ f^{n_\sigma} = f \circ \sigma_A$ for each function symbol $\sigma \in \Sigma$. The category of $\mathbb{T}$-algebras is denoted by $\mathsf{Alg}(\mathbb{T})$.

Given a functor F on a category A, an F-algebra (denoted by (A, α) or simply α) is a morphism $\alpha{:}FA \to A$ in A. A morphism $f{:}\alpha \to \alpha'$ between two F-algebras is a morphism $f{:}A \to A'$ in A such that $f \circ \alpha = \alpha' \circ Ff$. The category of F-algebras is denoted by $\mathsf{Alg}(F)$. Dually, given a functor T on a category X, a T-*coalgebra* (denoted by (X, ξ) or just ξ) is a morphism $\xi{:}X \to TX$ in X. A morphism $f{:}\xi \to \xi'$ between two T-coalgebras is a morphism $f{:}X \to X'$ in X such that $Tf \circ \xi = \xi' \circ f$. The category of T-coalgebras is denoted by $\mathsf{Coalg}(T)$.

A category A, or more precisely, a functor $U{:}\mathsf{A} \to \mathsf{Set}$ is *monadic* if A is (isomorphic to) a category $\mathsf{Alg}(\mathbb{T})$ of $\mathbb{T}$-algebras over an algebraic theory $\mathbb{T} = (\Sigma, E)$ and, moreover, U has left adjoint. The latter condition implies that a monadic category has free algebras. For example, the category of distributive lattices DLat is monadic [16]. Indeed $\mathsf{DLat} \cong \mathsf{Alg}(\mathbb{D})$, where $\mathbb{D} = (\Sigma_\mathbb{D}, E_\mathbb{D})$ is the algebraic theory of distributive lattices with a signature $\Sigma_\mathbb{D}$ consisting of two function symbols of arity 0 (namely $\top$ and $\bot$) and two binary function symbols (namely $\wedge$ and $\vee$). The set of equations $E_\mathbb{D}$ is as expected. Because the forgetful functor $U{:}\mathsf{DLat} \to \mathsf{Set}$ is monadic, it has a left adjoint denoted by F throughout. Hence FX is the free distributive lattice over X and UFX is the set of equivalence classes of $\Sigma_\mathbb{D}$-terms over X modulo the equations $E_\mathbb{D}$.

We conclude by recalling a useful technique to describe objects of monadic categories (for more see [39]). Let $U{:}\mathsf{A} \to \mathsf{Set}$, be a monadic functor with left adjoint F. A *presentation* $\mathsf{A}\langle G \mid R \rangle$ of an object of A consists of a set of *generators* G and a set of *relations* $R \subseteq UFG \times UFG$. A morphism $f{:}FG \to A$ in A *satisfies the relations* R if $(t, s) \in R$ implies

$Uf(t) = Uf(s)$. An object A is *presented* by $\mathsf{A}\langle G \mid R\rangle$ if

$$\begin{tikzcd} FG \arrow[d, "\imath_A"'] \arrow[dr, "f"] & \\ A \arrow[r, dashed, "f^+"'] & A' \end{tikzcd}$$

- A comes with an insertion of generators $\imath_A{:}FG \to A$ satisfying the relations R,
- for all $A' \in \mathsf{A}$ and all $f{:}FG \to A'$ satisfying the relations R there is a unique $f^+{:}A \to A'$ with $f^+ \circ \imath_A = f$.

For example, each distributive lattice D can be presented in DLat by taking as generators the set $\{\widehat{d} \mid d \in D\}$ and as relations the set containing the pairs $(\widehat{\top}, \top)$, $(\widehat{\bot}, \bot)$, and $((\widehat{d_1 \wedge d_2}), \widehat{d_1} \wedge \widehat{d_2})$, $(\widehat{d_1 \vee d_2}, \widehat{d_1} \vee \widehat{d_2})$ for all $d_1, d_2 \in D$.

The next proposition shows that also the converse holds.

Proposition 3.1 *Every presentation $\mathsf{A}\langle G \mid R\rangle$ presents an object in A.*

Proof: The proof relies on the fact that A, as a category monadic over Set, has coequalizers [23]. The object presented by $\mathsf{A}\langle G \mid R\rangle$ is given by the coequalizer

$$\begin{tikzcd} FR \arrow[r, shift left, "\pi_1^\sharp"] \arrow[r, shift right, "\pi_2^\sharp"'] & FG \arrow[r, "\imath_A"] & A. \end{tikzcd}$$

where $\pi_1^\sharp, \pi_2^\sharp$ come from the projections $\pi_1, \pi_2{:}R \to UFG$. ◻

4 Distributive lattices with operators

The algebraic theory $\mathbb{D} = (\Sigma_{\mathbb{D}}, E_{\mathbb{D}})$ of distributive lattices can be extended with signature $\Omega_{\mathbb{O}}$ for operators over distributive lattices and a set of identities $I_{\mathbb{O}}$ relating the new operators $\omega \in \Omega_{\mathbb{O}}$ with the function symbols of the theory of distributive lattices. The corresponding algebras are called *distributive lattice with operators*. In what follows we formally define an algebraic theory for operators over distributive lattices, and introduce a framework relating the category of algebras over the extended theory with a category of coalgebras over spectral spaces.

Definition 4.1 ([5]) *Let U be the forgetful functor from DLat to Set with left adjoint F. A theory $\mathbb{O}$ of operators over distributive lattices consists of*

1. *a signature Ω of operations $\omega \in \Omega$ with arities n_ω which gives rise to a functor $G_\Omega{:}\mathsf{Set} \to \mathsf{Set}$, $X \mapsto \coprod_{\omega\in\Omega} X^{n_\omega}$,*
2. *a set I of identities containing pairs of elements in $UFG_\Omega UFV$, for some set V of variables.*

Note that $I \subseteq UFG_\Omega UFV \times UFG_\Omega UFV$ means that the terms appearing in identities may freely use the finite joins and finite meets of distributive lattices but do not contain nested occurrences of operations from Ω. We will discuss this restriction in the conclusions.

For each algebraic theory $\mathbb{O} = (\Omega_\mathbb{O}, I_\mathbb{O})$ of operators over distributive lattices, let $\mathbb{T} = (\Sigma, E)$ be the algebraic theory where $\Sigma = \Sigma_\mathbb{D} + \Omega_\mathbb{O}$ is the disjoint union of the signatures, and $E = E_\mathbb{D} + I_\mathbb{O}$ is the disjoint union of the equations. The equations in $E_\mathbb{D}$ and $I_\mathbb{O}$ are understood as equations over $\Sigma_\mathbb{D} + \Omega_\mathbb{O}$. [2] We define the category $\mathsf{DLat}(\mathbb{O})$ of *distributive lattices with operators* in the algebraic theory $\mathbb{O}$ as the category of algebras $\mathsf{Alg}(\mathbb{T})$.

