Strongly Complete Logics for Coalgebras

Alexander Kurz *
University of Leicester, UK

Jiří Rosický †
Masaryk University, Brno, Czech Republic

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Abstract

Coalgebras for a functor T on a category $\mathcal{X}$ model many different types of transition systems in a uniform way. This paper focuses on a uniform account of finitary strongly complete specification languages for Set-based coalgebras.

We show how to associate a finitary logic to any finite-sets preserving functor T and prove the logic to be strongly complete under a mild condition on T. The proof is based on the following result. An endofunctor on a variety has a presentation by operations and equations iff it preserves sifted colimits.

1 Introduction

Coalgebras for a functor T on a category $\mathcal{X}$ model many different types of transition systems in a uniform way. Coalgebras are dual to algebras and the logic of algebras is equational logic. But then, what is the logic of coalgebras? Can logics for coalgebras be described in a uniform way, and their properties be established in a uniform manner?

Our approach to these questions is based on Stone duality. We think of Stone duality [15, 2] as relating a category of algebras $\mathcal{A}$ representing a propositional logic to a category of topological spaces $\mathcal{X}$ representing the state-based models of the logic. The duality is provided by two contravariant functors P and S,

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "P"] & \mathcal{A} \arrow[l, bend left, "S"] \end{tikzcd} . \tag{1}$$

P maps a space X to a propositional theory and S maps a propositional theory to its 'canonical model'. Abramsky [1] extended a basic Stone duality as in Diagram 1 by 'synchronising' dual constructions on both sides of Diagram 1, thus providing a description of domain theory in logical form. This suggests that the modal logic of a functor T should be given by its dual L on $\mathcal{A}$:

$$\begin{tikzcd} \mathcal{X} \arrow[loop left, "T"] \arrow[r, bend left, "P"] & \mathcal{A} \arrow[l, bend left, "S"] \arrow[loop right, "L"] \end{tikzcd} \tag{2}$$

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Then the category of L-algebras is dual to the category of T-coalgebras and the initial L-algebra provides a propositional theory characterising T-bisimilarity. Moreover, if L can be presented by generators and relations, one inherits a proof system from equational logic which is sound and strongly complete. Thus, logics for T-coalgebras arise from presentations of the dual of T by generators and relations. We characterise those functors L on varieties $\mathcal{A}$ that have a finitary presentation.

Whereas the result above gives us logics for coalgebras, our next aim is to prove a strong completeness result for finitary logics for Set-coalgebras. The approach indicated in Diagram 2 can be applied to Set-coalgebras, but as the dual of Set is the category CABA of *complete* atomic boolean algebras, the corresponding logics are infinitary. Our solution is to consider two Stone dualities:

$$\begin{tikzcd} \mathsf{Stone} \arrow[r, shift left] & \mathsf{BA} \arrow[l, shift left] \arrow[loop right, "L"] \end{tikzcd} \tag{3}$$

$$\begin{tikzcd} \mathsf{Set} \arrow[loop left, "T"] \arrow[r, shift left] & \mathsf{CABA} \arrow[l, shift left] \end{tikzcd}$$

The upper row is the duality between Stone spaces and Boolean algebras, accounting for (classical finitary) propositional logic. L describes an expansion of propositional logic by modal operators and axioms. The lower row is the duality where our Set-based T-coalgebras live. How can these two worlds be related?

The crucial observation is the following. BA is the Ind-completion of finite Boolean algebras, that is, the completion of finite Boolean algebras under filtered colimits; Set is the Ind-completion of finite sets; and finite sets are dual to finite Booleans algebras. In other words, $\mathsf{Set}^{\mathrm{op}}$ is the Pro-completion of finite Boolean algebras, that is, the completion of finite Boolean algebras under cofiltered limits.

$$\begin{tikzcd} \mathsf{BA} \arrow[loop left, "L"] \arrow[rr, bend right=10, "S"'] & & \mathsf{Set}^{\mathrm{op}} \arrow[ll, bend right=10, "P"'] \arrow[loop right, "T^{\mathrm{op}}"] \\ & \mathsf{BA}_\omega \cong \mathsf{Set}^{\mathrm{op}}_\omega \arrow[ul, "\hat{(-)}"] \arrow[ur, "\bar{(-)}"'] & \end{tikzcd} \tag{4}$$

If T preserves finite sets then we can associate a modal logic to T by defining L to be the continuous extension that agrees with T on finite sets. Moreover, we obtain a natural transformation

$$\delta : LP \to PT$$

giving the semantics to the logic by inducing a functor $\tilde{P} : \mathsf{Coalg}(T) \to \mathsf{Alg}(L)$. Similarly, if T weakly preserves cofiltered limits, we obtain

$$h : SL \to TS$$

giving rise to a map on objects $\tilde{S} : \mathsf{Alg}(L) \to \mathsf{Coalg}(T)$. From this, one obtains strong completeness: We can associate to any L-algebra A the coalgebra $\tilde{S}A$, which provides a counter example for each formula not holding in A.

Summary of Results Our main results are the following:

1. A functor on a variety $\mathcal{A}$ has a presentation iff it preserves sifted colimits.

2. Algebras over Ind-completions can be represented via algebras over Pro-completions.

3. To any functor on Stone that is determined by its action on finite Stone spaces, one can associate a finitary strongly complete modal logic that characterises T-bisimilarity.

4. To any functor on Set that preserves finite sets and weakly preserves cofiltered limits, one can associate a finitary strongly complete modal logic.

The first two results are of purely categorical nature and are treated in Sections 4 and 5. The next two results are essentially corollaries of the first two and are described in Sections 7 and 8. The last one generalises a result in modal logic known as bisimilarity-somewhere-else.

Comparison with other approaches In his seminal paper [24], Moss described a coalgebraic logic for any weak pullback preserving functor on sets, which to a large extent, answers our question for a parametric logic for coalgebras. But his solution has some drawbacks. First, the restrictions to sets and to weak-pullback preserving functors are essential to his approach. This prevents generalisations to logics for systems modelled in a domain theoretic (ie topological) setting. And it prevents extensions to situations where the modal law $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$ does not hold. This is typically the case in logics for games where one takes $\Box\varphi$ to mean that the player can play some move that restricts the opponent to moves after which φ holds. Second, Moss's logic does not provide modalities to decompose the structure of T, which is needed to allow for a flexible specification language. Related to this, there is no proof system and no completeness result.

To address these issues, attention was focused on special classes of functors given by a restricted number of type constructors for which logics were built in an ad hoc manner [22, 27, 14]. Pattinson [25] showed that these languages with their ad hoc modalities arise from modal operators given by certain natural transformations, called predicate liftings, associated with the functor T. Schröder [29] investigates the logics given by all predicate liftings of finite arity and shows that these logics are expressive for finitary functors T. This restriction to finitary functors excludes traditional transition systems. Moreover, it is not clear how this approach generalises to topological and domain theoretic settings.

Our approach does not suffer any of these drawbacks. On the other hand, for Set-functors, we restrict attention to those that preserve finite sets and weakly preserve cofiltered limits. As we will explain, this is justified by focussing on strong completeness results.

