Positive Fragments of Coalgebraic Logics

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Abstract. Positive modal logic was introduced in an influential 1995 paper of Dunn as the positive fragment of standard modal logic. His completeness result consists of an axiomatization that derives all modal formulas that are valid on all Kripke frames and are built only from atomic propositions, conjunction, disjunction, box and diamond.

In this paper, we provide a coalgebraic analysis of this theorem, which not only gives a conceptual proof based on duality theory, but also generalizes Dunn's result from Kripke frames to coalgebras of weak-pullback preserving functors.

For possible application to fixed-point logics, it is note-worthy that the positive coalgebraic logic of a functor is given not by all predicate-liftings but by all monotone predicate liftings.

Keywords: coalgebraic logic, duality, positive modal logic

1 Introduction

Consider modal logic as given by atomic propositions, Boolean operations, and a unary box, together with its usual axiomatisation stating that box preserves finite meets. In [10], Dunn answered the question of an axiomatisation of the positive fragment of this logic, where the positive fragment is given by atomic propositions, lattice operations, and unary box and diamond.

Here we seek to generalize this result from Kripke frames to coalgebras for a weak pullback preserving functor. Whereas Dunn had no need to justify that the positive fragment actually *adds* a modal operator (the diamond), the general situation requires a conceptual clarification of this step. And, as it turns out, what looks innocent enough in the familiar case is at the heart of the general construction.

In the general case, we start with a functor $T : \mathsf{Set} \to \mathsf{Set}$. From T we can obtain by duality a functor $L : \mathsf{BA} \to \mathsf{BA}$ on the category BA of Boolean algebras, so that the free L-algebras are exactly the Lindenbaum algebras of the

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modal logic. We are going to take the functor L itself as the category theoretic counterpart of the corresponding modal logic, a move that is similar in spirit to the one which takes monads as counterpart of equational theories. How should we construct the positive T-logic? Dunn gives us a hint in that he notes that in the same way as standard modal logic is given by algebras over BA, positive modal logic is given by algebras over the category DL of (bounded) distributive lattices. It follows that the positive fragment of (the logic corresponding to) L should be a functor $L' : \mathsf{DL} \to \mathsf{DL}$ which, in turn, by duality, should arise from a functor $T' : \mathsf{Pos} \to \mathsf{Pos}$ on the category Pos of posets and monotone maps.

The centre-piece of our construction is now the observation that any (finitary) functor $T : \mathsf{Set} \to \mathsf{Set}$ has a canonical extension to a functor $T' : \mathsf{Pos} \to \mathsf{Pos}$. Theorem 4.10 then shows that this construction $T \mapsto T' \mapsto L'$ indeed gives the positive fragment of L and so generalizes Dunn's theorem.

An important observation about the positive fragment is the following: given any Boolean formula, we can rewrite it as a positive formula with negation only appearing on atomic propositions. In other words, the translation β from positive logic to Boolean logic given by

$$\beta(\Diamond\phi) = \neg\Box\neg\beta(\phi) \tag{1}$$
$$\beta(\Box\phi) = \Box\beta(\phi) \tag{2}$$

induces a bijection (on equivalence classes of formulas taken up to logical equivalence). More algebraically, we can formulate this as follows.

Given a Boolean algebra $B \in \mathsf{BA}$, let LB be the free Boolean algebra generated by $\{\Box b \mid b \in B\}$ modulo the axioms of modal logic. Given a distributive lattice A, let $L'A$ be the free distributive lattice generated by $\{\Box a : a \in A\} \cup \{\Diamond a \mid a \in A\}$ modulo the axioms of positive modal logic. Further, let us denote by $W : \mathsf{BA} \to \mathsf{DL}$ the forgetful functor. Then the above observation that every modal formula can be written, up to logical equivalence, as a positive modal formula with negations pushed to atoms, can be condensed into the statement that the (natural) distributive lattice homomorphism

$$\beta_B : L'WB \to WLB \tag{3}$$

induced by (1), (2) is an isomorphism.

Our main results are the following. If T' is an extension of T and L, L' are the induced logics, then $\beta : L'W \to WL$ exists. If, moreover, T' is the induced extension (posetification) of T and T preserves weak pullbacks, then β is an isomorphism. Furthermore, in the same way as the induced logic L can be seen as the logic of all predicate liftings of T, the induced logic L' is the logic of all *monotone* predicate liftings of T. These results depend crucially on the fact that the posetification T' of T arises from the inclusion $\mathsf{Set} \to \mathsf{Pos}$ being dense, a result which only holds if we move to enriched category theory. On the algebraic side the move to Pos-enriched colimits guarantees that the modal operators are monotone. Accordingly, and recalling [19, Theorem 4.7] stating that a functor $L' : \mathsf{DL} \to \mathsf{DL}$ preserves ordinary sifted colimits if and only if it has a presentation

by operations and equations, we show here that $L' : \mathsf{DL} \to \mathsf{DL}$ preserves *enriched* sifted colimits if and only if it has a presentation by *monotone* operations and equations. To see the relevance of a presentation result specific to monotone operations, observe that in the example of positive modal logic it is indeed the case that both $\Box$ and $\Diamond$ are monotone.

2   On coalgebras and coalgebraic logic

I.   Coalgebras. A Kripke model (W, R, v) with $R \subseteq W \times W$ and $v : W \to 2^{\mathsf{AtProp}}$ can also be described as a coalgebra $W \to \mathcal{P}W \times 2^{\mathsf{AtProp}}$, where $\mathcal{P}W$ stands for the powerset of W. This point of view suggests to generalize modal logic from Kripke frames to coalgebras

$$\xi : X \to TX$$

where T may now be any functor $T : \mathsf{Set} \to \mathsf{Set}$. We get back Kripke models by putting $TX = \mathcal{P}X \times 2^{\mathsf{AtProp}}$. We also get the so-called bounded morphisms or p-morphisms as coalgebras morphisms, that is, as maps $f : X \to X'$ such that $Tf \circ \xi = \xi' \circ f$.

