Stone Coalgebras

CLEMENS KUPKE* ALEXANDER KURZ†‡ YDE VENEMA§

October 31, 2003

Abstract

We argue that the category of Stone spaces forms an interesting base category for coalgebras, in particular, if one considers the Vietoris functor as an analogue to the power set functor on the category of sets.

We prove that the so-called descriptive general frames, which play a fundamental role in the semantics of modal logics, can be seen as Stone coalgebras in a natural way. This yields a duality between modal algebras and coalgebras for the Vietoris functor.

Building on this idea, we introduce the notion of a Vietoris polynomial functor over the category of Stone spaces. For each such functor T we provide an adjunction between T-sorted Boolean algebras with operators and the Stone coalgebras for T. We also indentify the subcategories on which the adjunction restricts to an equivalence and show that the final T-coalgebra is the dual of the initial T-BAO.

Keywords coalgebra, Stone spaces, Vietoris topology, modal logic, descriptive general frames, Kripke polynomial functors

1 Introduction

Every coalgebra is based on a carrier which is an object in the so-called base category. Most of the literature on coalgebras either focuses on Set as the base category, or takes a very general perspective, allowing arbitrary base categories (possibly restricted by some constraints). The aim of this paper is to argue that, besides Set, the category Stone of Stone spaces is an interesting base category. We have a number of reasons for believing that *Stone coalgebras*, that is, coalgebras based on Stone, are of relevance.

To start with, in Section 3 we discuss interesting examples of Stone coalgebras, namely the ones that are associated with the *Vietoris functor* $\mathbb{V}$: Stone $\to$ Stone, a topological analogue of the power set functor on Set. $\mathbb{V}$ is a functorial extension of a well-known topological construction which associates with a topology its Vietoris topology [12]. This construction preserves a number of nice topological properties; in particular, it turns Stone spaces into Stone spaces [18]. As we will see further on, the category Coalg($\mathbb{V}$) of coalgebras for the Vietoris functor is of interest because it is isomorphic to the category DGF of descriptive general frames. This category in its turn is dual to that of modal algebras, and hence, unlike Kripke frames, descriptive general frames form a mathematically adequate semantics for modal logics [8].

*Institute for Logic, Language and Computation, University of Amsterdam, Plantage Muidergracht 24, NL–1018 TV Amsterdam, and Centrum voor Wiskunde en Informatica, P.O. Box 94079, NL–1090 GB Amsterdam.

†Department of Computer Science, University of Leicester, University Road, Leicester LE1 7RH, UK.

‡Partially supported by NWO/British Council.

§Institute for Logic, Language and Computation, University of Amsterdam, Plantage Muidergracht 24, NL–1018 TV Amsterdam.

The connection with modal logic thus forms a second reason as to why Stone coalgebras are of interest. Since coalgebras can be seen as a very general model of state-based dynamics, and modal logic as a logic for dynamic systems, the relation between modal logic and coalgebras is rather tight. Starting with the work of Moss [25], this has been an active research area [28, 17, 6, 27, 15, 9]. The relation between modal logic and coalgebras can be seen to dualise that between equational logic and algebra [20, 19], an important difference being that the relation with Set-based coalgebras seems to work smoothly only for modal languages that allow infinitary formulas. In the case of the Vietoris functor however, it follows from the duality between $\mathsf{Coalg}(\mathbb{V})$ and the category MA of modal algebras, that $\mathsf{Coalg}(\mathbb{V})$ provides a natural semantics for *finitary* modal logics. Although probably not widely known, this insight is in fact due to Abramsky [1].

In Sections 4 and 5 we further substantiate our case for Stone spaces as a coalgebraic base category, by considering so-called Vietoris polynomial functors as the Stone-based analogues of Kripke polynomial functors over Set [28]. Transferring the work of Jacobs [17] from the setting of Set-coalgebras to Stone-coalgebras, we establish, for each such functor T, a link between the category BAO_T of T-sorted Boolean algebras with operators and the category $\mathsf{Coalg}(T)$ of Stone coalgebras for T. In Section 4 we lay the foundations of this work, introducing the notions of a Vietoris polynomial functor (VPF), the algebraic and coalgebraic categories, and functors between these categories. Section 5 shows that these functors form an adjunction between the categories BAO_T and $\mathsf{Coalg}(T)$, for any VPF T. Although this adjunction is not a dual equivalence in general, we will see that each coalgebra can be represented by an algebra, more precisely, $\mathsf{Coalg}(T)^{\mathrm{op}}$ is (isomorphic to) a full coreflective subcategory of BAO_T. We identify the full subcategory of BAO_T on which the adjunction restricts to an equivalence and show that the initial T-BAO is dual to the final T-coalgebra.

Let us add two more observations on Stone-coalgebras. First, the duality of descriptive general frames and modal algebras shows that the (trivial) duality between the categories $\mathsf{Coalg}(T)$ and $\mathsf{Alg}(T^{\mathrm{op}})$ has non-trivial instances. Second, it might be interesting to note that Stone provides a meaningful example of a base category for coalgebras which is not locally finitely presentable.

Related Work Coalgebras over topological spaces have been a topic of study in denotational semantics for a long time. One of the main concerns of denotational semantics is to find, for a given type constructor $T : \mathcal{X} \to \mathcal{X}$, solutions to the equation $X \cong TX$. The typical situation is the following. $\mathcal{X}$ is a category of topological spaces as, for example, domains (see e.g. [4]) or (ultra)metric spaces (see e.g. [11, 32]), T is a functor, and the favoured solution of $X \cong TX$ is the final T-coalgebra $X \to TX$. The Vietoris functor is known in domain theory as the Plotkin powerdomain and its version on Stone has been considered in Abramsky [1]. The closely related category of Stone spaces with a countable base and their connection to SFP-domains have been investigated by Alessi, Baldan, and Honsell [5]. Comparing with Abramsky [2], our work might be seen as a variation based on the use of Stone spaces instead of SFP-domains. Motivated by a different perspective, coalgebras over Stone spaces have been considered recently also by Davey and Galati [10].

Acknowledgements We would like to thank the participants of the ACG-meetings at the Amsterdam Centrum voor Wiskunde en Informatica (CWI); we benefited in particular from discussions with Marcello Bonsangue, Alessandra Palmigiano, and Jan Rutten.We also greatly appreciated the comments by Samson Abramsky.

2 Preliminaries

We presuppose some familiarity with category theory, general topology, the (duality) theory of boolean algebras, and universal coalgebra. The main purpose of this section is to fix our notation and terminology.

Definition 2.1 Let C be a category and $T : \mathsf{C} \to \mathsf{C}$ an endofunctor. Then a *T-coalgebra* is a pair $(X, \xi : X \to TX)$ where X denotes an object of C and ξ a morphism of C. A T-coalgebra morphism $h : (X_1, \xi_1) \to (X_2, \xi_2)$ is a C-morphism $h : X_1 \to X_2$ satisfying $\xi_2 \circ h = Th \circ \xi_1$. The category $\mathsf{Coalg}(T)$ has T-coalgebras as its objects and T-coalgebra morphisms as arrows. Dually, we define a *T-algebra* to be a T^{op}-coalgebra and $\mathsf{Alg}(T) = (\mathsf{Coalg}(T^{\mathrm{op}}))^{\mathrm{op}}$. ◁

Example 2.2 A *Kripke frame* is a structure $\mathbb{F} = (X, R)$ such that R is a binary relation on X. Kripke frames can be seen as coalgebras for the power set functor $\mathcal{P}$ over Set: replace the binary relation R of a frame $\mathbb{F} = (X, R)$ with the map $R[\_] : X \to \mathcal{P}(X)$ given by $R[s] := \{t \in X \mid Rst\}$. In fact, Kripke frames (and models) form some of the prime examples of coalgebras; in particular, *bounded morphisms* between Kripke frames coincide with $\mathcal{P}$-coalgebra morphisms.