The observation that all logics given by predicate liftings correspond to a functor L on BA was made in [19]. That functors that have a presentation give rise to a logic for coalgebras was noted in [10]. Here we give a characterisation of the functors which have presentations. The process of taking a finite set preserving functor and extending it to BA, and hence to Stone, is related to a construction in Worrell [33] where a Set-functor is lifted to complete ultrametric spaces.

2 Algebras and Coalgebras

Given a functor L on a category $\mathcal{A}$, an L-algebra (notation: (A, α) or just α) is an arrow $\alpha : LA \to A$. A morphism $f : \alpha \to \alpha'$ is an arrow $f : A \to A'$ such that $f \circ \alpha = \alpha' \circ Lf$.

The category of algebras for a signature Σ and equations E is defined as usual (in particular, carriers are sets) and denoted by $\mathsf{Alg}(\Sigma, E)$. We say that a category $\mathcal{A}$, equipped with a forgetful functor $U : \mathcal{A} \to \mathsf{Set}$, has a *presentation* if there exists a signature Σ and equations E such that $\mathcal{A}$ is concretely isomorphic to $\mathsf{Alg}(\Sigma, E)$. $\mathcal{A}$ (or more precisely $U : \mathcal{A} \to \mathsf{Set}$) is *monadic* (over Set) iff $\mathcal{A}$ has such a presentation and $U : \mathcal{A} \to \mathsf{Set}$ has a left adjoint.

A functor is *finitary* if it preserves filtered colimits. An object K of a category $\mathcal{K}$ is *finitely presentable* if its hom-functor $\hom(K, -) : \mathcal{K} \to \mathsf{Set}$ is finitary. In Set, the finitely presentable objects are precisely the finite sets and in $\mathsf{Alg}(\Sigma, E)$ they are the algebras described by a finite set of generators and a finite set of relations.

A category monadic over Set is called a *variety* if it has a set of finitely presentable objects and every object is a filtered colimit of these. This is the case whenever all operations in Σ are of finite arity. We are particularly interested in the variety BA of Boolean algebras and in the variety DL of distributive lattices (with top and bottom elements).

Given a functor T on a category $\mathcal{X}$, a T*-coalgebra* (notation: (X, ξ) or just ξ) is an arrow $\xi : X \to TX$ in $\mathcal{X}$. A morphism $f : \xi \to \xi'$ is an arrow $f : X \to X'$ such that $Tf \circ \xi = \xi' \circ f$.

Throughout the paper it will be the case that $\mathcal{X}$ is the category Set or some category of topological spaces. It makes therefore sense to speak of the elements, or *states*, of some $X \in \mathcal{X}$. We say that two states x, x' of $\xi : X \to TX$ and $\xi' : X' \to TX'$ are behaviourally equivalent or *bisimilar* if there are coalgebra morphisms f, f' with $f(x) = f'(x')$. This notion of bisimilarity avoids the problems of Aczel and Mendler [4] bisimulations, which do not work properly if T does not preserve weak pullbacks. It goes back to Aczel and Mendler [4], who use it to generalise the final coalgebra theorem of Aczel [3] by removing the assumption of weak-pullback preservation.

3 Sifted Colimits Preserving Functors

Since a variety $\mathcal{A}$ can be built from its finitely presentable algebras by using filtered colimits, filtered colimits preserving functors $L : \mathcal{A} \to \mathcal{A}$ are fully determined by their values on finitely presentable algebras. The latter form a small part of $\mathcal{K}$ in the sense that, up to an isomorphism, there is only a set of them.

Filtered colimits are precisely those which commute in sets with finite limits. Thus they stem out from the doctrine of finite limits while varieties are given by the doctrine of finite products, see Lawvere [23]. It is therefore natural to consider colimits which commute in sets with finite products. These colimits are called *sifted colimits*. They were studied in [6] and the main result is that any variety can be built up from its strongly finitely presentable algebras by using sifted colimits. Here, an algebra A is *strongly finitely presentable* if $\hom(A, -) : \mathcal{A} \to \mathsf{Set}$ preserves sifted colimits. These algebras coincide with finitely presentable (regular) projective algebras, ie with retracts of finitely generated free algebras. Any filtered colimit is of course sifted. Another important kind of sifted colimits are reflexive coequalizers (a parallel pair of arrows f, g is reflexive if there is t with $ft = gt = id$). Reflexive coequalizers include coequalizers of equivalence relations.

Sifted colimits preserving functors $L : \mathcal{A} \to \mathcal{A}$ are fully determined by their values on finitely generated free algebras. Their algebraic character is documented by the next result; recall that for a functor L preserving filtered colimits, $\mathsf{Alg}(L)$ is only locally finitely presentable.

Theorem 3.1. *Let $\mathcal{A}$ be a variety and $L : \mathcal{A} \to \mathcal{A}$ preserve sifted colimits. Then $\mathsf{Alg}(L)$ is a variety.*

Proof. Analogous to [5, Remark 2.75] using [7, 1.4.19]. $\square$

The following result is a consequence of the fact that every finitely presentable algebra is a reflexive coequalizer of finitely generated free algebras.

Proposition 3.2. *Let $\mathcal{A}$ be a variety and $L : \mathcal{A} \to \mathcal{A}$ preserve filtered colimits and reflexive coequalizers. Then L preserves sifted colimits.*

Proof. Let $\mathcal{A}_{fp}$ be the full subcategory of $\mathcal{A}$ consisting of finitely presentable objects and $\mathcal{A}_{sfp}$ be the full subcategory of $\mathcal{A}$ consisting of strongly finitely presentable objects. Following [6, 2.3.(2)], $\mathcal{A}_{fp}$ is the (free) closure of $\mathcal{A}_{sfp}$ under reflexive coequalizers. Thus L is uniquely determined by its restriction L_{sfp} on $\mathcal{A}_{sfp}$. Since $\mathcal{A}$ is a (free) closure of $\mathcal{A}_{sfp}$ under sifted colimits, L_{sfp} has a unique extension $L' : \mathcal{A} \to \mathcal{A}$ preserving sifted colimits. Since both reflexive coequalizers and filtered colimits are sifted colimits, we have $L' = L$. Hence L preserves sifted colimits. $\square$

In some very simple but important varieties like sets or linear spaces, every finitely presentable algebra is projective. As a consequence we get the next result which, in particular, implies that $\mathsf{Alg}(L)$ is a variety.

Proposition 3.3. *Let $\mathcal{A}$ be a variety such that every finitely presentable algebra is projective. Then any functor $L : \mathcal{A} \to \mathcal{A}$ preserving filtered colimits preserves sifted colimits.*

The previous proposition can be extended to boolean algebras. In fact, the trivial Boolean algebra $\mathbb{1}$ is the only finitely presentable that is not projective. $\mathbb{1}$ is the reflexive coequalizer

$$F1 \underset{o}{\overset{i}{\rightrightarrows}} F0 \xrightarrow{s} \mathbb{1} \tag{5}$$

where F is the left adjoint to the forgetful functor $\mathsf{BA} \to \mathsf{Set}$, i maps the generator to the top, and o maps the generator to the bottom. If $L : \mathsf{BA} \to \mathsf{BA}$ preserves filtered colimits and the above coequalizer, then L preserves sifted colimits.