II.   Coalgebras and algebras. More generally, for any category $\mathcal{C}$ and functor $T : \mathcal{C} \to \mathcal{C}$, we have the category $\mathsf{Coalg}(T)$ of T-coalgebras with objects and morphisms as above. Dually, $\mathsf{Alg}(T)$ is the category where the objects $TX \xrightarrow{\alpha} X$ are arrows in $\mathcal{C}$ and where the morphisms $f : (X, \alpha) \to (X', \alpha')$ are arrows $f : X \to X'$ in $\mathcal{C}$ such that $f \circ \alpha = \alpha' \circ Tf$. It is worth noting that T-coalgebras over $\mathcal{C}$ are dual to T^{op}-algebras over $\mathcal{C}^{op}$.

III.   Duality of Boolean algebras and sets. The abstract duality between algebras and coalgebras becomes particularly interesting if we put it on top of a concrete duality, such as the dual adjunction between the category Set of sets and functions and the category BA of Boolean algebras. We denote by $P : \mathsf{Set}^{op} \to \mathsf{BA}$ the functor taking powersets and by $S : \mathsf{BA} \to \mathsf{Set}^{op}$ the functor taking ultrafilters. Alternatively, we can describe these functors by $PX = \mathsf{Set}(X, 2)$ and $SA = \mathsf{BA}(A, 2)$, which also determines their action on arrows (here 2 denotes the two-element Boolean algebra). P and S are adjoint, satisfying $\mathsf{Set}(X, SA) \cong \mathsf{BA}(A, PX)$. Restricting P and S to finite Boolean algebras/sets, this adjunction becomes a dual equivalence.

IV.   Boolean logics for coalgebras, syntax. What now are logics for coalgebras? We follow a well-established methodology in modal logic ([6]) and study modal logics via the associated category of modal algebras. More formally, given a modal logic $\mathcal{L}$ extending Boolean propositional logic and with associated category $\mathcal{A}$ of modal algebras, we describe $\mathcal{L}$ by a functor

$$L : \mathsf{BA} \to \mathsf{BA}$$

so that the category $\mathsf{Alg}(L)$ of algebras for the functor L coincides with $\mathcal{A}$. In particular, the Lindenbaum algebra of $\mathcal{L}$ will be the initial L-algebra.

Example 2.1. Let $T = \mathcal{P}$ be the powerset functor and $L : \mathsf{BA} \to \mathsf{BA}$ be the functor mapping an algebra A to the algebra LA generated by $\Box a$, $a \in A$, and quotiented by the relation stipulating that $\Box$ preserves finite meets, that is,

$$\Box\top = \top \qquad \Box(a \wedge b) = \Box a \wedge \Box b \tag{4}$$

$\mathsf{Alg}(L)$ is the category of modal algebras (Boolean algebras with operators), a result which appears to be explicitly stated first in [1].

V.   Boolean logics for coalgebras, semantics. The semantics of such a logic is described by a natural transformation

$$\delta : LP \to PT^{op}$$

Intuitively, each modal operator in LPX is assigned its meaning as a subset of TX. More formally, δ allows us to lift $P : \mathsf{Set}^{op} \to \mathsf{BA}$ to a functor $P^\sharp : \mathsf{Coalg}(T) \to \mathsf{Alg}(L)$, and if we take a formula ϕ to be an element of the initial L-algebra (the Lindenbaum algebra of the logic), then the semantics of ϕ as a subset of a coalgebra (X, ξ) is given by the unique arrow from that initial algebra to $P^\sharp(X, \xi)$.

Example 2.2. We define the semantics $\delta_X : LPX \to P\mathcal{P}^{op}X$ by, for $a \in PX$,

$$\Box a \mapsto \{b \in \mathcal{P}X \mid b \subseteq a\}. \tag{5}$$

It is an old result in domain theory that δ_X is an isomorphism for finite X ([1]). This implies completeness of the axioms (4) with respect to Kripke semantics.

VI.   Functors having presentations by operations and equations. One might ask when a functor $L : \mathsf{BA} \to \mathsf{BA}$ can legitimately be considered to give rise to a modal logic. For us, in this paper, a minimal requirement on L is that $\mathsf{Alg}(L)$ is a variety in the sense of universal algebra, that is, that $\mathsf{Alg}(L)$ can be described by operations and equations, the operations then corresponding to modal operators and the equations to axioms. This happens if L is determined by its action on finitely generated free algebras (see [19]). These functors are also characterized as functors having presentations by operations and equations, or as functors preserving sifted colimits. Most succinctly, they are precisely those functors that arise as left Kan-extensions along the inclusion functor of the full subcategory of BA consisting of free algebras on finitely many generators.

VII.   The (finitary, Boolean) coalgebraic logic of a Set-functor. The general considerations laid out above suggest to define the finitary (Boolean) coalgebraic logic associated to a given functor $T : \mathsf{Set} \to \mathsf{Set}$ as

$$\mathbf{L}Fn = PT^{op}SFn \tag{6}$$

where Fn denotes the free Boolean algebra over n generators, for n ranging over natural numbers. The semantics $\boldsymbol{\delta}$ is given by observing that natural transformations $\boldsymbol{\delta} : \mathbf{L}P \to PT$ are in bijection with natural transformations

$$\hat{\boldsymbol{\delta}} : \mathbf{L} \to PT^{op}S \tag{7}$$

so that we can define $\hat{\boldsymbol{\delta}}$ to be the identity on finitely generated free algebras.