Next we set-up a framework for relating categories of distributive lattices with operators to suitable categories of coalgebras over spectral spaces.

Definition 4.2 *Given a theory of operators $\mathbb{O} = (\Omega, I)$ and a functor $L{:}\mathsf{DLat} \to \mathsf{DLat}$, we say that a natural transformation $f{:}FG_\Omega U \to L$* satisfies the identities I *if for all distributive lattices D and all morphisms $v{:}FV \to D$ (mapping variables to closed terms) it holds*

$$(t, s) \in I \quad \Rightarrow \quad (f_D \circ FG_\Omega Uv)(t) = (f_D \circ FG_\Omega Uv)(s)\,.$$

Each theory of operators $\mathbb{O} = (\Omega, I)$ defines a canonical functor $L_\mathbb{O}{:}\mathsf{DLat} \to \mathsf{DLat}$ that comes equipped with a natural transformation $\imath$ satisfying the identities I. We define the functor $L_\mathbb{O}$ on a distributive lattice D as

$$FI \underset{\pi_2^\sharp}{\overset{\pi_1^\sharp}{\rightrightarrows}} FG_\Omega UFV \xrightarrow{FG_\Omega Uv} FG_\Omega UD \xrightarrow{\imath_D} L_\mathbb{O}D$$

where the $\pi_1^\sharp, \pi_2^\sharp$ come from the projections $\pi_1, \pi_2{:}I \to UFG_\Omega UFV$, and $\imath_D$ is the joint coequalizer with respect to all pairs $(FG_\Omega Uv \circ \pi_1^\sharp, FG_\Omega Uv \circ \pi_2^\sharp)$, for $v{:}FV \to D$. The universal property of $L_\mathbb{O}D$ gives the action of $L_\mathbb{O}$ on morphisms and the requested naturality of $\imath$.

The distributive lattice $L_\mathbb{O}D$ can be presented in DLat by taking as generators the set $G_\Omega UD$ of all Ω terms over D, and as relations the set of all pairs

$$(UFG_\Omega Uv \circ U\pi_1^\sharp(l), UFG_\Omega Uv \circ U\pi_2^\sharp(r))$$

for all $v{:}FV \to D$, and $(l, r) \in UFI$. More intuitively, we take as relations the set of all instantiations of the identities I obtained by substituting the variables with elements of D.

Theorem 4.3 ([5]) *Let $\mathbb{O} = (\Omega_\mathbb{O}, I_\mathbb{O})$ be a theory of operators for distributive lattices and $L_\mathbb{O}{:}\mathsf{DLat} \to \mathsf{DLat}$ its associated canonical functor. Then the category of distributive lattices with operators $\mathsf{DLat}(\mathbb{O})$ is isomorphic to the category $\mathsf{Alg}(L_\mathbb{O})$.*

Proof: Let Σ be $\Sigma_\mathbb{D} + \Omega_\mathbb{O}$, and E be $E_\mathbb{D} + I_\mathbb{O}$. Consider an $L_\mathbb{O}$ algebra $\alpha{:}L_\mathbb{O}D \to D$. The corresponding Σ-algebra A has carrier UD and the interpretation ω^A of the operations $\omega \in \Omega_\mathbb{O}$ is given by $(UD)^{n_\omega} \to UFG_{\Omega_\mathbb{O}}UD \overset{U\imath_D}{\to} ULD \overset{U\alpha}{\to} UD$, where the leftmost arrow is stemming from the composition of counit of the adjunction between DLat and Set with a

[2] Strictly speaking, $I_\mathbb{O}$ was defined on equivalence classes of $\Sigma_\mathbb{D}$-terms. Formally, one obtains the new $I_\mathbb{O}$, denoted $I'_\mathbb{O}$, as follows. Let $T_{\Sigma_\mathbb{D}}V$ be the set of $\Sigma_\mathbb{D}$-terms with variables in V. Consider a left-inverse m of the quotient $T_{\Sigma_\mathbb{D}}G_{\Omega_\mathbb{O}}T_{\Sigma_\mathbb{D}}V \to UFG_{\Omega_\mathbb{O}}UFV$ (m chooses a representative for each equivalence class). Then $I'_\mathbb{O} = \{(m(t), m(s)) \mid (t, s) \in I_\mathbb{O}\}$.

suitable projection associated with the functor $G_{\Omega_{\mathbb{O}}}$. The algebra A satisfies the equations $E_{\mathbb{D}}$ because D does, and it satisfies the identities $I_{\mathbb{O}}$ because $\imath_D$ does.

Conversely, every (Σ, E)-algebra A is also a distributive lattice. We then obtain, from the operations in $\Omega_{\mathbb{O}}$, a function $G_{\Omega_{\mathbb{O}}}UA \to UA$, i.e. a morphism $f{:}FG_{\Omega_{\mathbb{O}}}UA \to A$. Since A satisfies the equations $I_{\mathbb{O}}$ we obtain the required $L_{\mathbb{O}}$ algebra $f^+{:}L_{\mathbb{O}}A \to A$. □

To summarize the situation, for each theory $\mathbb{O} = (\Omega_{\mathbb{O}}, I_{\mathbb{O}})$ of operators for distributive lattices, we have the following diagram:

$$\begin{tikzcd}
& \mathsf{Alg}(L_{\mathbb{O}}) \arrow[d] & \cong & \mathsf{DLat}(\mathbb{O}) \\
\mathsf{Spec} \arrow[r, bend left, "\mathcal{K}"] & \mathsf{DLat} \arrow[l, bend left, "\mathcal{S}"] \arrow[loop right, "L_{\mathbb{O}}"] \arrow[d, bend left, "U"] & & \\
& \mathsf{Set} \arrow[u, bend left, "F"] & &
\end{tikzcd} \tag{2}$$

The ultimate goal of this paper is to give coalgebraic representations of distributive lattices with operators. The idea is to extend the duality generated by the functors $\mathcal{K}$ and $\mathcal{S}$ using a functor on Spec dual to $L_{\mathbb{O}}$, where a functor T on Spec is called the *dual* of $L_{\mathbb{O}}$ if there is a natural isomorphism $\delta{:}L_{\mathbb{O}}\mathcal{K} \to \mathcal{K}T$.