Proposition 3.4. *For any filtered colimit preserving functor $L : \mathsf{BA} \to \mathsf{BA}$ there is a sifted colimit preserving functor $L' : \mathsf{BA} \to \mathsf{BA}$ such that L and L' are isomorphic when restricted to the full subcategory of BA without $\mathbb{1}$. Moreover, $\mathsf{Alg}(L) = \mathsf{Alg}(L')$.*

Proof. Define $L' = L$ on the full subcategory of BA not containing $\mathbb{1}$, and $L'\mathbb{1}$ such that L' preserves the coequalizer s in (5). Since there are no arrows $\mathbb{1} \to A$ other than the identity, we only have to define L' on arrows $h : A \to \mathbb{1}$, $A \neq \mathbb{1}$. Choose an arrow $f : A \to F0$ and define $L'h = L's \circ L'f$. This does not depend on the choice of f. Indeed, for another arrow $g : A \to F0$, there is $k : A \to F1$ such that $i \circ k = f, o \circ k = g$. Finally, the proof that L' preserves sifted colimits is essentially the same as the one of Proposition 3.2. $\square$

The proposition shows that as far as we are concerned with algebras over BA, we can assume any finitary functor to preserve sifted colimits.

4 Presenting Functors on Varieties

We show that sifted colimits preserving functors are precisely those that can be presented by finitary operations and equations in finite sets of variables.

4.1 Presentations of algebras and functors

An algebra A in a variety $\mathcal{A}$ with forgetful functor $U : \mathcal{A} \to \mathsf{Set}$ is said to be presented by generators G and relations $R \subseteq UFG \times UFG$ if A is the coequalizer

$$FR \underset{\pi_2^\sharp}{\overset{\pi_1^\sharp}{\rightrightarrows}} FG \xrightarrow{q} A. \tag{6}$$

where F is left-adjoint to U and $\pi_1^\sharp, \pi_2^\sharp$ come from the projections from R to UFG. Each algebra A is presented by its canonical presentation which has as set of generators UA and as relations the kernel of the counit $\varepsilon_A : FUA \to A$.

Following [10], we define analogously the notion of a functor $L : \mathcal{A} \to \mathcal{A}$ having a presentation by operations and equations. The generators GUA of LA are given by a Set-functor $GX = \coprod_{k<\omega} G_k \times X^k$. The elements of G_k are the k-ary operations. Relations are now induced by equations over finite sets V of variables, which are instantiated using valuations $v : V \to UA$. The point of (7) below is that it generalises (6) in such a way that the generators and relations defining LA depend uniformly on A.

Definition 4.1 ([10]). A finitary presentation by operations and equations of a functor is a pair $\langle G, E \rangle$ where $G : \mathsf{Set} \to \mathsf{Set}$, $GX = \coprod_{k<\omega} G_k \times X^k$ and $E = (E_V)_{V \in \omega}$, $E_V \subseteq (UFGUFV)^2$. The functor L presented by $\langle G, E \rangle$ is the joint coequalizer

$$FE_V \underset{\pi_2^\sharp}{\overset{\pi_1^\sharp}{\rightrightarrows}} FGUFV \xrightarrow{FGUv} FGUA \xrightarrow{q_A} LA \tag{7}$$

where V ranges over finite cardinals and v over morphisms (valuations of variables) $FV \to A$.

Example 4.2. A modal algebra, or Boolean algebra with operator (BAO), is the algebraic structure required to interpret (classical) modal logic which consists of propositional logic plus a unary modal operator $\Box$ preserving finite conjunctions. Modal algebras are therefore algebras for the functor $L : \mathsf{BA} \to \mathsf{BA}$, where LA is defined by generators $\Box a$, $a \in A$, and relations $\Box\top = \top$, $\Box(a \wedge a') = \Box a \wedge \Box a'$. That is, $GX = X$, $E_V = \emptyset$ for $V \neq 2$, $E_2 = \{\Box\top = \top, \Box(v_0 \wedge v_1) = \Box v_0 \wedge \Box v_1\}$.

Remark 4.3.
1. That the generators appear now as a functor expresses that the same generators (the $\Box$ in the example above) are used for all LA. Similarly, the coequalizer (7) is expressed using equations in variables V, that is, the same relations are used for all LA. In $E_V \subseteq (UFGUFV)^2$ the inner UF allows for the conjunction in $\Box(v_0 \wedge v_1)$ whereas the outer UF allows for the conjunction in $\Box v_0 \wedge \Box v_1$. Finally, note that relationship between the operator $\Box$ and the boolean operators can not be expressed by a distributive law between L and UF as L is not defined on sets but only on algebras.
2. In the works of [25, 19, 29] 'modal axioms of rank 1' play a prominent role. These are exactly those which, considered as equations, are of the form $E_V \subseteq (UFGUFV)^2$.

Before we come to the main consequence of the definition, let us point out the useful fact:

Proposition 4.4. *Consider a functor L on a variety $\mathcal{A}$. If L has a presentation then L preserves surjective morphisms and injective morphisms.*

The main point of the definition is that one obtains a presentation of $\mathsf{Alg}(L)$ from a presentation of $\mathcal{A}$ and from a presentation of L.

Theorem 4.5 ([10]). *Let $\mathcal{A} \cong \mathsf{Alg}(\Sigma_\mathcal{A}, E_\mathcal{A})$ be a variety and $\langle \Sigma_L, E_L \rangle$ a finitary presentation of $L : \mathcal{A} \to \mathcal{A}$. Then $\mathsf{Alg}(\Sigma_\mathcal{A} + \Sigma_L, E_\mathcal{A} + E_L)$ is isomorphic to $\mathsf{Alg}(L)$, where equations in $E_\mathcal{A}$ and E_L are understood as equations over $\Sigma_\mathcal{A} + \Sigma_L$.*

Remark 4.6.
1. The logical significance of theorem is that it ensures that the Lindenbaum algebra for the signature $\Sigma_\mathcal{A} + \Sigma_L$ and the equations $E_\mathcal{A} + E_L$ is the initial L-algebra.
2. The special format of the equations is needed to guarantee that, given a presentation of a functor $\langle \Sigma, E \rangle$, the algebras for the presented functor satisfy E.

4.2 The characterisation theorem

Before turning to functors on an arbitrary variety we have a look at functors on Set.

It is well known how to present a finitary endofunctor H on Set by operations and equations: Any such H is a quotient

$$\coprod_{k<\omega} Hk \times X^k \xrightarrow{\tau_X} HX \tag{8}$$

$$(\ \sigma\ ,\ l\ ) \ \mapsto\ Hl(\sigma) \tag{9}$$

where $k = \{0, \dots k-1\}$ and $l : k \to X$. $\sigma \in Hk$ is an k-ary operation and the quotient gives the action of σ on a list of arguments $l : k \to X$.