Definition 2.3 A topological space $\mathbb{X} = (X, \tau)$ is a *Stone space* if it is compact Hausdorff and has a basis of clopen sets. $\mathsf{Clp}_{\mathbb{X}}$ will denote the set of clopen subsets of X. The category Stone has Stone spaces as objects and continuous functions as morphisms. ◁

We now turn to Stone duality.

Definition 2.4 The category of Boolean algebras and homomorphisms between them is denoted as BA. The Stone space $(\mathsf{Sp}\,\mathbb{B}, \tau_{\mathbb{B}})$ corresponding to a Boolean algebra $\mathbb{B}$ is given by the collection $\mathsf{Sp}\,\mathbb{B}$ of ultrafilters of $\mathbb{B}$ and the topology $\tau_{\mathbb{B}}$ generated by basic opens of the form $\{u \in \mathsf{Sp}\,\mathbb{B} \mid b \in u\}$ for any b in $\mathbb{B}$. We let $\mathbb{S}\mathsf{p}$ denote the functor that associates with a Boolean algebra its corresponding Stone space, and with a Boolean homomorphism its inverse image function.

Conversely the functor mapping a Stone space $\mathbb{X}$ to the Boolean algebra $\mathsf{Clp}_{\mathbb{X}}$ of its clopens, and a continuous morphism to its inverse image function, is denoted as $\mathbb{C}\mathsf{lp}$.

Furthermore, for any Boolean algebra $\mathbb{B}$ we define a map $i_{\mathbb{B}} : \mathbb{B} \to \mathbb{C}\mathsf{lp}\,\mathbb{S}\mathsf{p}\mathbb{B}$ given by $h(b) = \hat{b} := \{u \in \mathsf{Sp}\,\mathbb{B} \mid b \in u\}$, and for any Stone space $\mathbb{X}$ we define a map $\epsilon_{\mathbb{X}} : \mathbb{X} \to \mathbb{S}\mathsf{p}\mathbb{C}\mathsf{lp}\mathbb{X}$ fixed by $\epsilon_{\mathbb{X}}(x) := \{U \in \mathsf{Clp}_{\mathbb{X}} \mid x \in U\}$. ◁

Theorem 2.5 *The families of morphisms $(i_{\mathbb{B}})_{\mathbb{B} \in \mathsf{BA}}$ and $(\epsilon_{\mathbb{X}})_{\mathbb{X} \in \mathsf{Stone}}$ are natural isomorphisms. Hence, the functors $\mathbb{S}\mathsf{p} : \mathsf{BA} \to \mathsf{Stone}^{\mathrm{op}}$ and $\mathbb{C}\mathsf{lp} : \mathsf{Stone}^{\mathrm{op}} \to \mathsf{BA}$ induce a dual equivalence*

$$\mathsf{BA} \simeq \mathsf{Stone}^{\mathrm{op}}.$$

Definition 2.6 Let $\mathbb{X} = (X, \tau)$ be a topological space. We let $K(\mathbb{X})$ denote the collection of all closed subsets of X. Define the operations $[\ni], \langle\ni\rangle : \mathcal{P}(X) \to \mathcal{P}(K(\mathbb{X}))$ by

$$\begin{array}{lcl} [\ni]U & := & \{F \in K(\mathbb{X}) \mid F \subseteq U\}, \\ \langle\ni\rangle U & := & \{F \in K(\mathbb{X}) \mid F \cap U \neq \varnothing\}. \end{array}$$

Given a subset $Q \subseteq \mathcal{P}(X)$, define

$$V_Q := \{[\ni]U \mid U \in Q\} \cup \{\langle\ni\rangle U \mid U \in Q\}.$$

The *Vietoris space* $\mathbb{V}(\mathbb{X})$ associated with $\mathbb{X}$ is given by the topology $\upsilon_{\mathbb{X}}$ on $K(\mathbb{X})$ which is generated by V_τ as subbasis. ◁

In case the original topology is compact, then we might as well have generated the Vietoris topology in other ways. This has nice consequences for the case that the original topology is a Stone space.

Lemma 2.7 *Let $\mathbb{X} = (X, \tau)$ be a compact topological space and let $\mathcal{B}$ be a basis of τ that is closed under finite unions. Then the set $V_{\mathcal{B}}$ forms a subbasis for $\upsilon_{\mathbb{X}}$. In particular, if $\mathbb{X}$ is a Stone space, then the set $V_{\mathsf{Clp}_{\mathbb{X}}}$ forms a subbasis for $\upsilon_{\mathbb{X}}$.*

The Vietoris construction preserves various nice topological properties; proofs of this can be found in for instance [24].

Lemma 2.8 *Let $\mathbb{X} = (X, \tau)$ be a topological space.*

1. *If $\mathbb{X}$ is compact then $(K(\mathbb{X}), \upsilon_{\mathbb{X}})$ is compact.*
2. *If $\mathbb{X}$ is compact and Hausdorff, then $(K(\mathbb{X}), \upsilon_{\mathbb{X}})$ is compact and Hausdorff.*
3. *If $\mathbb{X}$ is a Stone space, then so is $(K(\mathbb{X}), \upsilon_{\mathbb{X}})$.*

3 General frames as coalgebras

In this section we discuss what are probably the prime examples of Stone coalgebras, namely those for the Vietoris functor $\mathbb{V}$. As we will see, the importance of these structures lies in the fact that the category $\mathsf{Coalg}(\mathbb{V})$ is isomorphic to the category of so-called *descriptive general frames*, and hence, dual to the category of modal algebras (all these notions will be defined below). We hasten to remark that when it comes down to the technicalities, this section contains little news; most of the results in this section can be obtained by exposing existing material from Esakia [13], Goldblatt [16], Johnstone [18], and Sambin and Vaccaro [31] in a new coalgebraic light. Moreover, it will turn out that the duality of descriptive general frames and modal algebras is an instance of the the general relationship between syntax and semantics as laid out by Abramsky in his domain theory in logical form [3]. He was also the first to observe the duality of $\mathsf{Coalg}(\mathbb{V})$ and modal algebras in [1].

Modal algebras and (descriptive) general frames play a crucial role in the theory of modal logic, providing an important class of structures interpreting modal languages. From a mathematical perspective they display much better behaviour than Kripke frames, since the latter provide too poor a tool to make the required distinctions between modal logics (in the technical sense, see for instance [8], Chapter 4). The algebraic semantics for modal logic does not suffer from this fundamental incompleteness result: every modal logic is determined by the class of modal algebras on which it is valid.