The natural isomorphism δ gives us the link between the algebraic structure of distributive lattices with operators and the relational structure of coalgebras. Indeed it allows us to extend the equivalence in Diagram (1) to an equivalence of algebras and coalgebras by lifting the functors $\mathcal{K}$ and $\mathcal{S}$. Explicitly, on objects, the lifted $\tilde{\mathcal{K}}{:}\mathsf{Coalg}(T) \to \mathsf{Alg}(L)$ and $\tilde{\mathcal{S}}{:}\mathsf{Alg}(L_{\mathbb{O}}) \to \mathsf{Coalg}(T)$ are given as

$$\begin{array}{rcl}
\tilde{\mathcal{K}}(X,\xi) & = & L\mathcal{K}(X) \xrightarrow{\delta_X} \mathcal{K}(TX) \xrightarrow{\mathcal{K}(\xi)} \mathcal{K}(X) \\
\tilde{\mathcal{S}}(D,\alpha) & = & \mathcal{S}(D) \xrightarrow{\mathcal{S}(\alpha)} \mathcal{S}(LD) \cong \mathcal{S}(L\mathcal{K}(\mathcal{S}(D))) \xrightarrow{(\mathcal{S}\delta\mathcal{S})_D} \mathcal{S}(\mathcal{K}(T\mathcal{S}(D))) \cong T\mathcal{S}(D)
\end{array}$$

We can thus fill, for each theory $\mathbb{O} = (\Omega_{\mathbb{O}}, I_{\mathbb{O}})$ of operators for distributive lattices, the upper left corner of Diagram (2) as follows:

$$\begin{tikzcd}
\mathsf{Coalg}(T) \arrow[r, bend left, "\tilde{\mathcal{K}}"] \arrow[d] & \mathsf{Alg}(L_{\mathbb{O}}) \arrow[l, bend left, "\tilde{\mathcal{S}}"] \arrow[d] & \cong & \mathsf{DLat}(\mathbb{O}) \\
\mathsf{Spec} \arrow[loop left, "T"] \arrow[r, bend left, "\mathcal{K}"] & \mathsf{DLat} \arrow[l, bend left, "\mathcal{S}"] \arrow[loop right, "L_{\mathbb{O}}"] \arrow[d, bend left, "U"] & & \\
& \mathsf{Set} \arrow[u, bend left, "F"] & &
\end{tikzcd}$$

If T is a dual functor of $L_{\mathbb{O}}$ then we say that category $\mathsf{Coalg}(T)$ is a *representation* of the category of distributive lattices with operators in the theory $\mathbb{O}$.

In the remainder of this section we give a few examples of functors T such that T-coalgebras represent distributive lattices with operators. Our discussion follows a common pattern. For each theory of operators $\mathbb{O}$ and each distributive lattice D, we give a presentation of the distributive lattice $L_{\mathbb{O}}D$ by generators and relations. Our method for establishing a representation theorem for $\mathsf{DLat}(\mathbb{O})$ goes via the definition of an isomorphism

$\delta_X{:}L_{\mathbb{O}}\mathcal{K}(X) \to \mathcal{K}T(X)$ in DLat, where T is an assignment between spectral spaces. The assignment T can be lifted to a functor using the isomorphism δ_X, the functor $L_{\mathbb{O}}$ and the unit η of the duality between Spec and DLat. More explicitly, we define the action of T on a morphism $f{:}X \to Y$ as follows

$$T(f) = TX \xrightarrow{\eta_{TX}} \mathcal{SK}TX \xrightarrow{\mathcal{S}\delta_X} \mathcal{S}L_{\mathbb{O}}\mathcal{K}X \xrightarrow{\mathcal{S}L_{\mathbb{O}}\mathcal{K}f} \mathcal{S}L_{\mathbb{O}}\mathcal{K}Y \xrightarrow{\mathcal{S}\delta_X^{-1}} \mathcal{SK}TY \xrightarrow{\mathcal{S}\eta_{TY}^{-1}} TY\,.$$

This way δ is a natural isomorphism between the functors $L_{\mathbb{O}}\mathcal{K}(X) \to \mathcal{K}T(X)$. By the discussion above we finally obtain the category $\mathsf{Coalg}(T)$ as representation of the category $\mathsf{DLat}(\mathbb{O})$.

4.1 Additive operators

Let us consider the theory of an additive operator $\mathbb{A} = (\Omega, I)$, where Ω contains a unary operation $\Diamond$. Further, taking $V = \{v_0, v_1\}$ as set of variables, and, writing "$\cdot = \cdot$" instead of "$(\cdot,\cdot)$", the set of identities I is given by

$$\Diamond\bot = \bot \quad \text{and} \quad \Diamond(v_0 \vee v_1) = \Diamond v_0 \vee \Diamond v_1\,.$$

The theory $\mathbb{A}$ induces a functor $L_{\mathbb{A}}{:}\mathsf{DLat} \to \mathsf{DLat}$, mapping each distributive lattice D to the distributive lattice

$$L_{\mathbb{A}}D \quad = \quad \mathsf{DLat}\langle \Diamond d{:}d \in D \mid \Diamond \text{ preserves finite joins}\rangle.$$

By Theorem 4.3, the category $\mathsf{DLat}(\mathbb{A})$ of distributive lattice with an additive unary operator is isomorphic to the category of algebras $\mathsf{Alg}(L_{\mathbb{A}})$. We want to find a functor on Spec dual to $L_{\mathbb{A}}$. First we note that points of the distributive lattice $L_{\mathbb{A}}D$ are related to sets of closed subsets of points of D: The lemma below will allow us to associate $f{:}L_{\mathbb{A}}D \to 2$ with the set $\mathcal{S}D \setminus \bigcup\{\Delta(d') \mid d' \in D \text{ and } f(\Diamond d') = \bot\}$.

Lemma 4.4 *Let D be a distributive lattice and $d \in D$. For every morphism $f{:}L_{\mathbb{A}}D \to 2$ we have*

$$f(\Diamond d) = \bot \;\text{ if and only if }\; \Delta(d) \subseteq \bigcup\{\Delta(d') \mid d' \in D \text{ and } f(\Diamond d') = \bot\}$$

Proof: The direction from left to right is obvious. For the converse we first notice that the right-hand side is a directed union of compact opens. Since $\Delta(d)$ is compact there exists $d' \in D$ with $f(\Diamond d') = \bot$ and $\Delta(d) \subseteq \Delta(d')$. Because $\Delta(-)$ is isomorphic as co-unit of the adjunction, $d \le d'$, from which it follows $f(\Diamond d) \le f(\Diamond d') = \bot$ because both f and $\Diamond$ preserve joins. $\square$

Next we recall the definition of the topology of the lower hyperspace [24, 26].