Now consider $L : \mathcal{A} \to \mathcal{A}$. The idea behind (8) can also be applied to the set-valued functor UL. Since (8) depends on considering arities k as objects in the domain of H, we replace k by the free algebras $Fk \in \mathcal{A}$. So Hk becomes $ULFk$ and $X^k = \mathsf{Set}(k, X)$ becomes $\mathcal{A}(Fk, A) \cong \mathsf{Set}(k, UA) = UA^k$, which leads us to consider

$$\coprod_{k<\omega} ULFk \times UA^k \xrightarrow{\varrho_A} ULA \tag{10}$$

$$(\ \sigma\ ,\ \ l\ ) \ \mapsto\ (UL\varepsilon_A \circ ULFl)(\sigma) \tag{11}$$

This gives us a signature with k-ary operations $ULFk$, which we describe by the functor

$$GY = \coprod_{k<\omega} ULFk \times Y^k. \tag{12}$$

To obtain the equations from (10), since sifted colimit preserving functors are determined by their action on free algebras, it is enough to consider the maps ϱ_{FV}. But the kernel of $\varrho_{FV} : GUFV \to ULFV$ determines only the set $ULFV$ and not the algebra LFV. The equations in variables V will therefore be given by the kernel of the adjoint transpose of ϱ

$$FGUFV \xrightarrow{\varrho^\sharp_{FV}} LFV.$$

Definition 4.7. Let L be an endofunctor on a variety $\mathcal{A}$. The finitary presentation $\langle G, E\rangle$ of L (Definition 4.1) is given by generators $G : \mathsf{Set} \to \mathsf{Set}$ as in (12) and equations $E = (E_V)_{V\in\omega}$ where E_V is the kernel $\pi_1, \pi_2 : E_V \longrightarrow UFGUFV$ of $U\varrho^\sharp_{FV}$, with ϱ as in (10).

We first show that the functor presented by $\langle G, E\rangle$ agrees with L on finitely generated free algebras Fn.

Lemma 4.8. *Consider $n \in \mathbb{N}$. Then*

$$FE_V \rightrightarrows FGUFV \xrightarrow{FGUv} FGUFn \xrightarrow{\varrho^\sharp_{Fn}} LFn \tag{13}$$

is a joint coequalizer.

Proof. We start by showing that

$$\begin{tikzcd} FE_n \arrow[r, shift left, "\pi_1^\sharp"] \arrow[r, shift right, "\pi_2^\sharp"'] & FGUFn \arrow[r, "\varrho^\sharp_{Fn}"] & LFn \end{tikzcd} \tag{14}$$

is a coequalizer. Let P be the kernel pair of $\varrho^\sharp_{Fn}$ in $\mathcal{A}$. $UP = E_n$. Moreover, the U-image of this kernel pair is a split coequalizer. From this, it follows that the U-image of (14) is split, hence (14) is a coequalizer in $\mathcal{A}$. Next we show that the joint coequalizer of (13) agrees with the coequalizer of (14). First, to see that all pairs in the kernel of $\varrho^\sharp_{Fn}$ are identified in the joint coequalizer it is enough to choose $V = n$ and $v = \mathrm{id}_V$. Conversely, by the definition of E_V, a pair (s,t) in $FGUFn$ is identified in the joint coequalizer only if there is V and $v : FV \to Fn$ such that (s,t) is the image under $UFGUv$ of a pair in the kernel of $\varrho^\sharp_{FV}$. So we have to show that the kernel of $\varrho^\sharp_{FV}$ is contained in the kernel of $\varrho^\sharp_{Fn} \circ FGUv$. But this follows from the commutativity of

$$\begin{tikzcd} FGUFn \arrow[r, "\varrho^\sharp_{Fn}"] & LFn \\ FGUFV \arrow[u, "FGUv"] \arrow[r, "\varrho^\sharp_{FV}"'] & LFV \arrow[u, "Lv"'] \end{tikzcd}$$

which is due to naturality of $\varrho^\sharp$. ☐

The characterisation theorem is now proved using that sifted colimits preserving functors are determined by their action on finitely generated free algebras.

Theorem 4.9. *An endofunctor on a variety has a finitary presentation by operations and equations if and only if it preserves sifted colimits.*

Proof. Suppose first that L has a presentation as in (7). Let $c_i : A_i \to A$ be a sifted colimit. We have to show that Lc_i is a sifted colimit. Given a cocone $d_i : LA_i \to L'$ we have to show that there is a unique k as depicted in

$$\begin{tikzcd} FE_V \arrow[r, shift left, "\pi_1^\sharp"] \arrow[r, shift right, "\pi_2^\sharp"'] & FGUFV \arrow[r] \arrow[dr, "FGUv^\sharp"'] & FGUA_i \arrow[r, "q_{A_i}"] \arrow[d, "FGUc_i"] & LA_i \arrow[d, "Lc_i"] \arrow[dr, "d_i"] & \\ & & FGUA \arrow[r, "q_A"'] \arrow[rr, bend right, "h"'] & LA \arrow[r, dotted, "k"'] & L' \end{tikzcd}$$

U preserves sifted colimits because $\mathcal{A}$ is a variety, G preserves sifted colimits because they commute with finite products, and F preserves all colimits. Therefore we have h with $d_i \circ q_{A_i} = h \circ FGUc_i$. Then k is obtained from the joint coequalizer q_A once we show that $h \circ FGUv^\sharp \circ \pi_1^\sharp = h \circ FGUv^\sharp \circ \pi_2^\sharp$ for all $v : V \to UA$. For this consider $v : V \to UA$. Since $\hom(V, -)$ preserves sifted colimits (V is finite) and U preserves sifted colimits, there is some A_j and some $w : V \to UA_j$ such that $v = Uc_j \circ v'$. It follows $v^\sharp = c_j \circ w^\sharp$, hence $FGUv^\sharp = FGUc_j \circ FGUw^\sharp$.

For the converse, let $\langle G, E \rangle$ be the presentation of a sifted colimits preserving functor L and denote by L' the functor presented by $\langle G, E \rangle$. It follows from Lemma 4.8 that $L'Fn \cong LFn$ on finitely generated free algebras Fn. We know from the first part of the proof that L' preserves sifted colimits. Hence L and L' are sifted colimits preserving functors which agree on finitely generated free algebras and, therefore, are isomorphic. $\square$

The theorem will give rise to finitary logics for functors in Sections 7 and 8. But let us note two immediate corollaries. First sifted colimit preserving functors preserve regular epis in general. Using that they have a presentation, one can also show:

Corollary 4.10. *A sifted colimit preserving functor L on a variety preserves monos.*

The import of this proposition is the following. If we build the Lindenbaum algebra of a logic using the initial algebra sequence $0 \to L0 \to L^2 0 \ldots$ then the corollary implies that all arrows in the sequence are injective. This means that logical equivalence of formulas of depth n can be decided at level n.