Definition 3.1 Let $\mathbb{B}$ and $\mathbb{B}'$ be boolean algebras; an operation $g : B \to B'$ on their carriers is said to *preserve finite meets* if $g(\top) = \top'$ and $g(b_1 \wedge b_2) = g(b_1) \wedge' g(b_2)$. A *modal algebra* is a structure $\mathbb{A} = (A, \wedge, -, \bot, \top, g)$ such that the reduct $(A, \wedge, -, \bot, \top)$ of $\mathbb{A}$ is a Boolean algebra, and $g : A \to A$ preserves finite meets. The category of modal algebras (with homomorphisms) is denoted by MA. $\lhd$

The intended meaning of g is to provide an interpretation of the modal operator $\Box$. Thinking of $a \in A$ as the interpretation of a modal formula φ, $g(a)$ provides the interpretation of $\Box\varphi$.

Example 3.2
1. If (X, R) is a Kripke frame then $(\mathcal{P}X, \cap, -, \emptyset, X, [R])$ is a modal algebra where $[R](a) = \{x \in X \mid x\,R\,y \;\Rightarrow\; y \in a\}$.
2. Let Prop be a set of propositional variables and $\mathcal{L}(\mathsf{Prop})$ be the set of modal formulas over Prop quotiented by $\varphi \equiv \psi \;\Leftrightarrow\; \vdash_{\mathbf{K}} \varphi \leftrightarrow \psi$ where $\vdash_{\mathbf{K}}$ denotes derivability in the basic modal logic $\mathbf{K}$ (see eg [8]). Then $\mathcal{L}(\mathsf{Prop})$—equipped with the obvious operations—is a modal algebra. In fact, $\mathcal{L}(\mathsf{Prop})$ is the modal algebra free over Prop and is called the Lindenbaum-Tarski algebra (over Prop).

Remark 3.3 Although not needed in the following, we indicate how modal formulas are evaluated in modal algebras. Let φ be a modal formula taking propositional variables from Prop and let $\mathbb{A} = (A, \wedge, -, \bot, \top, g)$ be a modal algebra. Employing the freeness of the modal algebra $\mathcal{L}(\mathsf{Prop})$ we can identify valuations of variables $v : \mathsf{Prop} \to A$ with algebra morphisms $\mathcal{L}(\mathsf{Prop}) \to \mathbb{A}$ and define $\mathbb{A} \models \varphi$ if $v([\varphi]_{\equiv}) = \top$ for all morphisms $v : \mathcal{L}(\mathsf{Prop}) \to \mathbb{A}$.

Modal algebras are fairly abstract in nature and many modal logicians prefer the intuitive, geometric appeal of Kripke frames. *General frames*, unifying the algebraic and the Kripke semantics in one structure, provide a nice compromise.

Definition 3.4 A *general frame* is a structure $\mathbb{G} = (G, R, A)$ such that (G, R) is a Kripke frame and A is a collection of so-called *admissible* subsets of G that is closed under the boolean operations and under the operation $\langle R \rangle : \mathcal{P}(G) \to \mathcal{P}(G)$ given by: $\langle R \rangle X := \{y \in G \mid Ryx \text{ for some } x \in X\}$.

A general frame $\mathbb{G} = (G, R, A)$ is called *differentiated* if for all distinct $s_1, s_2 \in G$ there is a 'witness' $a \in A$ such that $s_1 \in a$ while $s_2 \notin a$; *tight* if whenever t is not an R-successor of s, then there is a 'witness' $a \in A$ such that $t \in a$ while $s \notin \langle R \rangle a$; and *compact* if $\bigcap A_0 \neq \varnothing$ for every subset A_0 of A which has the finite intersection property. A general frame is *descriptive* if it is differentiated, tight and compact. $\lhd$

The term 'admissible' subset is explained by the semantic restriction that only allows those Kripke models on a general frame for which the extensions of the atomic formulae are admissible sets.

Example 3.5 1. Any Kripke frame (X, R) can be considered as a general frame $(X, R, \mathcal{P}X)$.

2. If $\mathbb{A} = (A, \wedge, -, \perp, \top, g)$ is a modal algebra then $(\mathsf{Sp}\,\mathbb{A}, R, \hat{A})$ where $R = \{(u, v) \mid a \in u \Rightarrow g(a) \in v\}$ and $\hat{A} = \{\{u \in \mathsf{Sp}\,\mathbb{A} \mid a \in u\} \mid a \in A\}$ is a descriptive general frame.

3. If $\mathbb{G} = (G, R, A)$ is a general frame then $(A, \cap, -, \emptyset, G, [R])$ is a modal algebra (where $[R]X = \{y \in G \mid Ryx \Rightarrow x \in X\}$).

The last two examples form the basis of the dual equivalence

$$\mathsf{MA} \simeq \mathsf{DGF}^{\mathrm{op}}$$

between the categories of modal algebras and descriptive general frames where the latter category is defined as follows.

Definition 3.6 (GF, DGF) A morphism $\theta : (G, R, A) \to (G', R', A')$ is a function from G to G' such that (i) $\theta : (G, R) \to (G', R')$ is a bounded morphism (see Example 2.2) and (ii) $\theta^{-1}(a') \in A$ for all $a' \in A'$. We let GF (DGF) denote the category with general frames (descriptive general frames, respectively) as its objects, and the general frame morphisms as the arrows. $\lhd$

Since Kripke frames (and models) form prime examples of coalgebras, the question naturally arises whether (descriptive) general frames can be seen as coalgebras as well. Our positive answer to this question is based on two crucial observations. First, the admissible sets of a descriptive frame form a basis for a Stone topology because descriptive general frames are compact, differentiated and the admissible sets are closed under boolean operations. Second, the tightness condition of descriptive general frames can be reformulated as the requirement that the relation is *point-closed*; that is, the successor set of any point is closed in the Stone topology. This suggests that if we are looking for a coalgebraic counterpart of a descriptive general frame $\mathbb{G} = (G, R, A)$, it should be of the form $R[\_] : (G, \tau) \to (K(G), \tau_?)$ where $K(G)$ is the collection of closed sets in the Stone topology τ on G and $\tau_?$ is some suitable topology on $K(G)$, which turns $K(G)$ again into a Stone space. A good candidate is the Vietoris topology: it is based on the closed sets of τ and it yields a Stone space if we started from one. Moreover, as we will see, choosing the Vietoris topology for $\tau_?$, continuity of the map $R[\_]$ corresponds to the admissible sets being closed under $\langle R \rangle$.

We will prove that the category of descriptive general frames and the category $\mathsf{Coalg}(\mathbb{V})$ of coalgebras for the Vietoris functor are in fact *isomorphic*. We first note that [12, Theorem 3.1.8] whenever $f : \mathbb{X} \to \mathbb{X}'$ is a continuous map between compact Hausdorff spaces, then the image map $f[\_]$ sends closed sets to closed sets. This motivates defining $\mathbb{V}(f) : K(\mathbb{X}) \to K(\mathbb{X}')$ by

$$\mathbb{V}(f)(F) := f[F] \; (= \{f(x) \mid x \in F\}). \tag{1}$$

Definition 3.7 The *Vietoris functor* on the category of Stone spaces is given on objects as in Definition 2.6 and on morphisms as in (1). $\lhd$

We now turn to the isomorphism between the categories DGF and $\mathsf{Coalg}(\mathbb{V})$. It is straightforward to verify that the following definition is correct, that is, indeed defines two functors.