More explicitly, $\mathbf{L}A$ can be represented as the free BA over $\{\sigma(a_1,\ldots a_n) \mid \sigma \in PT^{op}SFn, a_i \in A, n < \omega\}$ modulo appropriate axioms, with $\boldsymbol{\delta}_X : \mathbf{L}PX \to PT^{op}X$ given by $\boldsymbol{\delta}\sigma(a_1,\ldots a_n) = PT^{op}(\hat{a})(\sigma)$ where $\hat{a} : X \to SFn$ is the adjoint transpose of $(a_1,\ldots a_n) : n \to UPX$, with the forgetful functor $U : \mathsf{BA} \to \mathsf{Set}$ being right adjoint of F. [4] Of course, in concrete examples one is often able to obtain much more succinct presentations:

Proposition 2.3. *With $T = \mathcal{P}$, the functor $\mathbf{L}$ defined by* (6) *is isomorphic to the functor L of Example 2.1.*

VIII. Positive coalgebraic logic. It is evident that, at least for some of the developments above, not only the functor T, but also the categories Set and BA can be considered parameters. Accordingly, one expects that positive coalgebraic logic takes place over the category DL of (bounded) distributive lattices which in turn, is part of an adjunction $P' : \mathsf{Pos}^{op} \to \mathsf{DL}$, taking upsets, and $S' : \mathsf{DL} \to \mathsf{Pos}^{op}$, taking prime filters, or, equivalently, $P'X = \mathsf{Pos}(X, \mathbb{2})$ and $S'A = \mathsf{DL}(A, \mathbb{2})$ where $\mathbb{2}$ is, as before, the two-chain (possibly considered as a distributive lattice). Consequently, the 'natural semantics' of positive logics is 'ordered Kripke frames'. That is, we may define a logic for T'-coalgebras, with $T' : \mathsf{Pos} \to \mathsf{Pos}$, to be given by a natural transformation

$$\boldsymbol{\delta}' : \mathbf{L}'P' \to P'T'^{op} \tag{8}$$

where

$$\mathbf{L}'F'n = P'T'^{op}S'F'n \tag{9}$$

is a functor determined by its action on finitely generated free distributive lattices and $\boldsymbol{\delta}'$ is given by its transpose in the same way as (7).

Example 2.4. Let T' be the convex powerset functor $\mathcal{P}'$ and $L' : \mathsf{DL} \to \mathsf{DL}$ be the functor mapping a distributive lattice A to the distributive lattice $L'A$ generated by $\Box a$ and $\Diamond a$ for all $a \in A$, and quotiented by the relations stipulating that $\Box$ preserves finite meets, $\Diamond$ preserves finite joins, and

$$\Box a \wedge \Diamond b \leq \Diamond(a \wedge b) \qquad \Box(a \vee b) \leq \Diamond a \vee \Box b \tag{10}$$

The natural transformation $\delta'_X : L'P'X \to P'\mathcal{P}'^{op}X$ is defined by, for $a \in P'X$,

$$\Diamond a \mapsto \{b \in \mathcal{P}X \mid b \cap a \neq \emptyset\}, \tag{11}$$

the clause for $\Box a$ being the same as in (5).

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[4] Since elements in $PTSFn$ are in one-to-one correspondence with natural transformations $\mathsf{Set}(-, 2^n) \to \mathsf{Set}(T-, 2)$, also known as predicate liftings [25], we see that the logic L coincides with the logic of all predicate liftings of [27], with the difference that L also incorporates axioms. The axioms are important to us as otherwise the natural transformation β mentioned in the introduction might not exist.

Remark 2.5. $\mathsf{Alg}(L')$ is the category of positive modal algebras of Dunn [10] and we will show that it is isomorphic to $\mathsf{Alg}(\mathbf{L}')$ in Corollary 3.6. Again we have that for finite X, δ'_X is an isomorphism, a representation first stated in [12,13], the connection with modal logic being given by [30,26,1] and investigated from a coalgebraic point of view in [24].

3   On **Pos** and **Pos**-enriched categories

I.   The category **Pos** *of posets and monotone maps.* **Pos** is complete and co-complete (even locally finitely presentable [3]), limits being computed as in **Set**, while for colimits one has to quotient the corresponding colimits obtained in the category of preordered sets and monotone maps (however, directed colimits are computed as in **Set**, see [3]). **Pos** is also cartesian closed, with the internal hom $[X, Y]$ being the poset of monotone maps from X to Y, ordered pointwise.

This paper will consider categories *enriched* in **Pos** because this automatically takes care of the algebraic operations being monotone. Therefore when we say category, functor, natural transformation in what follows, we always mean the enriched concept. Thus a category has ordered homsets and functors are locally monotone, that is, they preserve the order on the homsets.

When we want to deal with non-enriched concepts, we always call them *ordinary*. Thus, for example, the category **Pos** has its underlying ordinary category Pos_o. Everything below with the subscript o is the underlying ordinary thing of the **Pos**-enriched thing.

In particular, we consider **Set** as discretely enriched over **Pos**. Then $D :$ **Set** $\to$ **Pos**, the discrete functor, is trivially **Pos**-enriched. There are two more **Pos**-categories appearing in this paper, namely **BA** and **DL**. The first one is considered discretely enriched, while in **DL** the enrichment is a consequence of the natural order induced by operations.

II.   Sifted weights and sifted (co)limits. The theory of (locally monotone) **Pos**-functors and their logics of monotone modal operators naturally leads to the world of ordered varieties. Since the details are only needed for the proofs (which had to be omitted for reasons of space) we note here only that our arguments are based on [21,8,22,17].