Using that sifted colimits preserving functors are closed under composition one immediately obtains:

Corollary 4.11. *Functor having a finitary presentation are closed under composition.*

5 Algebras on Ind and Pro Completions

As motivated in the introduction, we study the relation between two completions of a small category $\mathcal{C}$ with finite limits and colimits: the completion $(\hat{-}) : \mathcal{C} \to \mathsf{Ind}\mathcal{C}$ by filtered colimits and the completion $(\bar{-}) : \mathcal{C} \to \mathsf{Pro}\mathcal{C}$ by cofiltered limits. Consider

$$\begin{tikzcd} \mathsf{Ind}\mathcal{C} \arrow[rr, bend right=10, "\Sigma"'] & & \mathsf{Pro}\mathcal{C} \arrow[ll, bend right=10, "\Pi"'] \\ & \mathcal{C} \arrow[ul, "(\hat{-})"] \arrow[ur, "(\bar{-})"'] & \end{tikzcd} \tag{15}$$

where Σ is the left Kan-extension of $(\bar{-})$ along $(\hat{-})$, and Π is the right Kan-extension of $(\hat{-})$ along $(\bar{-})$. In particular we have

$$\Sigma\hat{C} = \bar{C} \qquad \Pi\bar{C} = \hat{C} \tag{16}$$

Example 5.1. 1. $\mathcal{C} = \mathsf{BA}_\omega$ (finite Boolean algebras = finitely presentable Boolean algebras), $\mathsf{Ind}\mathcal{C} = \mathsf{BA}$, $\mathsf{Pro}\mathcal{C} = \mathsf{Set}^{\mathrm{op}}$. ΣA is the set of ultrafilters over A and Π is (contravariant) powerset.

2. $\mathcal{C} = \mathsf{DL}_\omega$ (finite distributive lattices = finitely presentable distributive lattices), $\mathsf{Ind}\mathcal{C} = \mathsf{DL}$, $\mathsf{Pro}\mathcal{C} = \mathsf{Poset}^{\mathrm{op}}$. ΣA is the set of prime filters over A and Π gives the set of downsets.

Proposition 5.2. *Σ is left adjoint to Π.*

Proof. $\mathsf{Ind}\mathcal{C}$ is the subcategory of $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ of finite limit preserving functors, $(\hat{-})$ is the codomain restriction of the Yoneda embedding $\mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$, and $(\bar{-})$ is the codomain restriction of the dual of the Yoneda embedding $\mathcal{C}^{\mathrm{op}} \to \mathsf{Set}^{\mathcal{C}}$. We need a natural isomorphism

$$\frac{\Sigma A \to X}{A \to \Pi X}$$

Since Σ is filtered colimit preserving and Π cofiltered limit preserving, $\mathsf{Pro}\mathcal{C}$ is a (free) completion of $\mathcal{C}$ under cofiltered limits and $\mathsf{Ind}\mathcal{C}$ is a (free) cocompletion of $\mathcal{C}$ under filtered colimits, it suffices to have natural isomorphisms

$$\frac{\hom(C,-) \to \hom(D,-)}{\hom(-,C) \to \hom(-,D)}$$

(because $\hat{C} = \hom(-,C)$ and $\bar{C} = \hom(C,-)$). Since the first line is in $(\mathsf{Set}^{\mathcal{C}})^{\mathrm{op}}$, we need a natural isomorphism

$$\frac{\hom(D,-) \to \hom(C,-)}{\hom(-,C) \to \hom(-,D)}$$

where the both lines are in the corresponding functor categories. But this is evident. □

We want to present algebras over $\mathsf{Ind}\mathcal{C}$ by algebras over $\mathsf{Pro}\mathcal{C}$

$$\begin{tikzcd} \mathsf{Ind}\mathcal{C} \arrow[loop left, "H"] \arrow[r, bend right, "\Sigma"'] & \mathsf{Pro}\mathcal{C} \arrow[l, bend right, "\Pi"'] \arrow[loop right, "K"] \end{tikzcd} \tag{17}$$

where we assume that H and K agree on $\mathcal{C}$, that is,

$$\Sigma H\hat{C} = K\bar{C} \qquad \Pi K\bar{C} = H\hat{C} \tag{18}$$

The natural transformation $\delta : H\Pi \to \Pi K$. ΠX is a filtered colimit $\hat{C}_i \to \Pi X$. If H preserves filtered colimits we therefore obtain $H\Pi \to \Pi K$ as in

$$\begin{tikzcd} \Pi X & H\Pi X \arrow[r, "\delta_X"] & \Pi K X \\ \hat{C}_i \arrow[u, "c_i"] & H\hat{C}_i \arrow[u, "Hc_i"] \arrow[r, "="] & \Pi K\Sigma\hat{C}_i \arrow[u, "\Pi K c_i^\sharp"'] \end{tikzcd} \tag{19}$$

where $c_i^\sharp : \Sigma\hat{C}_i \to X$ is the transpose of $c_i : \hat{C}_i \to \Pi X$. δ allows us to lift Π to a functor

$$\begin{tikzcd} \mathsf{Alg}(H) & \mathsf{Alg}(K) \arrow[l, "\tilde{\Pi}"'] \end{tikzcd} \tag{20}$$

mapping a K-algebra (B,β) to the H-algebra $(\Pi B, \beta \circ \delta_B)$.

The transformation $h : K\Sigma \to \Sigma H$. h is defined as in the following diagram where the d_k are a filtered colimit and we assume that K weakly preserves filtered colimits.

$$\begin{tikzcd} A & K\Sigma A \arrow[r, "h_A"] & \Sigma H A \\ \hat{A}_k \arrow[u, "d_k"] & K\Sigma\hat{A}_k \arrow[u] \arrow[r, "="] & \Sigma H\hat{A}_k \arrow[u] \end{tikzcd} \tag{21}$$

h_A is not uniquely determined and we do not assume that it is natural. It allows to lift Σ to a map on objects

$$\begin{tikzcd} \mathsf{Alg}(H) \arrow[r, "\tilde{\Sigma}"'] & \mathsf{Alg}(K) \end{tikzcd} \tag{22}$$

Representing H-algebras as Π-images of K-algebras. Denote by ι the unit of the adjunction $\Sigma \dashv \Pi$. Our next theorem states that for all algebras $HA \to A$ the following diagram commutes

$$\begin{tikzcd}
A \arrow[d, "\iota_A"'] & & & HA \arrow[lll] \arrow[d, "H\iota_A"] \\
\Pi\Sigma A & \Pi\Sigma HA \arrow[l] & \Pi K\Sigma A \arrow[l, "\Pi h_A"'] & H\Pi\Sigma A \arrow[l, "\delta_{\Sigma A}"']
\end{tikzcd} \tag{23}$$

Theorem 5.3. *Assume in Diagram 17 that H preserves filtered colimits, that K weakly preserves filtered colimits, and that H and K agree on $\mathcal{C}$ (ie the equations (18) hold). Then for any H-algebra (A, α) it holds that $\iota_A : A \to \Pi\Sigma A$ is an H-algebra morphism $(A, \alpha) \to \tilde{\Pi}(\Sigma A, \alpha \circ h_A)$.*