Definition 3.8 We define the functor $\mathbb{C} : \mathsf{DGF} \to \mathsf{Coalg}(\mathbb{V})$ via

$$(G, R, A) \quad \mapsto \quad (G, \sigma_A) \xrightarrow{R[\text{-}]} \mathbb{V}(G, \sigma_A)$$

Here σ_A denotes the Stone topology generated by taking A as a basis. Conversely, there is a functor $\mathbb{D} : \mathsf{Coalg}(\mathbb{V}) \to \mathsf{DGF}$ given by

$$(\mathbb{X}, \gamma) \quad \mapsto \quad (X, R_\gamma, \mathsf{Clp}_{\mathbb{X}})$$

where R_γ is defined by $R_\gamma s_1 s_2$ iff $s_2 \in \gamma(s_1)$. On morphisms both functors act as the identity with respect to the underlying Set-functions. $\lhd$

Theorem 3.9 *The functors $\mathbb{C}$ and $\mathbb{D}$ form an isomorphism between the categories DGF and $\mathsf{Coalg}(\mathbb{V})$.*

Proof. The theorem can be easily proven by just spelling out the definitions. QED

Remark 3.10 For a set-coalgebra (X, ξ), a valuation of propositional variables $p \in \mathsf{Prop}$ is a function $X \to \prod_{\mathsf{Prop}} 2$ where 2 is the two-element set of truth-values. For a Stone-coalgebra $(\mathbb{X}, \xi)$, a valuation is a continuous map $v : \mathbb{X} \to \prod_{\mathsf{Prop}} 2$ where 2 is taken with the discrete topology. The continuity of v is equivalent to the statement that the propositional variables take their values in admissible sets. Indeed, writing $\pi_p : \prod_{\mathsf{Prop}} 2 \to 2$ ($p \in \mathsf{Prop}$) for the projections, continuity of v is equivalent to $v^{-1}(\pi_p^{-1}(\{1\}))$ clopen for all $p \in \mathsf{Prop}$. Observing that $v^{-1}(\pi_p^{-1}(\{1\})) = \{x \in X \mid v(x)_p = 1\}$ is the extension of p the claim now follows from the fact that the clopens coincide with the admissible sets.

Let us note two corollaries of Theorem 3.9. Using $\mathsf{MA} \simeq \mathsf{DGF}^{\mathrm{op}}$ and $(\mathsf{Coalg}(\mathbb{V}))^{\mathrm{op}} = \mathsf{Alg}(\mathbb{V}^{\mathrm{op}})$, it follows that $\mathsf{MA} \simeq \mathsf{Alg}(\mathbb{V}^{\mathrm{op}})$. With $\mathsf{Stone}^{\mathrm{op}} \simeq \mathsf{BA}$ we obtain the following.

Corollary 3.11 *There is a functor $H : \mathsf{BA} \to \mathsf{BA}$ such that the category of modal algebras MA is equivalent to the category $\mathsf{Alg}(H)$ of algebras for the functor H.*

Proof. With the help of the contravariant functors $\mathbb{C}\mathsf{lp} : \mathsf{Stone} \to \mathsf{BA}$, $\mathbb{S}\mathsf{p} : \mathsf{BA} \to \mathsf{Stone}$, we let $H = \mathbb{C}\mathsf{lp}\mathbb{V}\mathbb{S}\mathsf{p}$. The claim now follows from the observation that $\mathsf{Alg}(H)$ is dual to $\mathsf{Coalg}(\mathbb{V})$: An algebra $HA \xrightarrow{\alpha} A$ corresponds to the coalgebra $\mathbb{S}\mathsf{p}A \xrightarrow{\mathbb{S}\mathsf{p}\alpha} \mathbb{S}\mathsf{p}HA \cong \mathbb{V}\mathbb{S}\mathsf{p}A$ and a coalgebra $\mathbb{X} \xrightarrow{\xi} \mathbb{V}\mathbb{X}$ corresponds to the algebra $H\mathbb{C}\mathsf{lp}\mathbb{X} \cong \mathbb{C}\mathsf{lp}\mathbb{V}\mathbb{X} \xrightarrow{\mathbb{C}\mathsf{lp}\xi} \mathbb{C}\mathsf{lp}\mathbb{X}$. QED

An explicit description of H not involving the Vietoris functor is given by the following proposition.

Proposition 3.12 *Let $H : \mathsf{BA} \to \mathsf{BA}$ be the functor that assigns to a Boolean algebra the free Boolean algebra over its underlying meet-semilattice. Then $\mathsf{Alg}(H)$ is isomorphic to the category of modal algebras MA.*

Proof. We use the well-known fact that MA is isomorphic to the category MPF which is defined as follows. An object of MPF is an endofunction $A \xrightarrow{m} A$ on a Boolean algebra A that preserves finite meets (i.e. binary meets and the top-element). A morphism $f : (A \xrightarrow{m} A) \longrightarrow (A' \xrightarrow{m'} A')$ is a Boolean algebra morphism $f : A \to A'$ such that $m' \circ f = f \circ m$. We also write $\mathsf{BA}_\wedge$ for the category with Boolean algebras as objects and finite meet preserving functions as morphisms.

To prove that $\mathsf{Alg}(H)$ and MPF are isomorphic categories, we first show that $\mathsf{BA}(HA,A) \cong \mathsf{BA}_\wedge(A,A)$,or slightly more general and precise, $\mathsf{BA}(HA,B) \cong \mathsf{BA}_\wedge(IA,IB)$ where $I : \mathsf{BA} \hookrightarrow \mathsf{BA}_\wedge$. (Here we denote, for a category C and objects A, B in C, the set of morphisms between A and B by $\mathsf{C}(A,B)$.) Indeed, consider the forgetful functors $U : \mathsf{BA} \to \mathsf{SL}$, $V : \mathsf{BA}_\wedge \to \mathsf{SL}$ to the category SL of meet-semilattices with top element and the left adjoint F of U. Using our assumption $H = FU$, we calculate $\mathsf{BA}(HA,B) = \mathsf{BA}(FUA,B) \cong \mathsf{SL}(UA,UB) \cong \mathsf{SL}(VIA,VIB) \cong \mathsf{BA}_\wedge(IA,IB)$. The isomorphisms $\varphi_A : \mathsf{BA}(HA,A) \to \mathsf{BA}_\wedge(A,A)$, $A \in \mathsf{BA}$, give us an isomorphism φ between the objects of $\mathsf{Alg}(H)$ and MPF. On morphisms, we define φ to be the identity. This is well-defined because the isomorphisms $\mathsf{BA}(HA,B) \cong \mathsf{BA}_\wedge(IA,IB)$ are natural in A and B. QED

Remark 3.13 Detailing the construction of a free Boolean algebra over its underlying meet-semilattice, we see that, given a boolean algebra $\mathbb{A} = (A, \wedge, -, \bot, \top)$, $H\mathbb{A}$ is the free boolean algebra generated by $\{\Box a \mid a \in A\}$ (the insertion of generators being $\Box : \mathbb{A} \to H\mathbb{A}$, $a \mapsto \Box a$) and satisfying the equations $\Box\top = \top$, $\Box(a \wedge b) = \Box a \wedge \Box b$. That is, the functor H describes how to obtain modal logic by adding an operator to boolean logic. As observed above this functor is the Stone dual of the Vietoris functor. This observation was made earlier by Abramsky in [1] and is an instance of the general relationship between syntax and semantics as laid out in his domain theory in logical form [3].

As another corollary to the duality we obtain that $\mathsf{Coalg}(\mathbb{V})$ has cofree coalgebras.