In the non-enriched setting, a functor on a variety preserves ordinary sifted colimits iff it preserves filtered colimits and reflexive coequalizers. In the **Pos**-enriched setting, a functor on an ordered variety preserves (enriched) sifted colimits iff it preserves filtered colimits and reflexive coinserters. We recall that the coinserter ([16]) of a parallel pair of arrows $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in a **Pos**-category con-sists of an object $\mathsf{coins}(f, g)$ and of an arrow $\pi : Y \to \mathsf{coins}(f, g)$ with $\pi f \leq \pi g$, with the following universal property: for any $q : Y \to Z$ with $qf \leq qg$, there is a unique $h : \mathsf{coins}(f, g) \to Z$ with $h\pi = q$. Moreover, this assignment is monotone, in the sense that given $q, q' : Y \to Z$ with $q \leq q'$, $qf \leq qg$ and $q'f \leq q'g$, the

corresponding unique arrows $h, h' : \mathsf{coins}(f, g) \to Z$ satisfy $h \leq h'$. The coinserter is called reflexive if f and g have a common right inverse. By switching the arrows, one obtain the dual notion of a (coreflexive) inserter.

III. Functors preserving sifted colimits and their equational presentation. Denote by Set_f the category of finite sets and maps and by ι the composite $\mathsf{Set}_f \hookrightarrow \mathsf{Set} \xrightarrow{D} \mathsf{Pos}$. Then Pos is the free cocompletion of Set_f under (enriched) sifted colimits [21].

A functor $\mathcal{T} : \mathsf{Pos} \to \mathsf{Pos}$ is called *strongly finitary* if one of the equivalent conditions below holds: (i) $\mathcal{T}$ is isomorphic to the left Kan extension along ι of its restriction, that is $\mathcal{T} \cong \mathsf{Lan}_\iota(\mathcal{T}\iota)$; (ii) $\mathcal{T}$ preserves sifted colimits.

Recall that there are monadic (enriched) adjunctions $F \dashv U : \mathsf{BA} \to \mathsf{Set}$, $F' \dashv U' : \mathsf{DL} \to \mathsf{Pos}$, where U and U' are the corresponding forgetful functors. We denote by $\mathbf{J} : \mathsf{BA}_{\mathit{ff}} \to \mathsf{BA}$ and $\mathbf{J}' : \mathsf{DL}_{\mathit{ff}} \to \mathsf{DL}$ the inclusion functors of the full subcategories spanned by the algebras which are free on finite (discrete po)sets.

Lemma 3.1. $\mathbf{J}$ *and* $\mathbf{J}'$ *exhibit* BA*, respectively* DL*, as the free cocompletions under sifted colimits of* $\mathsf{BA}_{\mathit{ff}}$ *and* $\mathsf{DL}_{\mathit{ff}}$*. In particular, these functors are dense.*

Corollary 3.2. *A functor* $L : \mathsf{BA} \to \mathsf{BA}$ *has the form* $\mathsf{Lan}_{\mathbf{J}}(L\mathbf{J})$ *iff it preserves (ordinary) sifted colimits. A functor* $L' : \mathsf{DL} \to \mathsf{DL}$ *has the form* $\mathsf{Lan}_{\mathbf{J}'}(L'\mathbf{J}')$ *iff it preserves sifted colimits.*

Theorem 3.3. *Suppose* $L : \mathsf{BA} \to \mathsf{BA}$ *and* $L' : \mathsf{DL} \to \mathsf{DL}$ *preserve sifted colimits. Then they both have an equational presentation.*

Remark 3.4. The (proof of the) above theorem actually shows that every functor $L' : \mathsf{DL}_{\mathit{ff}} \to \mathsf{DL}$ (i.e., every L' preserving sifted colimits) has a presentation in the form of a coequalizer

$$\widehat{H_\Gamma} \rightrightarrows \widehat{H_\Sigma} \longrightarrow L'$$

for some strongly finitary signatures Γ and Σ, i.e. some locally monotone functors $\Gamma, \Sigma : |\mathsf{Set}_f| \to \mathsf{Pos}$, where $|\mathsf{Set}_f|$ is the skeleton of the category of finite sets. Here, $\widehat{H_\Sigma}$ is defined as follows: given $\Sigma : |\mathsf{Set}_f| \to \mathsf{Pos}$, $H_\Sigma : \mathsf{Set}_f \to \mathsf{Pos}$ is the *polynomial strongly finitary functor*

$$H_\Sigma n = \coprod_{k \in |\mathsf{Set}_f|} \mathsf{Set}_f(k, n) \bullet \Sigma k$$

and it extends to a strongly finitary $H_\Sigma : \mathsf{Pos} \to \mathsf{Pos}$ by sifted colimits. In the above formula, $\coprod$ and $\bullet$ refers to the coproduct, respectively copower in the category Pos. The resulting $\widehat{H_\Sigma} : \mathsf{DL}_{\mathit{ff}} \to \mathsf{DL}$ is thus given, at a free distributive lattice with finite discrete set of generators, by

$$\widehat{H_\Sigma}(F'Dn) = F'H_\Sigma U'(F'Dn)$$

(see Remark 3.16 of [28]) and, again, it is extended to an endofunctor on DL by means of sifted colimits.

We say that a functor $\mathsf{DL} \to \mathsf{DL}$ has a presentation by monotone operations and equations if it has a presentation by operations and equations in the sense of [7], such that, moreover, all operations are monotone. We then obtain the following enriched version of [19, Theorem 4.7], characterizing *enriched* sifted colimits preserving functors in terms of presentations with *monotone* operations.