Definition 4.5 *For a spectral space X, we define the lower hyperspace $\mathcal{L}(X)$ to be the set of all closed subsets of X taken with the topology generated by the sub-basic sets*

$$L_o = \{c \in \mathcal{L}(X) \mid c \cap o \neq \emptyset\}$$

for each $o \in \mathcal{K}(X)$.

Spectral spaces are closed under the lower hyperspace construction, that is, if X is a spectral space then so is $\mathcal{L}(X)$ [35].

We can now state and prove the essential ingredient for a coalgebraic representation of distributive lattices with additive operators.

Theorem 4.6 *For each spectral space X, $L_{\mathbb{A}}\mathcal{K}(X)$ is isomorphic in* DLat *to $\mathcal{K}\mathcal{L}(X)$.*

Proof: Let X be a spectral space and o be a compact open of X. The assignment $\Diamond o \mapsto L_o$ extends to a morphism $\gamma{:}FG_{\mathcal{K}(X)} \to \mathcal{K}\mathcal{L}(X)$, where $G_{\mathcal{K}(X)}$ is the set of generators of $L_{\mathbb{A}}\mathcal{K}(X)$. Since $L_\emptyset = \emptyset$ and $L_{o_1 \cup o_2} = L_{o_1} \cup L_{o_2}$, the morphism γ satisfies the relations of $L_{\mathbb{A}}\mathcal{K}(X)$. Hence we obtain a canonical morphism $\gamma^+{:}L_{\mathbb{A}}\mathcal{K}(X) \to \mathcal{K}\mathcal{L}(X)$ in DLat, that is, by definition, surjective as a function.

Next we prove that γ^+ is an isomorphism. Because $\Diamond$ preserves joins, an element in $L_{\mathbb{A}}\mathcal{K}(X)$ is the finite meet of elements in $G_{\mathcal{K}(X)}$. Suppose $\bigwedge_I \Diamond o_i \not\leq \bigwedge_J \Diamond o_j$, for some finite index sets I and J, with all o_i's and o_j's in $\mathcal{K}(X)$. By Corollary 2.4, there is a morphism $f{:}L_{\mathbb{A}}\mathcal{K}(X) \to 2$ in DLat such that $f(\bigwedge_I \Diamond o_i) = \top$ but $f(\bigwedge_J \Diamond o_j) = \bot$, that is, $f(\Diamond o_i) = \top$ for all $i \in I$ and there is a $k \in J$ such that $f(\Diamond o_k) = \bot$. Consider now the set

$$S = \bigcup\{u \in \mathcal{K}(X) \mid f(\Diamond u) = \bot\}\,.$$

Its complement is closed, and hence in $\mathcal{L}\mathcal{K}(X)$. Furthermore, by Lemma 4.4, $o_k \subseteq S$ whereas $o_i \not\subseteq S$ for all $i \in I$. Hence $X \setminus S \in \bigcap_I L_{o_i}$ but $X \setminus S \notin L_{o_k}$. Therefore $\bigcap_I L_{o_i} \not\subseteq \bigcap_J L_{o_j}$. $\square$

We thus have a duality between the category $\mathsf{DLat}(\mathbb{A})$ of distributive lattices with a unary additive operator and the category $\mathsf{Coalg}(\mathcal{L})$ of $\mathcal{L}$-coalgebras over spectral spaces.

Recall that a join-hemimorphism is an n-ary operator on a distributive lattice additive on each of its arguments [11, 12]. We leave it to the reader to verify that the functor induced by the theory of a join-hemimorphism on distributive lattices is dual to the functor $\mathcal{L}\prod_n$. More generally, the following result holds.

Corollary 4.7 *For a theory $\mathbb{O}$ with a signature Ω of operators additive in each of their arguments, the functor $L_{\mathbb{O}}$ is dual to the functor $\mathcal{L}\coprod_{\omega\in\Omega}\prod_{n_\omega}$.*

4.2 Multiplicative operators

Next we consider the theory of a unary multiplicative operator $\mathbb{M} = (\Omega, I)$, where Ω contains a unary operation $\Box$, and the set of identities I over the set of variables $V = \{v_0, v_1\}$ is given by

$$\Box\top = \top \quad \text{and} \quad \Box(v_0 \wedge v_1) = \Box v_0 \wedge \Box v_1\,.$$

The theory $\mathbb{M}$ induces the functor $L_{\mathbb{M}}{:}\mathsf{DLat} \to \mathsf{DLat}$, mapping each distributive lattice D to the distributive lattice

$$L_{\mathbb{M}}D \quad = \quad \mathsf{DLat}\langle \Box d{:}d \in D \mid \Box \text{ preserves finite meets}\rangle\,.$$

By Theorem 4.3 we have that the category $\mathsf{DLat}(\mathbb{M})$ of distributive lattices with unary multiplicative operators is isomorphic to the category of algebras $\mathsf{Alg}(L_{\mathbb{M}})$.

As for additive operators, we have the following lemma, relating points of the distributive lattice $L_{\mathbb{M}}D$ with saturated subsets of the representation of D: The lemma below will allow us to associate $f{:}L_{\mathbb{M}}D \to 2$ with $\bigcap\{\Delta(d') \mid d' \in D \text{ and } f(\Box d') = \top\}$. (Recall that a subset q of a spectral space is *saturated* if q is the intersection of some compact opens.)

Lemma 4.8 *Let D be a distributive lattice and $d \in D$. For every morphism $f{:}L_{\mathbb{M}}D \to 2$ we have*

$$f(\Box d) = \top \text{ if and only if } \bigcap\{\Delta(d') \mid d' \in D \text{ and } f(\Box d') = \top\} \subseteq \Delta(d)$$

Proof: The direction from left to right is obvious. For the converse we first notice that right hand side intersection is a filtered intersection of compact open subsets of $\mathcal{S}(D)$. Hence we can apply the Hoffman-Mislove theorem [13, 39] to show that there exists $d' \in D$ with $f(\Box d') = \top$ and $\Delta(d') \subseteq \Delta(d)$. Since $\Delta(-)$ is injective, $d' \leq d$, from which it follows $\top = f(\Box d') \leq f(\Box d)$ because both f and $\Box$ preserve finite meets $\Box$

Compact saturated subsets are used in the definition of the upper hyperspace [24, 26].