Corollary 3.14 *The forgetful functor* $\mathsf{Coalg}(\mathbb{V}) \to \mathsf{Stone}$ *has a right adjoint.*

Proof. Consider the forgetful functors $R : \mathsf{MA} \to \mathsf{BA}$, $U : \mathsf{MA} \to \mathsf{Set}$, $V : \mathsf{BA} \to \mathsf{Set}$. Since U and V are monadic, R has a left adjoint. Hence, by duality, $\mathsf{Coalg}(\mathbb{V}) \to \mathsf{Stone}$ has a right adjoint. QED

Finally, let us see how arbitrary general general frames can be seen as coalgebras.

Remark 3.15 (General Frames as Coalgebras) Stone spaces provide a convenient framework to study descriptive general frames since the admissible sets can be recovered from the topology: each Stone space $\mathbb{X} = (X, \tau)$ has a *unique* basis that is closed under the boolean operations. Making a generalisation to arbitrary general frames, we can still work in a coalgebraic framework, but we have to make two adjustments.

First, we work directly with admissible sets instead of with topologies: the category RBA (referential or represented Boolean algebras) has objects (X, A) where X is a set and A a set of subsets of X closed under boolean operations. It has morphisms $f : (X,A) \to (Y,B)$ where f is a function $X \to Y$ such that $f^{-1}(b) \in A$ for all $b \in B$.

And second, in the absence of tightness, the relation of the general frame will no longer be point-closed. Hence, its coalgebraic version has the full power set as its codomain. For $\mathbb{X} = (X, A) \in \mathsf{RBA}$ let $\mathbb{W}(\mathbb{X}) = (\mathcal{P}(X), v_{\mathbb{X}})$ where $v_{\mathbb{X}}$ is the Boolean algebra generated by $\{\{F \in \mathcal{P}X \mid F \cap a \neq \varnothing\} \mid a \in A\}$. On morphisms let $\mathbb{W}(f) = \mathcal{P}(f)$. This clearly defines an endofunctor on the category RBA, and the induced category $\mathsf{Coalg}(\mathbb{W})$ is the coalgebraic version of general frames:

$$\text{There is an isomorphism between } \mathsf{GF} \text{ and } \mathsf{Coalg}(\mathbb{W}) \tag{2}$$

The crucial observation in the proof of (2) is that, for $\mathbb{X} = (X, A) \in \mathsf{RBA}$ and R a relation on X, we have that A is closed under $\langle R \rangle$ iff $R[\_] : X \to \mathcal{P}X$ is a RBA-morphism $\mathbb{X} \to \mathbb{W}(\mathbb{X})$. This follows from the fact that $\langle R \rangle a = (R[\_])^{-1}(\{F \in \mathcal{P}X \mid F \cap a \neq \varnothing\})$.

4 Vietoris Polynomial Functors: from coalgebras to algebras and back

In this section we introduce the notion of a Vietoris polynomial functor (VPF) as a natural analogue for the category Stone of Stone spaces of what the so-called Kripke polynomial functors [28, 17] are for Set. This section can be therefore seen as a first application of the observation that coalgebras over Stone can be used as semantics for (coalgebraic) modal logics. Much of the work in this section consists of transferring the work by Jacobs in [17] to the topological setting. After introducing the Vietoris polynomial functors, we define, for each VPF, the category BAO_T of T-sorted Boolean algebras with operators and their morphisms. We then link the categories BAO_T and $\mathsf{Coalg}(T)$ by functors $\mathcal{A} : \mathsf{Coalg}(T)^{\mathrm{op}} \to \mathsf{BAO}_T$ and $\mathcal{C} : \mathsf{BAO}_T \to \mathsf{Coalg}(T)^{\mathrm{op}}$.

Definition 4.1 The collection of *Vietoris polynomial functors*, in brief: VPFs, over Stone is inductively defined as follows:

$$T \ ::= \ \mathbb{I} \mid \mathbb{Q} \mid T_1 + T_2 \mid T_1 \times T_2 \mid T^D \mid \mathbb{V}T.$$

Here $\mathbb{I}$ is the identity functor on the category Stone; $\mathbb{Q}$ denotes a finite Stone space (that is, the functor $\mathbb{Q}$ is a constant functor); '+' and '×' denote disjoint union and binary product, respectively; and, for an arbitrary set D, T^D denotes the functor sending a Stone space $\mathbb{X}$ to the D-fold product of $T(\mathbb{X})$.

Associated with this we inductively define the notion of a *path*:

$$p \ ::= <>\mid \pi_1 \cdot p \mid \pi_2 \cdot p \mid \kappa_1 \cdot p \mid \kappa_2 \cdot p \mid [\mathrm{ev}(d)] \cdot p \mid \mathbb{V} \cdot p.$$

By induction on the complexity of paths we now define when two VPFs T_1 and T_2 are related by a path p, notation: $T_1 \overset{p}{\leadsto} T_2$:

$$\begin{array}{rcccl}
T & \overset{<>}{\leadsto} & T & & \\
T_1 \times T_2 & \overset{\pi_i \cdot p}{\leadsto} & T' & \text{if} & T_i \overset{p}{\leadsto} T' \\
T_1 + T_2 & \overset{\kappa_i \cdot p}{\leadsto} & T' & \text{if} & T_i \overset{p}{\leadsto} T' \\
T^D & \overset{[\mathrm{ev}(d)] \cdot p}{\leadsto} & T' & \text{if} & T \overset{p}{\leadsto} T' \text{ and } d \in D \\
\mathbb{V}T & \overset{\mathbb{V} \cdot p}{\leadsto} & T' & \text{if} & T \overset{p}{\leadsto} T'.
\end{array}$$

Finally, for a VPF T we define $\mathsf{Ing}(T)$ to be the category with the set $\mathrm{Ing}(T) := \{S \mid \exists p.T \overset{p}{\leadsto} S\} \cup \{\mathbb{I}\}$ as the set of objects and the paths as morphisms between them. ◁

Remark 4.2 All of our results could have been generalised to a setting that allows infinite constants and infinite topologised sums as in Gehrke [14]. We confine ourselves to the functors of Definition 4.1 in order to stay as close as possible to existing work on Kripke polynomial functors.

We will now define *the boolean algebras with operators associated with a VPF*. The definition of a so-called T-BAO may look slightly involved, but it is based on a simple generalisation of the concept of a modal algebra. The generalisation is that instead of dealing with one single Boolean algebra, we will be working with a *family* $(\Phi(S))_{S \in \mathrm{Ing}(T)}$ of Boolean algebras. As before, we let $\mathsf{BA}_\wedge$ denote the category with Boolean algebras as objects and finite-meet preserving operations as morphisms.

Definition 4.3 (T-BAO) Let T be a VPF. A *T-sorted Boolean algebra with operators*, T-BAO, consists of a functor $\Phi : \mathsf{Ing}(T)^{\mathrm{op}} \longrightarrow \mathsf{BA}_\wedge$, together with an additional map $\mathsf{next} : \Phi(T) \to \Phi(\mathbb{I})$ which preserves all Boolean operations.