Corollary 3.5. *A functor $L' : \mathsf{DL} \to \mathsf{DL}$ has a presentation by monotone operations and equations if and only if L' is the Pos-enriched left Kan extension of its restriction to finitely generated free distributive lattices.*

As in Proposition 2.3, we now obtain that

Corollary 3.6. *If T' is the the convex powerset functor, then the functor L' of Example 2.4 is isomorphic to the sifted colimits preserving functor $\mathbf{L}'$ whose restriction to DL_{ff} is $P'T'^{op}S'$ as in (8).*

IV.   The Pos-extension of a Set-functor. In order to relate Set and Pos-functors, we recall from [4] the following

Definition 3.7. *Let T be an endofunctor on Set. A Pos-endofunctor T' is said to be a Pos-extension of T if it is locally monotone and if the square*

$$
\begin{array}{ccc}
\mathsf{Pos} & \xrightarrow{\;T'\;} & \mathsf{Pos} \\
{\scriptstyle D}\big\uparrow & \nwarrow\!{\scriptstyle \alpha} & \big\uparrow{\scriptstyle D} \\
\mathsf{Set} & \xrightarrow[\;T\;]{} & \mathsf{Set}
\end{array}
\tag{12}
$$

commutes up to an isomorphism $\alpha : DT \to T'D$.

   A Pos-extension T' is called the posetification of T if the above square exhibits T' as $\mathsf{Lan}_D DT$ (in the Pos-enriched sense), having α as its unit.

If T is finitary, then its posetification does exist. This can be seen by expressing $\mathsf{Lan}_D(DT)$ as a coend

$$
\mathsf{Lan}_D(DT)X = \int^{S \in \mathsf{Set}} [DS, X] \bullet DTS
\tag{13}
$$

and taking into account that T is determined by its action on finite sets: explicitly, the coend becomes

$$
\mathsf{Lan}_D(DT)X = \int^{n \in \mathsf{Set}_f} [Dn, X] \bullet DTn
\tag{14}
$$

which in turn is the following Pos-coequalizer

$$
\coprod_{m,n<\omega} \mathsf{Set}(m,n) \times Tm \times [Dn, X] \rightrightarrows \coprod_{n<\omega} Tn \times [Dn, X] \xrightarrow{\pi} \mathsf{Lan}_D(DT)X
\tag{15}
$$

In formulae above, $[-,-]$ denotes the internal hom in Pos and $\times$ refers to the (cartesian) product of posets, ordered component-wise.

Posetifications of (finitary) Set-functors are immediate examples of strongly finitary Pos-functors. Briefly, one can say that a Pos-functor is a posetification if it has a presentation by monotone operations and discrete arities. In fact, we can be much more precise: a functor $T' : \mathsf{Pos} \to \mathsf{Pos}$ is the posetification of a finitary Set-functor if it is strongly finitary and preserves discrete sets.

Example 3.8. 1. Let $T = \mathrm{Id}$ on Set. Then the discrete connected components functor and the upper-sets-functor are both extensions of T, while $\mathrm{Id} : \mathsf{Pos} \to \mathsf{Pos}$ is the posetification (recall that the discrete functor D is dense, see [9]).
2. If we take $T = \mathcal{P}_f$ to be the (finite) power-set functor, then its posetification is the (finitely generated) convex power-set functor, with the Egli-Milner order ([4,28]).
3. The collection of (finitary) Kripke polynomial Set-functors is inductively defined as follows: $T ::= \mathrm{Id} \mid T_{X_0} \mid T_0 + T_1 \mid T_0 \times T_1 \mid T^A \mid \mathcal{P}_f$, where T_{X_0} denotes the constant functor to the set X_0; $T_0 + T_1$ is the coproduct functor $X \mapsto T_0 X + T_1 X$; $T_0 \times T_1$ the product functor; and T^A denotes the exponent functor $X \mapsto (TX)^A$, with A finite.
   We have just seen above that the posetification of the identity functor is again the identity, while for the constant functor T_{X_0} is an easy exercise to check that the posetification is again a constant functor, this time to the discrete poset DX_0; the posetification of the coproduct functor $T_0 + T_1$ maps a poset X to the coproduct (in the category of posets) $T_0' X + T_1' X$, where T_0' and T_1' denote the posetifications of T_0, respectively T_1; and similarly for the product and exponent functors.
4. The finitary distribution functor is given by $\mathcal{D}X = \{d : X \to [0,1] \mid \sum_{x \in X} d(x) = 1,\ \mathsf{supp}(d) < \infty\}$, where $\mathsf{supp}(d) = \{x \in X \mid d(x) \neq 0\}$. For function $f : X \to Y$, we have $\mathcal{D}(f)(d)(y) = \sum y = f(x)d(x)$. Recall that $\mathcal{D}$ preserves weak pullbacks ([29]), thus the posetification $\mathcal{D}'$ can be described using the relation lifting ([4]). Explicitly, for a poset $(X, \leq)$, $\mathcal{D}'(X, \leq)$ has underlying set $\mathcal{D}X$, where for $d, d' \in \mathcal{D}$, the partial order reads $d \leq d'$ iff there is some $\omega \in \mathcal{D}(X \times X)$ such that $\omega(x,y) > 0 \Rightarrow x \leq y$, $\sum_{y \in X} \omega(x,y) = d(x)$ and $\sum_{x \in X} \omega(x,y) = d'(y)$.

V. Morphisms of logical connections. We recall the (enriched) logical connections (dual adjunctions, see [20]) between sets and Boolean algebras, and between posets and distributive lattices. Both should be seen as Pos-enriched, where for the first logical connection the enrichment is discrete. They are related as follows:

$$
\begin{array}{ccc}
\mathsf{Set}^{op} & \underset{P}{\overset{S}{\underset{\perp}{\rightleftarrows}}} & \mathsf{BA} \\
{\scriptstyle D^{op}}\big\downarrow & & \big\downarrow{\scriptstyle W} \\
\mathsf{Pos}^{op} & \underset{P'}{\overset{S'}{\underset{\perp}{\rightleftarrows}}} & \mathsf{DL}
\end{array}
\tag{16}
$$

In the top row of the above diagram, recall again that P is the contravariant powerset functor, while S maps a Boolean algebra to its set of ultrafilters. The bottom row has P' mapping a poset to the distributive lattice of its upper-sets, and S' associating to each distributive lattice the poset of its prime filters. About the pair of functors connecting the two logical connections: D was introduced earlier as the discrete functor, while W is the functor associating to each Boolean algebra its underlying distributive lattice.