Definition 4.9 *For a spectral space X we define the upper hyperspace $\mathcal{U}(X)$ to be the set of all compact saturated subset of X taken with the topology generated by the basic sets*

$$U_o = \{q \in \mathcal{U}(X) \mid q \subseteq o\}$$

for each $o \in \mathcal{K}(X)$.

If X is a spectral space then $\mathcal{U}(X)$ is a Scott domain (taken with the Scott topology), and hence spectral [39].

The proof of the theorem below follows the same line of reasoning as that of Theorem 4.6. However the similarity is only apparent, as in one the additivity of the operators is translated into closed sets whereas in the next theorem, the multiplicativity of the operators is translated into (compact) upward-closed sets, and not into (compact) opens as one would expect. Moreover this similarity breaks down even more if one considers spaces that are not sober [3].

Theorem 4.10 *For each spectral space X, $L_{\mathbb{M}}\mathcal{K}(X)$ is isomorphic in DLat to $\mathcal{KU}(X)$.*

Proof: Let X be a spectral space and o be a compact open of X. The assignment $\Box o \mapsto U_o$ extends to a morphism $\delta{:}FG_{\mathcal{K}(X)} \to \mathcal{KU}(X)$, where $G_{\mathcal{K}(X)}$ is the set of generators of $L_{\mathbb{M}}\mathcal{K}(X)$. Since $U_X = \mathcal{U}(X)$ and $U_{o_1 \cap o_2} = U_{o_1} \cap U_{o_2}$, the morphism δ satisfies the relations of $L_{\mathbb{M}}\mathcal{K}(X)$. Hence we obtain a canonical morphism $\delta^+{:}L_{\mathbb{M}}\mathcal{K}(X) \to \mathcal{KU}(X)$ that, by definition, surjective as a function.

Next we prove that δ^+ is an isomorphism in DLat. Because $\Box$ preserves meets, an element in $L_{\mathbb{M}}\mathcal{K}(X)$ is the finite join of elements in $G_{\mathcal{K}(X)}$. Suppose $\bigvee_I \Box o_i \not\leq \bigvee_J \Box o_j$, for some finite index sets I and J, and with all o_i's and o_j's in $\mathcal{K}(X)$. By Corollary 2.4, there is a a morphism $f{:}L_{\mathbb{M}}\mathcal{K}(X) \to 2$ in DLat such that $f(\bigvee_I \Box o_i) = \top$ but $f(\bigvee_J \Box o_j) = \bot$. Hence there is $k \in I$ such that $f(\Box o_i) = \top$, and $f(\Box o_j) = \bot$ for all $j \in J$. Consider now the set $S = \bigcap\{u \in \mathcal{K}(X) \mid f(\Box u) = \top\}$. It is a compact saturated and hence in $\mathcal{U}(\mathcal{K}(X))$. Furthermore, by Lemma 4.8, $S \subseteq o_k$ whereas $S \not\subseteq o_j$, for all $j \in J$. In other words, $S \in U_{o_k}$ but $S \notin \bigcup_J U_{o_j}$. Therefore $\bigcup_I U_{o_i} \not\subseteq \bigcup_J U_{o_j}$. $\Box$

We thus have a duality between the category $\mathsf{DLat}(\mathbb{M})$ of distributive lattices with a unary multiplicative operator and the category $\mathsf{Coalg}(\mathcal{U})$ of $\mathcal{U}$-coalgebras over spectral spaces.

Corollary 4.11 *For a theory $\mathbb{O}$ with a signature Ω of operators multiplicative on each of their arguments, the functor $L_{\mathbb{O}}$ is dual to the functor $\mathcal{U}\coprod_{\omega\in\Omega}\prod_{n_\omega}$.*

4.3 Additive and multiplicative operators, together

The above theories $\mathbb{A}$ of a unary additive operator and $\mathbb{M}$ of a unary multiplicative operator can be combined in a single complex theory that relates the two operators. We define it by $\mathbb{V} = (\Omega, I)$, where the set Ω contains two unary operations $\Diamond$ and $\Box$, and the set of identities I over the variables v_0, v_1 is given by

$$\begin{array}{ll} \Diamond\bot = \bot & \Box\top = \top \\ \Diamond(v_0 \vee v_1) = \Diamond v_0 \vee \Diamond v_1 & \Box(v_0 \wedge v_1) = \Box v_0 \wedge \Box v_1 \\ \Box(v_0 \vee v_1) = \Box v_0 \vee (\Box(v_0 \vee v_1) \wedge \Diamond v_1) & \Box v_0 \wedge \Diamond v_1 = \Box v_0 \wedge \Diamond(v_0 \wedge v_1)\,. \end{array}$$

The theory $\mathbb{V}$ induces the functor $L_{\mathbb{V}}{:}\mathsf{DLat} \to \mathsf{DLat}$, mapping each distributive lattice D to the Vietoris locale of D [17]:

$$\begin{array}{rcll} L_{\mathbb{V}}D & = & \mathsf{DLat}\langle \Box d, \Diamond d{:}d \in D \mid & \Box \text{ preserves finite meets} \\ & & & \Diamond \text{ preserves finite joins} \\ & & & \Box(d \vee d') = \Box d \vee (\Box(d \vee d') \wedge \Diamond d') \\ & & & \Box d \wedge \Diamond d' = \Box d \wedge \Diamond(d \wedge d')\,\rangle\,. \end{array}$$

By Theorem 4.3 we have that the category $\mathsf{DLat}(\mathbb{V})$ is isomorphic to the category of algebras $\mathsf{Alg}(L_{\mathbb{V}})$.

Recall that a subset S of a topological space X is *convex closed* if $S = {\uparrow}S \cap \overline{S}$, where ${\uparrow}S$ is the upper closure of S with respect to the order defined by its topology, and $\overline{S}$ is the topological closure of S [24].

Definition 4.12 *For a spectral space X we define the Vietoris hyperspace $\mathcal{V}(X)$ to be the set of all compact convex subsets of X taken with the topology generated by the sub-basic sets U_o and L_o for each $o \in \mathcal{K}(X)$.*

If X is a spectral space then $\mathcal{V}(X)$ is also spectral [17, 39].

Theorem 4.13 *For each spectral space X, $L_{\mathbb{V}}\mathcal{K}(X)$ is isomorphic in* DLat *to $\mathcal{K}\mathcal{V}(X)$.*

Proof: Let X be a spectral space and o be a compact open of X. The assignments $\Box o \mapsto U_o$ and $\Diamond o \mapsto L_o$ extends to a morphism $\rho{:}FG_{\mathcal{K}(X)} \to \mathcal{K}\mathcal{V}(X)$, where $G_{\mathcal{K}(X)}$ is the set of generators of $L_{\mathbb{V}}\mathcal{K}(X)$. It is not hard to see that ρ satisfies the relations of $L_{\mathbb{V}}\mathcal{K}(X)$. Hence we obtain a canonical morphism $\rho^+{:}L_{\mathbb{V}}\mathcal{K}(X) \to \mathcal{K}\mathcal{V}(X)$ that is clearly surjective as a function.