This functor is supposed to meet the conditions (1) $\Phi(\mathbb{Q}) = \mathsf{Clp}_{\mathbb{Q}}$, (2) the functions $\Phi(\pi_i)$ and $\Phi([\mathrm{ev}(d)])$ are Boolean homomorphisms, and (3) the functions $\Phi(\kappa_i)$ induced by the injection paths satisfy (3a) $\Phi(\kappa_1)(\bot) = -\Phi(\kappa_2)(\bot)$ and (3b) $-\Phi(\kappa_i)(\bot) \wedge \Phi(\kappa_i)(-\alpha) \leq -\Phi(\kappa_i)(\alpha)$. ◁

Example 4.4 Let $\mathbb{A} = (A, \wedge, -, \bot, \top, g)$ be a modal algebra, cf. Definition 3.1. This algebra gives can be represented by two different $\mathbb{VI}$-BAOs. Note that $\mathrm{Ing}(\mathbb{VI}) = \{\mathbb{I}, \mathbb{VI}\}$ and $\mathbb{VI} \overset{\mathbb{V}}{\rightsquigarrow} \mathbb{I}$.

1. $\Phi(\mathbb{I}) := \mathbb{A}$, $\Phi(\mathbb{VI}) := \mathbb{A}$, $\Phi(\mathbb{V}) := g$, and $\mathsf{next} = id$.
2. $\Phi'(\mathbb{I}) := \mathbb{A}$, $\Phi'(\mathbb{VI}) := H\mathbb{A}$ (cf. Proposition 3.12), $\Phi'(\mathbb{V}) : \Phi'(\mathbb{A}) \hookrightarrow \Phi'(\mathbb{VI})$ the (meet-preserving) inclusion of generators, and next' the unique Boolean algebra morphism satisfying $\mathsf{next}' \circ \Phi'(\mathbb{V}) = g$.

We will see that (Φ', next') is the $\mathbb{VI}$-BAO obtained by considering the algebra (Φ, next) from (1) as a $\mathbb{VI}$-coalgebra and translating it back to an algebra, that is, in the notation we are about to introduce, $(\Phi', \mathsf{next}') = \mathcal{AC}(\Phi, \mathsf{next})$.

We leave it for the reader to verify that the following is the natural generalisation of the notion of a homomorphism between modal algebras.

Definition 4.5 (BAO_T) A *morphism* between T-BAOs $(\Phi', \mathsf{next}') \to (\Phi, \mathsf{next})$ is a natural transformation $t : \Phi' \to \Phi$ such that for each ingredient S of T the component $t_S : \Phi'(S) \to \Phi(S)$ preserves the Boolean structure, $t_{\mathbb{Q}} = \mathsf{id}_{\mathsf{Clp}_{\mathbb{Q}}}$ for all constants $\mathbb{Q} \in \mathsf{Ing}(T)$, and $t_{\mathbb{I}}$ and t_T satisfy $\mathsf{next} \circ t_T = t_{\mathbb{I}} \circ \mathsf{next}'$. This yields the category BAO_T. $\lhd$

It is not difficult to transform a T-coalgebra into a T-BAO; basically, we are dealing with a sorted version of Stone duality, together with a path-indexed predicate lifting. We omit the fairly straightforward proofs.

Lemma and Definition 4.6 *Let T be a VPF and let $\mathbb{X}$ be a Stone space. Then the following definition on the complexity of paths*

$$
\begin{array}{rcll}
\alpha^{<>} & := & \alpha & \\
\alpha^{\pi_1 \cdot p} & := & \pi_1^{-1}(\alpha^p) & \\
\alpha^{\pi_2 \cdot p} & := & \pi_2^{-1}(\alpha^p) & \\
\alpha^{\kappa_1 \cdot p} & := & \kappa_1(\alpha^p) \cup \kappa_2 S_2(X) & \textit{for } T_2 = S_1 + S_2 \\
\alpha^{\kappa_2 \cdot p} & := & \kappa_1 S_1(X) \cup \kappa_2(\alpha^p) & \textit{for } T_2 = S_1 + S_2 \\
\alpha^{[\mathsf{ev}(d)] \cdot p} & := & \pi_d^{-1}(\alpha^p) & \\
\alpha^{\mathbb{V} \cdot p} & := & \{\beta \mid \beta \subseteq \alpha^p \textit{ and } \beta \textit{ closed}\} & (= [\ni]\alpha)
\end{array}
$$

provides, for any path $T_1 \overset{p}{\rightsquigarrow} T_2$, a so-called predicate lifting

$$(\_)^p : \mathsf{Clp}_{T_2 \mathbb{X}} \to \mathsf{Clp}_{T_1 \mathbb{X}}.$$

Lemma 4.7 *For each Vietoris polynomial functor T, each T-coalgebra $(\mathbb{X}, c)$ gives rise to a T-BAO, namely, the 'complex algebra' functor $\mathcal{A}(\mathbb{X}, c) : \mathsf{Ing}(T)^{\mathrm{op}} \to \mathsf{BA}_{\wedge}$ given by*

$$
\begin{array}{rcl}
S & \mapsto & \mathbb{C}\mathsf{lp}\, S(\mathbb{X}) \\
\left(S_1 \overset{p}{\rightsquigarrow} S_2\right) & \mapsto & \left((\_)^p : \mathbb{C}\mathsf{lp}\, S_2(\mathbb{X}) \to \mathbb{C}\mathsf{lp}\, S_1(\mathbb{X})\right),
\end{array}
$$

accompanied by the map $\mathsf{next} : \mathbb{C}\mathsf{lp}(T\mathbb{X}) \to \mathbb{C}\mathsf{lp}(\mathbb{X})$ given by $\mathsf{next} := c^{-1}$.

Conversely, with each T-BAO Φ we may associate a T-coalgebra $\mathcal{C}(\Phi)$. Assume that T has the identity functor as an ingredient; given our results in the previous section, it seems fairly obvious that we should take the dual Stone space $\mathbb{S}\mathsf{p}\,\Phi(\mathbb{I})$ as the carrier of this dual coalgebra. However, how to obtain T-coalgebra structure on this? Applying duality theory to the Boolean algebras obtained from Φ only seems to provide information on the spaces $\mathbb{S}\mathsf{p}\,\Phi(S)$, whereas we need to work with $S(\mathbb{S}\mathsf{p}\,\Phi(\mathbb{I}))$ in order to define a T-coalgebra. Fortunately, in the next lemma and definition we show that there exists a map r which produces the S-structure. The definition of r is as in [17]; what we have to show is that it works also in the topological setting.