Proof. To show that Diagram 23 commutes, we have to show that ι_A is an H-algebra morphism. Since ι is natural $\iota_{HA} = \Pi h_A \circ \delta_{\Sigma A} \circ H\iota_A$ does suffice, see the upper row of Diagram 24, where

$$\begin{tikzcd}
HA \arrow[r, "H\iota_A"'] \arrow[rrrrr, bend left, "\iota_{HA}"] & H\Pi\Sigma A \arrow[rr, "\delta_{\Sigma A}"'] & & \Pi K\Sigma A \arrow[rr, "\Pi h_A"'] & & \Pi\Sigma A \\
H\hat{A}_k \arrow[u, "Hd_k"] \arrow[rrrrr, bend right, "\iota_{H\hat{A}_k}"'] & H\hat{C}_i \arrow[u, "Hc_i"'] \arrow[r, "="] & \Pi K\Sigma\hat{C}_i \arrow[ur, "\Pi Kc_i^\sharp"'] & & \Pi K\Sigma\hat{A}_k \arrow[ul, "\Pi K\Sigma d_k"] \arrow[r, "="] & \Pi\Sigma H\hat{A}_k \arrow[u, "\Pi\Sigma Hd_k"']
\end{tikzcd} \tag{24}$$

$c_i : \hat{C}_i \to \Pi\Sigma A$ and $d_k : \hat{A}_k \to A$ are filtered colimits. The left-hand quadrangle is Diagram 19 and the right-hand quadrangle is the Π-image of Diagram 21. The outer quadrangle commutes, hence ι_{HA} is the unique arrow from the colimiting cocone Hd_k to the cocone $\Pi\Sigma Hd_k \circ \iota_{H\hat{A}_k}$. We have to show that $\Pi h_A \circ \delta_{\Sigma A} \circ H\iota_A$ is also an arrow between these cocones.

The d_k form the colimiting cocone for the diagram $\hat{(-)}{\downarrow}A$, the c_i for $\hat{(-)}{\downarrow}\Pi\Sigma A$. There is a functor $l : \hat{(-)}{\downarrow}A \to \hat{(-)}{\downarrow}\Pi\Sigma A$ taking $k : B \to A$ to $l(k) = \iota_A \circ k : B \to \Pi\Sigma A$. We have $\hat{C}_{l(k)} = A_k$ and $\iota_A \circ d_k = c_{l(k)}$. It follows that the $H\hat{A}_k = H\hat{C}_{l(k)}$ form a subdiagram of the $H\hat{C}_i$ and that any cocone over the $H\hat{C}_{l(k)}$ induces a cocone over the $H\hat{A}_k$; it is therefore enough to show that Diagram 25 commutes and that the lower line is $\iota_{H\hat{C}_{l(k)}}$. The diagram commutes since the triangle in the middle

$$\begin{tikzcd}
HA \arrow[r, "H\iota_A"'] & H\Pi\Sigma A \arrow[rr, "\delta_{\Sigma A}"'] & & \Pi K\Sigma A \arrow[rr, "\Pi h_A"'] & & \Pi\Sigma A \\
& H\hat{C}_{l(k)} \arrow[ul, "Hd_k"] \arrow[u, "Hc_{l(k)}"'] \arrow[r, "="] & \Pi K\Sigma\hat{C}_{l(k)} \arrow[ur, "\Pi Kc_{l(k)}^\sharp"'] \arrow[rr, dotted, no head, "\mathrm{id}"] & & \Pi K\Sigma\hat{A}_k \arrow[ul, "\Pi K\Sigma d_k"] \arrow[r, "="] & \Pi\Sigma H\hat{A}_k \arrow[u, "\Pi\Sigma Hd_k"']
\end{tikzcd} \tag{25}$$

commutes, which follows from

$$\begin{tikzcd}
\Pi K\Sigma\hat{C}_{l(k)} \arrow[r, "\Pi K\Sigma c_{l(k)}"] & \Pi K\Sigma\Pi\Sigma A \arrow[r, "\Pi K\eta\Sigma A"] & \Pi K\Sigma A \\
\Pi K\Sigma\hat{A}_k \arrow[u, "\mathrm{id}"] \arrow[r, "\Pi K\Sigma d_k"'] & \Pi K\Sigma A \arrow[u, "\Pi K\Sigma\iota_A"'] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

where the right-hand triangle is one of the triangle equalities of the adjunction given by $\iota : Id \to \Pi\Sigma$ and $\eta : Id \to \Sigma\Pi$ and the upper row is $\Pi K c^{\sharp}_{l(k)}$. Finally, using the other triangle equality and that the units are isos on finite objects, the lower line is $(\Pi\eta_{K\Sigma\hat{C}_{l(k)}})^{-1} = \iota_{\Pi K\Sigma\hat{C}_{l(k)}} = \iota_{H\hat{C}_{l(k)}}$. $\square$

This theorem will give rise to completeness of Set-coalgebras in Section 8. To illustrate the power of the theorem we derive some corollaries.

1. (Stone [30]) Choose $\mathcal{C} = \mathsf{BA}_\omega$, H and K to be the identity. Then $\mathsf{Ind}\mathcal{C}$, $\mathsf{Pro}\mathcal{C}$, Σ and Π are as in Example 5.1 and we obtain: Every boolean algebra A can be embedded into a powerset, with Boolean operations receiving their set-theoretic interpretation.

2. (Jónsson and Tarski [17]) $\mathcal{C} = \mathsf{BA}_\omega$ as above but take H to be the functor L from Example 4.2 and K to be powerset. With this data, our theorem states that every Boolean algebra with operators can be embedded into a complete Boolean algebra whose carrier is a powerset.

3. (Stone [31]) Choose $\mathcal{C} = \mathsf{DL}_\omega$. Then $\mathsf{Ind}\mathcal{C}$, $\mathsf{Pro}\mathcal{C}$, Σ and Π are as in Example 5.1 and we obtain: Every distributive lattice A can be embedded into the completely distributive lattice of subsets $\Pi\Sigma A$.

4. (Gehrke and Jónsson [11]) This is the generalisation of (2.) to distributive lattices. For H one takes the Vietoris functor of Johnstone [16], restricted to DL.

6 A Brief Review of Stone Duality

For most of the paper, we need from this subsection only the fact that there is a category Stone of topological spaces which is dually equivalent to BA. The reason for giving a more abstract account, is that we will occasionally mention distributive lattices and want to indicate possible extensions of our results.

To treat different Stone dualities simultaneously, one considers them as arising from the adjunction of topological spaces Top and frames Frm [15]. Frm captures the algebraic properties of a topology, namely, a frame is a distributive lattice with infinite joins that distribute over finite meets. There are contravariant functors

$$\begin{tikzcd}
\mathsf{Top} \arrow[r, bend left, "P"] & \mathsf{Frm} \arrow[l, bend left, "S"]
\end{tikzcd}$$

$P(X, \mathcal{O}) = \mathcal{O}$, $P(f) = f^{-1}$, and $S(A) = \mathsf{Frm}(A, \mathbb{2})$, where $\mathbb{2}$ is the two element frame (consisting of $\bot, \top$). $S(f) = \lambda s \in S(A) \,.\, s \circ f$. The functors P and S are adjoint on the right, that is, there is a bijection, natural in X and A,

$$\mathsf{Top}(X, SA) \cong \mathsf{Frm}(A, PX).$$

We are interested in subcategories of Top and Frm on which the adjunction restricts to a dual equivalence, or duality for short. $PX \cong A$ then means that A is an expressive and complete propositional theory for X in the following sense

- for $x \neq y \in X$ there is $a \in PX$ separating x and y,
- for $a \not\leq b \in A$ there is $x \in SA \cong X$ such that $x(a) = 1$ and $x(b) = 0$.