Next we prove that ρ^+ is an isomorphism in DLat. First of all we note that elements in $FG_{\mathcal{K}(X)}$ are finite joins of finite meets of generators in $G_{\mathcal{K}(X)}$. Using the laws of distributive lattices, because $\Box$ preserves finite meets, $\Diamond$ preserves finite joins, and $\mathcal{K}(X)$ is closed under finite unions and finite intersections, we have that every element in $FG_{\mathcal{K}(X)}$ is identified to an element of the form $\bigvee_I(\Box o_i \wedge \bigwedge_{J_i} \Diamond o_j)$ for finite sets I and J_i's, $(i \in I)$, and compact opens o_i's and o_j's. Assume

$$d = \bigvee_I(\Box o_i \wedge \bigwedge_{J_i} \Diamond o_j) \not\leq \bigvee_N(\Box o_n \wedge \bigwedge_{M_n} \Diamond o_m) = d'$$

By Corollary 2.4, there is a function $f{:}L_{\mathbb{V}}\mathcal{K}(X) \to 2$ such that $f(d) = \top$ but $f(d') = \bot$. Because f is a morphism in DLat, the above means that

1. there exists $i_0 \in I$ such that $f(\Box o_{i_0}) = \top$ and $f(\Diamond o_j) = \top$ for all $j \in J_{i_0}$;

2. for all $n \in N$, $f(\Box o_n) = \bot$ or there exists $m_0 \in M_n$ such that $f(\Diamond o_{m_0}) = \bot$.

Consider now the sets $Q = \bigcap\{u \in \mathcal{K}(X) \mid f(\Box u) = \top\}$, $O = \bigcup\{u \in \mathcal{K}(X) \mid f(\Diamond u) = \bot\}$ and $C = Q \cap (X \setminus O)$. The set Q is compact saturated (and hence upper closed), O is open and C is compact because it is the intersection of a compact set with a closed one. Since $X \setminus O$ is closed, $\overline{C} \subseteq X \setminus O$. Similarly, because Q is upper closed, $\uparrow C \subseteq Q$. Hence $\uparrow C \cap \overline{C} \subseteq C$. Since the other inclusion trivially holds, we have that C is a convex closed set. Therefore $C \in \mathcal{V}(\mathcal{K}(X))$.

By Lemma 4.8 $C \subseteq Q \subseteq o_{i_0}$, that is $C \in U_{o_{i_0}}$. Further, for each $u \in \mathcal{K}(X)$ such that $f(\Box u) = \top$ and for each $j \in J_{i_0}$ we have

$$\top = f(\Box u) \wedge f(\Diamond o_j) = f(\Box u \wedge \Diamond o_j) \leq f(\Diamond(u \cap o_j))\,,$$

where the last inequality follows from the relations in $L_{\mathbb{V}}\mathcal{K}(X)$ and because f is monotone. Hence, by Lemma 4.4, $u \cap o_j \not\subseteq O$ for all $u \in \mathcal{K}(X)$ such that $f(\Box u) = \top$ and for all $j \in J_{i_0}$. By definition, we thus have $Q \cap o_j \not\subseteq O$, (or, equivalently, $Q \cap (X \setminus O) \cap o_j \neq \emptyset$) for all $j \in J_{i_0}$. Therefore $C \in \bigcap_{J_{i_0}} L_{o_j}$, from which we finally obtain that $C \in \bigcup_I (U_{o_i} \cap \bigcap_{J_i} L_{o_j})$.

Similarly, by Lemma 4.4, if $f(\Diamond o_{m_0}) = \bot$ for some $m_o \in M$ then $o_{m_0} \subseteq O$, from which it follows that $Q \cap (X \setminus O) \cap o_{m_0} = \emptyset$, or, equivalently, $C \notin L_{o_{m_0}}$. On the other hand, for each $u \in \mathcal{K}(X)$ such that $f(\Diamond u) = \bot$ and for each $n \in N$ with $f(\Box o_n) = \bot$ we have

$$f(\Box(o_n \cup u)) \leq f(\Box o_n \vee \Diamond u) = f(\Box o_n) \vee f(\Diamond u) = \bot\,,$$

where the first inequality follows from the relations in $L_{\mathbb{V}}\mathcal{K}(X)$ and because f is monotone. Hence, by Lemma 4.8, $Q \not\subseteq o_n \cup u$ for all $u \in \mathcal{K}(X)$ such that $f(\Diamond u) = \bot$ and for all $n \in N$ with $f(\Box o_n) = \bot$. By definition, this is equivalent to say $Q \not\subseteq o_n \cup O$ (or, equivalently, $Q \cap (X \setminus O) \not\subseteq o_n$) for all $n \in N$ with $f(\Box o_n) = \bot$. Therefore $C \notin U_{o_n}$ for each $n \in N$ with $f(\Box o_n) = \bot$, showing that $C \notin \bigcup_M (U_{o_m} \cap \bigcap_{N_m} L_{o_n})$.

Summarizing, we have thus seen that

$$\rho^+(d) = \bigcup_I (U_{o_i} \cap \bigcap_{J_i} L_{o_j}) \not\subseteq \bigcup_M (U_{o_m} \cap \bigcap_{N_m} L_{o_n}) = \rho^+(d')\,,$$

proving that ρ^+ is order preserving and hence an isomorphism in DLat. $\Box$

As a corollary we obtain a duality between the category $\mathsf{DLat}(\mathbb{V})$ of distributive lattices with two unary operators and the category $\mathsf{Coalg}(\mathcal{V})$ of $\mathcal{V}$-coalgebras over spectral spaces.

5 Ordered algebras and monotone operators

An ordered algebra (posalg in short) is an algebra with a partially ordered carrier set and monotone operators [10]. More specifically, for an algebraic theory $\mathbb{T} = (\Sigma, E)$, a $\mathbb{T}$-*posalg* is a poset A together with a corresponding monotone function $\sigma_A{:}A^{n_\sigma} \to A$ for each function symbol $\sigma \in \Sigma$, such that, independently of the way we substitute elements of A for the variables, each equation in E holds in A as an identity. A homomorphism between two $\mathbb{T}$-posalg A and B is a monotone function $f{:}A \to B$ such that $\sigma_B \circ f^{n_\sigma} = f \circ \sigma_A$ for each function symbol $\sigma \in \Sigma$. The category of $\mathbb{T}$-posalg is denoted by $\mathsf{PosAlg}(\mathbb{T})$.