It is easy to see that the pair (D^{op}, W) is a *morphism of adjunctions* in the sense of [23]. This means that the equalities

$$P'D^{op} = WP, \quad D^{op}S = S'W, \quad \epsilon'D^{op} = D^{op}\epsilon \tag{17}$$

hold, where ϵ and ϵ' are the counits of $S \dashv P$ and $S' \dashv P'$, respectively.

4   Positive coalgebraic logic

We now expand the propositional logics BA and DL by modal operators. We start with a Set-endofunctor T in the top left-hand corner of (16). We are mostly interested in the case where $T' : \mathsf{Pos} \to \mathsf{Pos}$ is the posetification of T (Definition 3.7) and $L : \mathsf{BA} \to \mathsf{BA}$ and $L' : \mathsf{DL} \to \mathsf{DL}$ are (the functors of) the associated logics as in (6) and (9), in which case we denote the logics by boldface letters $\mathbf{L}$ and $\mathbf{L}'$.

But some of the following holds under the weaker assumptions that T' is some extension of T and that L and L' are some logics for, respectively, T and T'. We therefore let T be a Set-endofunctor and T' be an extension of T to Pos as in (12). Logics for T, T' are given by functors $L : \mathsf{BA} \to \mathsf{BA}$ and $L' : \mathsf{DL} \to \mathsf{DL}$ and natural transformations

$$\delta : LP \to PT^{op} \qquad \delta' : L'P' \to P'T'^{op}.$$

Intuitively, δ and δ' assign to the syntax given by (presentations of) L and L' the corresponding semantics in subsets or upper sets. To compare L and L' we need the isomorphism $\alpha : DT \to T'D$ saying that T' extends T, and also the relation $WP = P'D$ from (17) (which formalizes the trivial observation that taking upsets of a discrete set is the same as taking all subsets). Referring back to the introduction, we now make the following

Definition 4.1. *We say that a logic (L', δ') for T' is* a positive fragment of *the logic (L, δ) for T, if there is a natural transformation $\beta : L'W \to WL$ with $W\delta \circ \beta P = P'\alpha^{op} \circ \delta'D^{op}$, or, in diagrams*
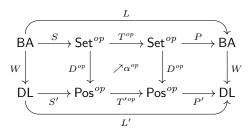
$$\begin{array}{ccc}
\mathsf{Set}^{op} \xrightarrow{P} \mathsf{BA} \xrightarrow{W} \mathsf{DL} & & \mathsf{Set}^{op} \xrightarrow{D^{op}} \mathsf{Pos}^{op} \xrightarrow{P'} \mathsf{DL} \\
\downarrow{\scriptstyle T^{op}} \quad {\scriptstyle \swarrow \delta} \; {\scriptstyle L} \; {\scriptstyle \swarrow \beta} \; \downarrow{\scriptstyle L'} & = & \downarrow{\scriptstyle T^{op}} \quad {\scriptstyle \swarrow \alpha^{op}T'^{op}} \; {\scriptstyle \swarrow \delta'} \; \downarrow{\scriptstyle L'} \\
\mathsf{Set}^{op} \xrightarrow{P} \mathsf{BA} \xrightarrow{W} \mathsf{DL} & & \mathsf{Set}^{op} \xrightarrow{D^{op}} \mathsf{Pos}^{op} \xrightarrow{P'} \mathsf{DL}
\end{array} \tag{18}$$

We call (L', δ') the (maximal) positive fragment *of (L, δ) if β is an isomorphism.*

Recall that we defined the logics $\mathbf{L}, \mathbf{L}'$ induced by T and an extension T' as $\mathbf{L} = PTS$ and $\mathbf{L}' = P'T'^{op}S'$ on finitely generated free objects. Our desired result is to prove that a certain canonically given $\beta : \mathbf{L}'W \to W\mathbf{L}$ is an isomorphism. The difficulty, as well as the need for the proviso that T preserves weak pullbacks, stems from the fact that in DL (as opposed to BA) the class of functors determined on finitely generated free algebras is strictly smaller than the class of functors determined on finitely presentable (=finite) algebras. As stepping stones, therefore, we first investigate what happens in the cases where the functors L, L' are determined on all algebras and on finitely presentable algebras, before we turn to the situation of functors determined on strongly finitely presentable (=finitely generated free) algebras.

I.   The case of $L' = P'T'^{op}S'$ on all algebras. We shall associate to any extension $\alpha : DT \to T'D$ the pairs (L, δ) and (L', δ') corresponding to T and T' respectively, with $L = PT^{op}S$ and $\delta = PT^{op}\epsilon : PT^{op}SP \to PT^{op}$, $L' = P'T'^{op}S'$ and δ' being defined analogously. Now the following is a consequence of (D^{op}, W) being a morphism of adjunctions (see (17)). We then immediately obtain an isomorphism β:

Proposition 4.2. *Given an extension $\alpha : DT \to T'D$, the isomorphism*

$$
\begin{array}{ccccccc}
 & & & L & & & \\
 & \overbrace{} & & & & & \\
\mathsf{BA} & \xrightarrow{\ S\ } & \mathsf{Set}^{op} & \xrightarrow{\ T^{op}\ } & \mathsf{Set}^{op} & \xrightarrow{\ P\ } & \mathsf{BA} \\
\ \downarrow{\scriptstyle W} & & \ \downarrow{\scriptstyle D^{op}} & {\scriptstyle \nearrow \alpha^{op}} & \ \downarrow{\scriptstyle D^{op}} & & \ \downarrow{\scriptstyle W} \\
\mathsf{DL} & \xrightarrow{\ S'\ } & \mathsf{Pos}^{op} & \xrightarrow{\ T'^{op}\ } & \mathsf{Pos}^{op} & \xrightarrow{\ P'\ } & \mathsf{DL} \\
 & \underbrace{} & & & & & \\
 & & & L' & & &
\end{array}
$$

exhibits $L' = P'T'^{op}S'$ as the maximal positive fragment of $L = PT^{op}S$.