Lemma and Definition 4.8 *(r_Φ) Let T be a VPF and let (Φ, next) be a T-BAO. Then the following definition by induction on the structure of ingredient functors of T*

$$\begin{array}{rcl}
r_\Phi(\mathbb{I})(U) & := & U \\
r_\Phi(\mathbb{Q})(U) & := & a \quad \textit{where} \bigcap U = \{a\} \\
r_\Phi(S_1 \times S_2)(U) & := & \big\langle r_\Phi(S_1)(\Phi(\pi_1)^{-1}(U)), r_\Phi(S_2)(\Phi(\pi_2)^{-1}(U))\big\rangle \\
r_\Phi(S_1 + S_2)(U) & := & \left\{ \begin{array}{l} \kappa_1 r_\Phi(S_1)(\Phi(\kappa_1)^{-1}(U)) \;\textit{if}\; -\Phi(\kappa_1)(\bot) \in U \\ \kappa_2 r_\Phi(S_2)(\Phi(\kappa_2)^{-1}(U)) \;\textit{if}\; -\Phi(\kappa_2)(\bot) \in U \end{array} \right. \\
r_\Phi(S^D)(U) & := & \lambda d \in D. r_\Phi(S)(\Phi(ev(d))^{-1}(U)) \\
r_\Phi(\mathbb{V}S)(U) & := & \big\{ r_\Phi(S)(V) \mid V \in \mathsf{Sp}\,\Phi(S) \;\textit{and}\; \Phi(\mathbb{V})^{-1}(U) \subseteq V \big\}
\end{array}$$

defines, for every $S \in \mathrm{Ing}(T)$ a continuous map

$$r_\Phi(S) : \mathbb{S}\mathsf{p}(\Phi(S)) \longrightarrow S(\mathbb{S}\mathsf{p}(\Phi(\mathbb{I}))).$$

Furthermore, the inverse image map next^{-1} is a continuous map

$$\mathsf{next}^{-1} : \mathbb{S}\mathsf{p}(\Phi(\mathbb{I})) \longrightarrow T(\mathbb{S}\mathsf{p}(\Phi(\mathbb{I}))).$$

Proof of 4.8. Let $S \in \mathsf{Ing}(T)$. Both claims (i.e. the one on well-definedness and the one on the continuity of $r_\Phi(S)$) are proven simultaneously by induction on S.

We only consider the case of the Vietoris functor: assume that $S = \mathbb{V}S'$. In order to show that $r_\Phi(S)$ is well-defined, take an arbitrary $U \in \mathsf{Sp}\,\Phi(\mathbb{V}S')$ and consider the set $F := \big\{ V \mid V \in \mathsf{Sp}\,\Phi(S') \text{ and } \Phi(\mathbb{V})^{-1}(U) \subseteq V \big\}$. F is closed in $S'(\mathsf{Sp}\,\Phi(\mathbb{I}))$, because for any $V' \in S'(\mathsf{Sp}\,\Phi(\mathbb{I})) \setminus F$ there is an $a \in \Phi(\mathbb{V})^{-1}(U)$ such that $a \notin V'$, whence $F \subseteq \hat{a}$ and $V' \notin \hat{a}$: for every $V' \notin F$ we can find an open set containing V' and disjoint from F. But from F being closed and the inductive hypothesis on $r_\Phi(S')$ it follows that $r_\Phi(S')[F]$ is closed as well, so by definition, $r_\Phi(S)(U) = r_\Phi(S')[F]$ belongs to $K(S'(\mathsf{Sp}\,\Phi(\mathbb{I})))$. This proves that $r_\Phi(S)$ is well-defined.

We now turn to the continuity of $r_\Phi(S)$. It suffices to show that for an arbitrary clopen set $O \subseteq S'(\mathsf{Sp}\,\Phi(\mathbb{I}))$, all sets of the form $r_\Phi(\mathbb{V}S')^{-1}([\ni](O))$ and $r_\Phi(\mathbb{V}S')^{-1}(\langle\ni\rangle(O))$ are clopen. We only consider sets of the first kind:

$$\begin{array}{rcl}
r_\Phi(\mathbb{V}S')^{-1}([\ni](O)) & = & \{U \in \mathsf{Sp}\,\Phi(\mathbb{V}S') \mid r_\Phi(\mathbb{V}S')(U) \in [\ni](O)\} \\
& = & \big\{U \mid \{r_\Phi(S')(V) \mid \Phi(\mathbb{V})^{-1}(U) \subseteq V\} \subseteq O\big\} \\
& = & \big\{U \mid \{V \mid \Phi(\mathbb{V})^{-1}(U) \subseteq V\} \subseteq r_\Phi(S')^{-1}(O)\big\}
\end{array}$$

According to the induction hypothesis, $r_\Phi(S')^{-1}(O)$ is a clopen set, say with $b \in \Phi(S')$ such that $r_\Phi(S')^{-1}(O) = \hat{b}$. This leads us to

$$\begin{array}{rcl}
r_\Phi(\mathbb{V}S')^{-1}([\ni](O)) & = & \big\{U \mid \{V \mid \Phi(\mathbb{V})^{-1}(U) \subseteq V\} \subseteq \hat{b}\big\} \\
& = & \big\{U \mid \forall V \in \mathsf{Sp}\,\Phi(S') \,.\, \big(\Phi(\mathbb{V})^{-1}(U) \subseteq V \to b \in V\big)\big\} \\
& \overset{(!)}{=} & \{U \mid \Phi(\mathbb{V})(b) \in U\}
\end{array}$$

which shows that $r_\Phi(\mathbb{V}S')^{-1}([\ni](O))$ is clopen. The proof of (!) is standard in the representation theory of modal algebras.

Finally, the claim on the map next^{-1} is a simple consequence of Stone duality. QED

The above lemma allows us to define a T-coalgebra for a given T-BAO.

Definition 4.9 Let T be a VPF and let (Φ, next) be a T-BAO. We define the coalgebra $\mathcal{C}(\Phi, \mathsf{next})$ as the structure $(\mathbb{S}\mathsf{p}(\Phi(\mathbb{I})), r_\Phi(T) \circ \mathbb{S}\mathsf{p}(\mathsf{next}))$. ◁

The maps $\mathcal{A}$ and $\mathcal{C}$ that allow us to move from a given T-BAO to a T-coalgebra and vice versa, can be extended to functors. Fix a Vietoris polynomial functor T, and let $f : (\mathbb{X}, c) \to (\mathbb{X}', c')$ be a $\mathsf{Coalg}(T)$-morphism. Then we define $\mathcal{A}(f) : \mathcal{A}(\mathbb{X}', c') \to \mathcal{A}(\mathbb{X}, c)$ as follows. For each $S \in \mathrm{Ing}(T)$ let $\mathcal{A}(f)(S) := \mathbb{C}\mathsf{lp}(S(f))$. Naturality of $\mathcal{A}(f)$ can be proven by induction on paths and the additional condition in Definition 4.5 concerning the next functions is fulfilled because f is a T-coalgebra homomorphism.

Conversely, given a BAO_T-morphism $t : (\Phi, \mathsf{next}) \to (\Phi', \mathsf{next}')$, define the map $\mathcal{C}(t) : \mathsf{Sp}(\Phi'(\mathbb{I})) \to \mathsf{Sp}(\Phi(\mathbb{I}))$ to be the inverse image map of $t_{\mathbb{I}} : \Phi(\mathbb{I}) \to \Phi'(\mathbb{I})$. We leave it to the reader to verify that $\mathcal{C}(t)$ is in fact a $\mathsf{Coalg}(T)$ morphism between $\mathcal{C}(\Phi, \mathsf{next})$ and $\mathcal{C}(\Phi', \mathsf{next}')$ (cf. the proof of Proposition 5.3 in [17]).