Clearly every $\mathbb{T}$-algebra is a $\mathbb{T}$-posalg. The converse holds, for example, for the theory $\mathbb{D}$ of distributive lattices, because every partial order on a distributive lattice that makes its operations monotone coincides with the partial order of the distributive lattice itself. It follows that $\mathsf{DLat} \cong \mathsf{PosAlg}(\mathbb{D})$. Let PoSet be the category of posets with monotone functions as morphisms. The forgetful functor $U{:}\mathsf{DLat} \to \mathsf{PoSet}$ has a left adjoint denoted by F, mapping a poset P to the distributive lattice FP presented by taking as generators the underlying set of P and as relations the pairs $p \wedge p' = p$ for all $p \leq p'$ in P. Further, products and coproducts in PoSet are computed as in Set.

Definition 5.1 *Let U be the forgetful functor from DLat to PoSet with left adjoint F. A theory $\mathbb{O}$ of monotone operators over distributive lattices consists of*

1. *a signature Ω of operations $\omega \in \Omega$ with arities n_ω which gives rise to a functor $G_\Omega{:}\mathsf{PoSet} \to \mathsf{PoSet}$, $X \mapsto \coprod_{\omega\in\Omega} X^{n_\omega}$,*
2. *a set I of identities containing pairs of elements of the underlying set of $UFG_\Omega UFV$, for some poset V of variables.*

For each algebraic theory $\mathbb{O} = (\Omega_\mathbb{O}, I_\mathbb{O})$ of monotone operators over distributive lattices we define the category $\mathsf{DLat}(\mathbb{O})$ of *distributive lattices with monotone operators* as the category of $\mathsf{PosAlg}(\mathbb{T})$. Here $\mathbb{T} = (\Sigma_\mathbb{D} + \Omega_\mathbb{O}, E_\mathbb{D} + I_\mathbb{O})$, where the equations in $E_\mathbb{D}$ and $I_\mathbb{O}$ are understood as equations over $\Sigma_\mathbb{D} + \Omega_\mathbb{O}$.

Definition 5.2 *Given a theory of monotone operators $\mathbb{O} = (\Omega, I)$ and a functor $L{:}\mathsf{DLat} \to \mathsf{DLat}$, we say that a natural transformation $f{:}FG_\Omega U \to L$* satisfies the identities I *if for all distributive lattices D and all morphism $v{:}FV \to D$ (mapping variables to closed terms) it holds*

$$(t,s) \in I \quad \Rightarrow \quad (f_D \circ FG_\Omega Uv)(t) = (f_D \circ FG_\Omega Uv)(s)\,.$$

In a similar way as we have already seen in the previous section, a theory of monotone operators $\mathbb{O} = (\Omega, I)$ defines a canonical functor $L_\mathbb{O}{:}\mathsf{DLat} \to \mathsf{DLat}$ that comes equipped with a natural transformation $\imath$ satisfying the identities I. We define the functor $L_\mathbb{O}$ on a distributive lattice D as

$$\begin{tikzcd} FI \arrow[r, shift left, "\pi_1^\sharp"] \arrow[r, shift right, "\pi_2^\sharp"'] & FG_\Omega UFV \arrow[r, "FG_\Omega Uv"] & FG_\Omega UD \arrow[r, "\imath_D"] & L_\mathbb{O} D \end{tikzcd}$$

where the $\pi_1^\sharp, \pi_2^\sharp$ come from the projections $\pi_1, \pi_2{:}I \to UFG_\Omega UFV$, and $\imath_D$ is the joint coequalizer with respect to to all pairs $(FG_\Omega Uv \circ \pi_1^\sharp, FG_\Omega Uv \circ \pi_2^\sharp)$, for $v{:}FV \to D$. The universal property of $L_\mathbb{O} D$ gives the action of $L_\mathbb{O}$ on morphisms and the naturality of $\imath$. With a proof similar to that of Theorem 4.3, we have that the category of distributive lattices with monotone operators $\mathsf{DLat}(\mathbb{O})$ is isomorphic to the category $\mathsf{Alg}(L_\mathbb{O})$. Therefore we can give a coalgebraic representation of a category of distributive lattices with monotone operators $\mathsf{DLat}(\mathbb{O})$ by finding a dual functor $T{:}\mathsf{Spec} \to \mathsf{Spec}$.

5.1 Monotone operators

Let us consider the theory of monotone operators $\mathbb{P} = (\Omega, I)$, where Ω contains a single unary monotone operation $\bigcirc$ and there are no identities, that is, $I = \emptyset$. Then $G_\Omega P = P$ and, for each distributive lattice D, $L_\mathbb{P} D$ can be presented by

$$\mathsf{DLat}\langle \bigcirc a{:}a \in D \mid \bigcirc \text{ preserves order}\rangle$$

The representation of the distributive lattice $L_{\mathbb{P}}D$ is obtained by consecutively applying the lower and upper hyperspace construction.

Theorem 5.3 *For every spectral space X, $\mathcal{LU}(X) \cong \mathcal{S}L_{\mathbb{P}}\mathcal{K}(X)$.*

Proof: For $S \in \mathcal{LU}(X)$ define $\gamma(S){:}\mathcal{K}(X) \to 2$ by

$$\gamma(S)(o) = \top \text{ iff } o \in S\,.$$

Note that if $o_1 \subseteq o_2$ and $o_1 \in S$ for some lower closed subset of $\mathcal{U}(X)$, then also $o_2 \in S$. Hence $\gamma(S)$ is monotone, i.e. it satisfies the relations of $L_{\mathbb{P}}\mathcal{K}(X)$. We can therefore extend $\gamma(S)$ to a distributive lattice morphism $\gamma^+(S) \in L_{\mathbb{P}}\mathcal{K}(X) \to 2$, that is $\gamma^+{:}\mathcal{LU}(X) \to \mathcal{S}L_{\mathbb{P}}\mathcal{K}(X)$. To prove that γ^+ is continuous we see that, for each $o \in \mathcal{K}(X)$,

$$\begin{array}{rcl} {\gamma^+}^{-1}(\Delta(o)) & = & \{S \in \mathcal{LU}(X) \mid \gamma(S) \in \Delta(o)\} \\ & = & \{S \in \mathcal{LU}(X) \mid \gamma(S)(o) = \top\} \\ & = & \{S \in \mathcal{LU}(X) \mid o \in S\}\,. \end{array}$$

But $o \in S \in \mathcal{LU}(X)$ if and only if $S \cap U_o \neq \emptyset$. Indeed, $o \in U_o$ by definition, hence if $o \in S$ then $S \cap U_o \neq \emptyset$. Conversely, if $q \in U_o$ then $q \subseteq o$. Thus, by lower closure of S, if $q \in S$, then also $o \in S$. It follows that ${\gamma^+}^{-1}(\Delta(o)) = L_{U_o}$.