II.   The case of $\bar{L}' = P'T'^{op}S'$ on finitely presentable algebras. A similar result holds if we define logics via $PT^{op}SA$ for finitely presentable A, as we are going to show now. To this end, we use the subscript $(-)_f$ to denote the restriction to finite[5] objects as e. g. when writing the dense inclusions $I : \mathsf{Set}_f \to \mathsf{Set}$, $I' : \mathsf{Pos}_f \to \mathsf{Pos}$, $J : \mathsf{BA}_f \to \mathsf{BA}$ and $J' : \mathsf{DL}_f \to \mathsf{DL}$. Note that we have the following commuting diagram

$$
\begin{array}{ccc}
S_f \dashv P_f & \xrightarrow{\ (D_f^{op}, W_f)\ } & S_f' \dashv P_f' \\
{\scriptstyle (I^{op}, J)}\downarrow & & \downarrow{\scriptstyle (I'^{op}, J')} \\
S \dashv P & \xrightarrow[\ (D^{op}, W)\ ]{} & S' \dashv P'
\end{array}
\tag{19}
$$

---

[5] As Pos is locally finitely presentable as closed category, and ordinary categories $\mathsf{Set}_o, \mathsf{DL}_o, \mathsf{BA}_o$ are also locally finitely presentable, it follows that the finitely presentable objects in all the above categories are precisely the same as in the ordinary case, i.e. the ones for which the underlying set is finite.

in the category of transformations of adjoints.

Define $(\bar{L}, \bar{\delta})$ for T as $\bar{L} = \mathsf{Lan}_J(PT^{op}SJ)$ and $\bar{\delta} = \bar{L}P \to PT^{op}$ as the adjoint transpose of $\bar{L} \to PT^{op}S$ arising from the universal property of the left Kan extension $\bar{L}$. By construction, $\bar{L}$ is finitary and is given by $PT^{op}S$ on finite(ly presentable) Boolean algebras. Similarly, obtain $(\bar{L}', \bar{\delta}')$ for T'.

Since W is left adjoint,[6] $\mathsf{Lan}_J(PT^{op}SJ)$ is preserved by W. Thus, to define an (iso)morphism $\bar{\beta} : \bar{L}'W = \mathsf{Lan}_{J'}(P'T'^{op}S'J')W \to W\bar{L} = \mathsf{Lan}_J(PT^{op}SJ)W$, it suffices to take the restriction along J of the isomorphism of Proposition 4.2, namely $\bar{\beta}_f : L'J'W_f = P'T'^{op}S'J'W_f \cong WPT^{op}SJ$,

Recall the definition of $\mathbf{L}$ from (6). Since every finitely presentable non-trivial Boolean algebra is a retract of a finitely generated free algebra, we can take $\mathbf{L} = \bar{L}$ (see eg [19, Prop 3.4]). To summarize, we have

Proposition 4.3. *The isomorphism $\bar{\beta}$ exhibits $\bar{L}' = P'T'^{op}S'J'$ as the maximal positive fragment of $(\mathbf{L}, \delta)$.*

The proposition does not yet give us the desired result, as $\bar{L}'$ is not necessarily determined by its action on finitely generated free algebras and, therefore, need not give rise to a variety of modal algebras. Paragraph III. will investigate when $\bar{L}'$ does actually have this additional property.

III.   The case of $\mathbf{L}' = P'T'^{op}S'$ on finitely generated free algebras. Recall that we denoted by $\mathbf{J} : \mathsf{BA}_{ff} \to \mathsf{BA}$ and $\mathbf{J}' : \mathsf{DL}_{ff} \to \mathsf{DL}$ the inclusion functors of the full subcategories spanned by the algebras which are free on finite discrete posets.

Definition 4.4. *Let T' be a Pos-endofunctor. We define the logic for T' to be the pair $(\mathbf{L}', \boldsymbol{\delta}')$, where:*

- $\mathbf{L}' : \mathsf{DL} \to \mathsf{DL}$ *is a Pos-functor preserving sifted colimits, whose restriction to free finitely generated distributive lattices is $P'T'^{op}S'\mathbf{J}'$, that is, $\mathbf{L}' = \mathsf{Lan}_{\mathbf{J}'}(P'T'^{op}S'\mathbf{J}')$.*
- $\boldsymbol{\delta}' : \mathbf{L}'P' \to P'T'^{op}$ *is the adjoint transpose of $\mathbf{L}' \to P'T'^{op}S'$ given by the universal property of the left Kan extension $\mathbf{L}'$.*

Remark 4.5. By the above definition, $\mathbf{L}'$ preserves sifted colimits. Thus, by Corollary 3.5, $\mathbf{L}'$ has an equational presentation by monotone operations, which in turn gives rise to a positive modal logic concretely given in terms of modal operators and axioms.

Recall that $\bar{L}' = P'T'^{op}S'$ on finitely presentable (=finite) algebras and that $\mathbf{L}' = P'T'^{op}S'$ on finitely generated free algebras.

Theorem 4.6. *Let T be a Set-endofunctor and T' a Pos-extension of T which preserves coreflexive inserters. Then $(\bar{L}', \bar{\delta}')$ and $(\mathbf{L}', \boldsymbol{\delta}')$ coincide. In particular, it follows that $\mathbf{L}'$ is the maximal positive fragment of $\mathbf{L}$.*

---

[6] The (enriched) right adjoint of W sends a distributive lattice A to the Boolean algebra of complemented elements in A (also known as the center of A).