We read the second property as: if a does not imply b then there is a counter-example x. These two properties are ultimately responsible for our expressiveness and completeness results.

Two examples of relevant subcategories are the category Stone of Stone spaces, which is dual to BA. And the category Spec of spectral spaces (coherent spaces in [15]), which is dual to DL. For many more examples see Abramsky and Jung [2] and Johnstone [15].

7 Adequate Modal Logics for Coalgebras

In this section we show how we can associate to a functor T a modal logic that is adequate for T coalgebras in the sense that it is sound, complete, and characterises bisimilarity. We do this in two steps.

1. First, abstracting from syntax, we simply consider as propositions of the logic the elements of the initial L algebra, where L is the dual of T. We call this logic the abstract logic of T-coalgebras.
2. Second, we obtain a syntax and a proof system for the abstract logic from a presentation of the functor L. We call these logics the concrete logics of T-coalgebras.

The point of the separation is that the results we prove about the concrete logics do not depend on the chosen presentation and are conducted solely on the level of the abstract logics.

7.1 Abstract modal logics

Consider

$$\begin{tikzcd}
\mathsf{Coalg}(T) \arrow[r, shift left, "\tilde{P}"] \arrow[d] & \mathsf{Alg}(L) \arrow[l, shift left, "\tilde{S}"] \arrow[d] \\
\mathcal{X} \arrow[loop left, "T"] \arrow[r, bend left, "P"] \arrow[d] & \mathcal{A} \arrow[loop right, "L"] \arrow[l, bend left, "S"] \arrow[d, "U"] \\
\mathsf{Set} & \mathsf{Set}
\end{tikzcd} \tag{26}$$

where we assume that

- the dual equivalence between $\mathcal{X}$ and $\mathcal{A}$ arises from the adjunction of Top and Frm by restricting to subcategories $\mathcal{X}$ and $\mathcal{A}$,
- L is *dual* to T, that is, there is an isomorphism

$$\delta : LP \to PT$$

- $\mathcal{A}$ is a variety (see p.3).

δ allows us to extend the equivalence between $\mathcal{X}$ and $\mathcal{A}$ to an equivalence between $\mathsf{Coalg}(T)$ and $\mathsf{Alg}(L)$, where $\tilde{P}$ maps a coalgebra (X, ξ) to $(PX, P\xi \circ \delta_X)$. We can therefore consider $\mathsf{Alg}(L)$ as providing a logic for $\mathsf{Coalg}(T)$:

Definition 7.1. The algebra of propositions in variables V is the free L-algebra $\mathsf{Prop}(V)$ over V. Given a coalgebra (X, ξ), we write $[\![-]\!]_{(X,\xi,h)}$ for the morphism $\mathsf{Prop}(V) \to \tilde{P}(X, \xi)$ determined by the valuation $h : V \to UPX$. The semantics of a proposition φ is $[\![\varphi]\!]_{(X,\xi,h)} \subseteq X$.

We define

$$\mathsf{Coalg}(T) \models (\varphi \vdash \psi)$$

if $[\![\varphi]\!]_{(X,\xi,h)} \subseteq [\![\psi]\!]_{(X,\xi,h)}$ for all coalgebras and all valuations. For a collection Γ of 'sequents' $\varphi_i \vdash \psi_i$, we write $\Gamma \models (\varphi \vdash \psi)$ if $\mathsf{Coalg}(T) \models \Gamma \Rightarrow \mathsf{Coalg}(T) \models (\varphi \vdash \psi)$.

Remark 7.2. Because distributive lattices have no implication, the notation $(\varphi \vdash \psi)$ is needed to logically encode the order $\leq$ of the lattices. This is as in [1].

Proposition 7.3. *The logic for T-coalgebras given in the previous definition*

1. *respects bisimilarity: propositions are invariant under bisimilarity*
2. *is expressive: any two non-bisimilar states are distinguished by some proposition*
3. *$\Gamma \models (\varphi \vdash \psi)$ iff $\varphi \leq \psi$ in the quotient of $\mathsf{Prop}(V)$ wrt the equations $\{\varphi_i \wedge \psi_i = \varphi_i \mid (\varphi_i \vdash \psi_i) \in \Gamma\}$.*

The proposition is an immediate consequence of Stone duality.

7.2 Concrete logics

We restrict our attention now to the duality of BA and Stone. In particular, in Diagram 26, P maps a Stone space to its basis of clopens and S maps a Boolean algebra to the set of ultrafilters over A.

We will show that a modal logic enjoying the properties of Proposition 7.3 can be associated to any functor $T : \mathsf{Stone} \to \mathsf{Stone}$ that is determined by its action on finite Stone spaces.

Let $L = PTS$ be the dual of T. That T is determined by its action on finite Stone spaces means that L preserves filtered colimits. By Proposition 3.4, we can assume L to preserve sifted colimits. It follows from Theorem 4.9 that L has a presentation $\langle \Sigma_L, E_L \rangle$ and from Theorem 4.5 that the free L-algebras $\mathsf{Prop}(V)$ are the Lindenbaum algebras of the equational logic $(\Sigma_{\mathsf{BA}} + \Sigma_L, E_{\mathsf{BA}} + E_L)$.

To translate this equational logic into a modal logic is a standard procedure [9]. Each term in operation symbols from $\Sigma_{\mathsf{BA}} + \Sigma_L$ is considered as a formula. Equations $s = t$ are rendered as $s \leftrightarrow t$. Conversely, any formula s can be read as an equation $s = \top$ where $\top \in \Sigma_{\mathsf{BA}}$. To summarise:

Theorem 7.4. *Let $T : \mathsf{Stone} \to \mathsf{Stone}$ be a functor preserving cofiltered limits. Then T has a sound and strongly complete modal logic that characterises bisimilarity.*

Example 7.5. 1. Stone coalgebras for functors built according to

$$T \ ::= \ K \mid Id \mid T \times T \mid T + T \mid T^N \mid \mathcal{P}T$$

(K a constant, N a finite constant, $\mathcal{P}$ powerspace) were considered in [21]. All these functors preserve cofiltered limits and the above theorem summarises most aspects of that paper.

2. Given a Stone space $(X, \mathcal{O}X)$ define $\mathcal{H}(X, \mathcal{O}X) = (\mathcal{H}X, \mathcal{H}\mathcal{O}X)$ as $\mathcal{H}X = \{h \subseteq PX\}$. $\mathcal{H}\mathcal{O}X$ is generated by the sets $\Box a = \{h \in \mathcal{H}X \mid a \in h\}$. Then $\mathcal{H}$ is a functor on Stone. Its dual has a simple presentation: One unary operation $\Box$ and no equations.