Next define, for $f \in \mathcal{S}L_{\mathbb{P}}\mathcal{K}(X)$,

$$\eta(f) = \{q \in \mathcal{U}(X) \mid \bigwedge_{q \subseteq o} f(o) = \top\}\,.$$

For $q_1, q_2 \in \mathcal{U}(X)$, if $q_1 \supseteq q_2$ and $\bigwedge_{q_2 \subseteq o} f(o) = \top$, then also $\bigwedge_{q_1 \subseteq o} f(o)$. Hence $\eta(f)$ is lower closed, that is, $\eta{:}\mathcal{S}L_{\mathbb{P}}\mathcal{K}(X) \to \mathcal{LU}(X)$. To show that η is a continuous function, we have, for each $o \in \mathcal{K}(X)$,

$$\begin{array}{rcl} \eta^{-1}(L_{U_o}) & = & \{f \in \mathcal{S}L_{\mathbb{P}}\mathcal{K}(X) \mid \eta(f) \cap U_o \neq \emptyset\} \\ & = & \{f \in \mathcal{S}L_{\mathbb{P}}\mathcal{K}(X) \mid o \in \eta(f)\} \\ & = & \{f{:}L_{\mathbb{P}}\mathcal{K}(X) \to 2 \mid f(o) = \top\} \\ & = & \Delta(o)\,, \end{array}$$

where the second equality holds because $o \in S \in \mathcal{LU}(X)$ if and only if $S \cap U_o \neq \emptyset$, as we have already seen above.

The function η is inverse of γ. Indeed, for $f \in \mathcal{S}L_{\mathbb{P}}\mathcal{K}(X)$ and $o \in \mathcal{K}(X)$ we have

$$\begin{array}{rcl} \gamma(\eta(f))(o) = \top & \iff & o \in \eta(f) \\ & \iff & \bigwedge_{o \subseteq o'} f(o') = \top \\ & \iff & f(o) = \top \quad [f \text{ is monotone}] \end{array}$$

and also, for $S \in \mathcal{LU}(X)$ and $q \in \mathcal{U}(X)$,

$$\begin{array}{rcl} q \in \eta(\gamma(S)) & \iff & \bigwedge_{q \subseteq o} \gamma(S)(o) = \top \\ & \iff & \forall q \subseteq o. o \in S \\ & \iff & q \in S \end{array}$$

where the last implication from right to left holds because S is lower closed, whereas the implication from left to right holds because $\mathcal{U}(X)$ is a Scott domain, thus S is Scott closed. Since the set $\{o \mid q \subseteq o\}$ is directed in $\mathcal{U}(X)$, it follows that its least upper bound q must be in S. □

Since the continuous image of compact opens is compact, as a corollary we obtain an isomorphism in DLat between $\mathcal{KLU}(X)$ and $L_{\mathbb{P}}\mathcal{K}(X)$. Thus it follows that the category of distributive lattices with a unary monotone operator is isomorphic to the category of $\mathcal{LU}$-coalgebras.

6 Conclusion and future directions

In this paper we presented a framework for a coalgebraic representation of distributive lattices with operators. We have applied our method to several theories of operators, including additive, multiplicative and monotone operators.

An immediate investigation is to apply our framework to similar operators but contravariant in their arguments, as studied, for example, in [36]. One intriguing way to define suitable coalgebraic representations could be to re-consider the definitions of the sub-basic opens of the lower, upper and Vietoris hyperspaces by indexing them with closed sets rather than with opens.

Although we have considered here only operators on distributive lattices, our framework could be applied to any category of algebras over Set (with minor changes for infinitary algebras to take into account equations involving infinitary operations) for which a dual category is known [5]. For example, to consider additional properties such as completeness of the lattice operations and of the operators, we can take the category of algebraic completely distributive lattices (the canonical extensions of distributive lattices [7, 8, 9]) as starting point of our investigations. Its dual is well-known: the category of PoSet.

Let us point out that our approach allows us to treat theories with arbitrary nesting of operators in their identities, although, according to Definitions 4.1 and 5.1, terms appearing in equations may not contain nested occurrences of operations from the theory $\mathbb{O}$ of operators. Intuitively, this restriction arises from our interest in a representation via coalgebras for a functor T dual to $L_{\mathbb{O}}$. In contrast to coalgebras for a comonad, a T-coalgebra encodes the operational view of what a system can perform in one single step [34]. From this point of view, our format of the equation is not a restriction, but formalizes that we do not need nested modalities to describe a single step (nested modalities describe sequences of steps), see [5]. In other words, identities with nested operators are not identities on distributive lattices, but rather on the algebras for the induced functor from the simple identities. Additional identities with nested operators can be dealt with without problems. They specify particular equationally/modally definable full subcategories of algebras/coalgebras for dual functors.

For example, it is routine to see that the addition of the nested identity $\Box v_0 \vee v_0 = v_0$ (i.e. $\Box v_0 \leq v_0$) on the theory $\mathbb{M}$ of a unary multiplicative operator characterizes those $L_{\mathbb{M}}$ algebras that are represented by reflexive $\mathcal{U}$-coalgebras on spectral spaces, that is, coalgebras $\alpha{:}X \to \mathcal{U}(X)$ such that $x \in \alpha(x)$ for all $x \in X$. More interestingly, the theory of distributive lattices with a monadic universal quantifier, obtained by adding the nested equations

$$\Box v_0 \vee v_0 = \Box v_0 \ \text{ and } \ \Box(v_0 \vee \Box v_1) = \Box v_0 \vee \Box v_1$$

to the theory of $\mathbb{M}$ of a unary multiplicative operator, characterizes those $L_{\mathbb{M}}$ algebras that are represented by $\mathcal{U}$-coalgebras $\alpha{:}X \to \mathcal{U}(X)$ with α an equivalence relation on X such that $\alpha(o) \in \mathcal{K}(X)$ for each $o \in \mathcal{K}(X)$ (i.e. α is an open equivalence relation on X) and the quotient of X with respect to α is a $\mathcal{T}_0$ space [2].

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