Remark 4.7. The isomorphism $(\bar{L}, \bar{\delta}) \cong (\mathbf{L}, \boldsymbol{\delta})$ of the corresponding Boolean logic for Set-functors was established in [19]. (Recall that $\mathbf{L}$ was introduced in (6), while $\bar{L}$ appeared in Paragraph II. above.)

Proposition 4.8. *If T' is a* Pos*-endofunctor (thus locally monotone) which preserves exact squares, then it preserves embeddings and coreflexive inserters.*

The reader should think of an exact square as being the Pos-enriched analogue of a weak pullback (see [11], [5] or [4] for the precise definition).

Proposition 4.9 ([4]). *Let T be any finitary* Set*-functor and T' its posetification. Then T preserves weak pullbacks if and only if T' preserves exact squares.*

As a consequence of all the results of this section, we obtain

Theorem 4.10. *Let $T :$ Set $\to$ Set be a finitary weak-pullback preserving functor and $T' :$ Pos $\to$ Pos its posetification. Let $(\mathbf{L}, \boldsymbol{\delta})$ and $(\mathbf{L}', \boldsymbol{\delta}')$ be the associated logics of T and T', that is $\mathbf{L} = \mathrm{Lan}_{\mathbf{J}}(PT^{op}S\mathbf{J})$ and $\mathbf{L}' = \mathrm{Lan}_{\mathbf{J}'}(P'T'^{op}S'\mathbf{J}')$. Then $(\mathbf{L}', \boldsymbol{\delta}')$ is the maximal positive fragment of $(\mathbf{L}, \boldsymbol{\delta})$.*

Example 4.11. For $T = \mathrm{Id}$, the corresponding finitary logics is $\mathbf{L} = \mathrm{Id}$ on BA, with trivial semantics $\boldsymbol{\delta} : LP \to PT^{op}$. It allows the extension $T' = DC$, the discrete connected components functor. Notice that T' does not preserve embeddings, neither coreflexive inserters. The corresponding logic $\mathbf{L}'$ is given by the constant functor to the distributive lattice $\mathbb{2}$. Thus $\beta : \mathbf{L}'W \to W\mathbf{L}$ fails to be an isomorphism (it is just the unique morphism from the initial object).

Our introductory example of positive modal logic is now regained as an instance of this theorem.[7] It can also easily be adapted to Kripke polynomial functors. More interesting is the case of the probability distribution functor. We know from the theorem above that it has a maximal positive fragment, but an explicit description still needs to be worked out.

5 Monotone predicate liftings

In this section we show that the logic of the posetification T' of T coincides with the logic of all monotone predicate liftings of T.

Recall that a predicate lifting [25,27] of arity n for T is an ordinary natural transformation $\heartsuit : \mathsf{Set}_o(-, 2^n) \to \mathsf{Set}_o(T-, 2)$,[8] or, using the ordinary adjunction $D_o \dashv V : \mathsf{Pos}_o \to \mathsf{Set}$, an ordinary natural transformation

$$\heartsuit : \mathsf{Pos}_o(D_o-, [n, \mathbb{2}]) \to \mathsf{Pos}_o(D_oT-, \mathbb{2})$$

---

[7] A minor issue here is that modal logic usually takes as semantics coalgebras for the (non-finitary) powerset, whereas for the posetification to exist we sofar assumed T to be finitary. There are two solutions to this. One is to note that going from T to its finitary coreflection T_ω and then to its posetification T'_ω does not change the functors $\mathbf{L}, \mathbf{L}'$ on the algebraic side. The second is to prove that the posetification exists despite the functor not being accessible.

[8] Equivalently, it can be described as an element $\heartsuit \in \mathsf{Set}(T(2^n), 2)$.

It is called *monotone* if it lifts to a natural transformation

$$\heartsuit : \mathsf{Pos}(D-, [Dn, \mathbb{2}]) \to \mathsf{Pos}(DT-, \mathbb{2})$$

By identifying a predicate lifting with an map $\heartsuit : T(2^n) \to 2$, the above says that $\heartsuit$ is monotone if for all $\overline{a_1} \le \overline{a_2} : D_o X \to [D_o n, \mathbb{2}]$, we have that $\overline{\heartsuit \circ Ta_1} \le \overline{\heartsuit \circ Ta_2}$, where $\overline{f} : D_o X \to Y$ denotes the adjoint transpose of $f : X \to VY$.

Consider now a Pos-functor T' (locally monotone!) and a finite poset p. By mimicking the above, we define a predicate lifting for T' of arity p as being a natural transformation[9]

$$\heartsuit : \mathsf{Pos}(-, [p, \mathbb{2}]) \to \mathsf{Pos}(T'-, \mathbb{2})$$

Proposition 5.1. *Let T be a Set-functor and $T' : \mathsf{Pos} \to \mathsf{Pos}$ an extension. Then:*

1. *There is an injection from the set of predicate liftings of T' of arity p into the set of monotone predicate liftings of T of arity Vp.*
   In particular, the set of predicate liftings of T' of discrete arity n embeds into the monotone predicate liftings of T.
2. *In case T' is the posetification of T, the above mapping is a bijection.*

As a corollary, we obtain

Corollary 5.2. *Let T be a finitary Set-functor. If the posetification T' of T preserves embeddings, then the logic of all monotone predicate liftings of T is expressive.*

Remark 5.3. We know from [4] that if T preserves weak pullbacks then T' preserves embeddings. So the above theorem applies to weak-pullback preserving functors. This result was obtained in [18, Cor 6.9] already in a different way.

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[9] Which can be identified with $\heartsuit \in \mathsf{Pos}(T'([p, \mathbb{2}]), \mathbb{2})$.

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