3. Define $\mathcal{H}^{\uparrow}(X) = \{h \subseteq PX \mid h \text{ upward closed }\}$. Then $\mathcal{H}^{\uparrow}$ is a functor on Stone. Its dual is presented by a unary operator $\Box$ and an equation saying that $\Box$ is monotone. $\mathcal{H}^{\uparrow}$-coalgebras were studied in [12].

8 The Finitary Modal Logic of Set-Coalgebras

The aim of this section is to associate a strongly complete modal logic to suitable functors $T : \mathsf{Set} \to \mathsf{Set}$. As we are interested here in classical propositional logic the logic will be given by a functor $L_T : \mathsf{BA} \to \mathsf{BA}$. That is we are concerned with the following situation

$$\begin{tikzcd} \mathsf{BA} \arrow[loop left, "L_T"] \arrow[r, bend right, "S"'] & \mathsf{Set} \arrow[l, bend right, "P"'] \arrow[loop right, "T"] \end{tikzcd} \tag{27}$$

where S maps an algebra to the set of its ultrafilters and P is the contravariant powerset. Note that S and P take a meaning here that differs from the previous section.

Assuming that T preserves finite sets, we define L_T to be $L_T A = PTSA$ on finite BAs and then extend L_T continuously to all of BA. As L_T preserves filtered colimits, we can associate a modal logic to it, as explained in Section 7. This logic is sound and strongly complete for Stone-coalgebras for the dual $\bar{T}$ of L_T. Here, we show that strong completeness also holds wrt T-coalgebras.

Note that Diagram 27 is an instance of Diagram 17. From (19) we obtain a natural transformation $\delta : L_T P \to PT$ which in turn yields, as in (20), a functor $\tilde{P} : \mathsf{Coalg}(T) \to \mathsf{Alg}(L_T)$. $\tilde{P}$ now induces a semantics exactly as in Definition 7.1. But we cannot use Proposition 7.3 to prove completeness as we do not have a dual equivalence between BA and Set. We proceed as follows:

Suppose $\Gamma \not\vdash \varphi$. Let A be the free L_T algebra quotiented by Γ. By Theorem 5.3, there is a T-coalgebra on SA such that $\iota_A : A \to PSA$ is an L_T-algebra morphism. ι_A maps all propositions in Γ to all of SA, but φ only to a proper subset. Therefore there is an element in SA satisfying Γ and refuting φ. We have shown:

Theorem 8.1. *Let $T : \mathsf{Set} \to \mathsf{Set}$ preserve finite sets and weakly preserve cofiltered limits. Then T has a sound and strongly complete modal logic.*

Remark 8.2. 1. The weak preservation of cofiltered limits means, in particular, that all projections in the final sequence are onto. The only example of a functor we are aware of that does not satisfy this condition is the finite powerset functor, see [33]. And indeed, standard modal logic is strongly complete wrt Kripke frames, but not wrt finitely branching ones.

2. The probability distribution functor [32] does not preserve finite sets. And indeed, modal logics for probabilistic transition systems, see eg [13], are not strongly complete. Similarly for $TX = K \times X$ where K is an infinite constant.

3. In contrast, we can extend our result to functors $X \mapsto (TX)^K$ for infinite K if T preserves finite sets. Indeed, T^K is a cofiltered limit of the functors T^{K_i} where K_i ranges over the finite subsets of K. We can now apply the theorem to obtain logics $L_{T^{K_i}}$ and then extend the result to the colimit of the $L_{T^{K_i}}$ and the limit of the T^{K_i}. This allows us to include functors such as $(\mathcal{P}X)^K \cong \mathcal{P}(K \times X)$, K infinite (which give rise to labelled transition systems).

4. In [20] it was shown that one can have such a theorem if a suitable h as in (21) exists. Here we gave conditions under which this is indeed the case.

Example 8.3. 1. The functors built according to

$$T \ ::= \ N \mid \mathit{Id} \mid T \times T \mid T + T \mid T^K \mid \mathcal{P}T$$

(K a constant, N a finite constant, $\mathcal{P}$ powerset) were studied in [26, 14]. Their completeness results are extended here to strong completeness.

2. The double contravariant powerset functor $2^{2^{-}}$ does not preserve weak pullbacks [28] and therefore cannot be treated by Moss's coalgebraic logic [24]. But it does satisfy the assumptions of the theorem and $L_{2^{2^-}}$ has a particularly simple presentation: one unary operation symbol $\Box$ and no equations.

3. Similarly, but more importantly, the subfunctor UpX of 2^{2^X}, which takes as values upward closed sets of subsets, does not preserve weak pullbacks [12]. L_{Up} can be presented by one unary operator $\Box$ and one equation expressing that $\Box$ is monotone. Coalgebras for that functor are also known as monotone predicate transformers. They provide a natural semantics for logics of 2-player games, mentioned in the introduction.

9 Conclusion

Summary The purpose of the paper was to associate a finitary modal logic to a functor T, so that the logic is strongly complete wrt T-coalgebras. We took up the idea, well-established in domain theory [2], that a logic for the solution of a domain equation $X \cong TX$ is given by a presentation of the dual L of T. We characterised those functors on a variety that have a presentation (Theorem 4.9).

In a second move, we related two pairs of Stone dualities, one for the logic and one for the semantics. Distilling the essence of the algebraic completeness proof of modal logic via the Jónsson-Tarski Theorem, our second main contribution is Theorem 5.3 relating algebras on Ind and Pro-completions. It yields strong completeness for a large class of Set-functors, see Example 8.3.

One of the main aspects of this work is that it makes use of the notion of the presentation of a functor in order to separate syntax and semantics. The syntax is given by the presentation, the semantics in terms of natural transformations between functors. This led to a syntax-independent proof of Theorem 8.1.

An important point is that we do not need the assumption that T is finitary. This assumption is powerful when working with T-algebras, but it is much less so for T-coalgebras. Similarly, we do not need that T preserves weak pullbacks. Each of these assumptions would exclude fundamental examples.

Further, we find it important not to restrict our attention to Set-coalgebras. In all of domain theory, the systems are based on topological spaces. In fact, in any situation where one wants to incorporate a notion of admissible or observable subset, one is quickly led to a topological setting.

Future work Our approach can be extended to cover, on the semantic side, coalgebras over presheaves, and on the algebraic side, many-sorted algebras. This will allow us to obtain results about logics for name-passing calculi.

Can our characterisation theorem be extended to treat infinitary logics?

If $\mathsf{Alg}(L)$ is a variety, does $L : \mathcal{A} \to \mathcal{A}$ then preserve sifted colimits (converse of Theorem 3.1)? It is true for $\mathcal{A} = \mathsf{Set}$ but the proof in [8, III.4.9] does not generalise.

The reason for getting strong completeness is that in a Stone duality any algebra can be represented by a space (see eg Theorem 8.1). For completeness, it is enough if free algebras can be represented. An example for this situation is propositional logic with countable conjunctions where strong completeness fails. Algebraically, this means that only the free countably complete Boolean algebras can be represented as algebras of subsets [18]. It will be interesting to extend our approach to settings like these.

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