Lemma 4.10 *If we extend $\mathcal{A}$ and $\mathcal{C}$ as described above we obtain functors*

$$\mathcal{A} : \mathsf{Coalg}(T)^{\mathrm{op}} \to \mathsf{BAO}_T \quad \textit{and} \quad \mathcal{C} : \mathsf{BAO}_T \to \mathsf{Coalg}(T)^{\mathrm{op}}.$$

5 Vietoris Polynomial Functors: representation and duality theorems

In the previous section we encountered functors

$$\mathcal{A} : \mathsf{Coalg}(T)^{\mathrm{op}} \to \mathsf{BAO}_T \qquad \text{and} \qquad \mathcal{C} : \mathsf{BAO}_T \to \mathsf{Coalg}(T)^{\mathrm{op}}.$$

Here we will study these functors in more detail, and show that in fact they provide an adjunction between the categories BAO_T and $\mathsf{Coalg}(T)$. We will define two families of morphisms, $\alpha_\Phi : \mathcal{AC}\Phi \to \Phi$ in BAO_T, and $\gamma_{(\mathbb{X},c)} : (\mathbb{X}, c) \to \mathcal{CA}(\mathbb{X}, c)$ in $\mathsf{Coalg}(T)^{\mathrm{op}}$; and prove that these are the unit and counit witnessing the fact that $\mathcal{A}$ is right adjoint to $\mathcal{C}$. Since the γ's will turn out to be isomorphisms, this will then show that $\mathsf{Coalg}(T)^{\mathrm{op}}$ is (isomorphic to) a full coreflective subcategory of BAO_T.

In contrast to the classical case of the duality $\mathsf{MA} \simeq \mathsf{DGF}^{\mathrm{op}}$, we do not obtain a dual equivalence between BAO_T and $\mathsf{Coalg}(T)$. This is due to the fact, which the reader might have noticed already, that the axiomatic definition of T-BAOs does not force a T-BAO Φ to respect T-structure. We take a closer look at this, characterising the largest full subcategory of BAO_T on which the adjunction restricts to an equivalence. By showing that the initial algebra of BAO_T is *exact*, that is, belongs to this subcategory, we obtain the final T-coalgebra as its dual.

We start by proving that every T-coalgebra has an 'ultrafilter representation': it is isomorphic to its double dual. Recall from Definition 2.4 that for a Stone space $\mathbb{Y}$, $\epsilon_{\mathbb{Y}} : \mathbb{Y} \to \mathbb{S}\mathsf{p}\mathbb{C}\mathsf{lp}\mathbb{Y}$ denotes the homeomorphism fixed by $\epsilon_{\mathbb{Y}}(y) := \{a \in \mathsf{Clp}_{\mathbb{Y}} \mid y \in a\}$.

Theorem 5.1 *Let T be a Vietoris polynomial functor, and let $(\mathbb{X}, c)$ be a T-coalgebra. Then the map $\epsilon_{\mathbb{X}} : \mathbb{X} \to \mathsf{Sp}(\mathsf{Clp}_{\mathbb{X}})$ is a $\mathsf{Coalg}(T)$-isomorphism witnessing that*

$$(\mathbb{X}, c) \cong \mathcal{C}(\mathcal{A}(\mathbb{X}, c)).$$

Proof. We first show that for each sort $S \in \mathsf{Ing}(T)$ the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Sp}\,\mathcal{A}(\mathbb{X},c)(S) \arrow[r, "r_{\mathcal{A}(\mathbb{X},c)}(S)"] & S(\mathsf{Sp}\,\mathcal{A}(\mathbb{X},c)(\mathbb{I})) \\
S\mathbb{X} \arrow[u, "\epsilon_{S\mathbb{X}}"] \arrow[ur, "S(\epsilon_{\mathbb{X}})"'] &
\end{tikzcd}$$

The proof id by induction on S. We will only treat the Vietoris functor, since all other cases work exactly as in the proof of Lemma 5.6 in [17]. In order to prove the commutativity of the above diagram for $S = \mathbb{V}S'$, take an arbitrary $F \in \mathbb{V}S'(\mathbb{X})$. Then, unravelling the definitions of r, $\mathcal{A}$ and of $(\cdot)^{\mathbb{V}}$, we find

$$\begin{aligned}
r_{\mathcal{A}(\mathbb{X},c)}(S)(\epsilon_{S\mathbb{X}}(F)) &= r_{\mathcal{A}(\mathbb{X},c)}(S')\left[\{V \mid \mathcal{A}(\mathbb{X},c)(\mathbb{V})^{-1}(\epsilon_{S\mathbb{X}}(F)) \subseteq V\}\right] \\
&= r_{\mathcal{A}(\mathbb{X},c)}(S')\left[\{V \mid \{\alpha \in \mathsf{Clp}_{S'\mathbb{X}} \mid (\alpha)^{\mathbb{V}} \in \epsilon_{S\mathbb{X}}(F)\} \subseteq V\}\right] \\
&= r_{\mathcal{A}(\mathbb{X},c)}(S')\left[\{V \mid \{\alpha \in \mathsf{Clp}_{S'\mathbb{X}} \mid F \subseteq \alpha\} \subseteq V\}\right] \\
&\overset{(!)}{=} \{S'(\epsilon_{\mathbb{X}})(u) \mid u \in F\} \\
&= S(\epsilon_{\mathbb{X}})(F).
\end{aligned}$$

It is left to prove (!). For ($\supseteq$), take an arbitrary $u \in F$, and define $V_u := \epsilon_{S'\mathbb{X}}(u)$. Then for all $a \in \mathsf{Clp}_{S'\mathbb{X}}$ it holds that $F \subseteq a$ implies $u \in a$, which is equivalent to $a \in \epsilon_{\mathbb{X}}(u) = V_u$; in other words,$V_u$ satisfies the condition $\{\alpha \mid F \subseteq \alpha\} \subseteq V_u$. Also, by the inductive hypothesis we have that $S'(\epsilon_{\mathbb{X}})(u) = r_{\mathcal{A}(\mathbb{X},c)}(S')(\epsilon_{S_1\mathbb{X}}(u))$. Taking these observations together we see that $S'(\epsilon_{\mathbb{X}})(u) \in r_{\mathcal{A}(c)}(S')\left[\{V \mid \{\alpha \in \mathsf{Clp}_{S'\mathbb{X}} \mid F \subseteq \alpha\} \subseteq V\}\right]$.

For ($\subseteq$), let $V \in \mathsf{Sp}\,\mathcal{A}(\mathbb{X},c)(S')$ be such that $\{\alpha \in \mathsf{Clp}_{S'\mathbb{X}} \mid F \subseteq \alpha\} \subseteq V$. By Stone duality we know that $\bigcap_{\alpha \in V} \alpha = \{u\}$ for exactly one $u \in S'\mathbb{X}$. This u must be an element of F, because $\bigcap_{\alpha \in V} \alpha \subseteq \bigcap\{\alpha \mid F \subseteq \alpha\} = F$ and we get $\epsilon_{S'\mathbb{X}}(u) = V$. By the induction hypothesis this is the same as saying $r_{\mathcal{A}(\mathbb{X},c)}(S')(V) = S'(\epsilon_X)(u)$, which proves the inclusion.

Now we proceed to prove the theorem: we calculate

$$\begin{aligned}
\mathcal{C}(\mathcal{A}(\mathbb{X},c)) \circ \epsilon_{\mathbb{X}} &= (r_{\mathcal{A}(\mathbb{X},c)}(T) \circ \mathbb{S}\mathsf{p}\mathbb{C}\mathsf{lp}(c)) \circ \epsilon_{\mathbb{X}} &&= r_{\mathcal{A}(\mathbb{X},c)}(T) \circ (\mathbb{S}\mathsf{p}\mathbb{C}\mathsf{lp}(c) \circ \epsilon_{\mathbb{X}}) \\
&= r_{\mathcal{A}(\mathbb{X},c)}(T) \circ (\epsilon_{T\mathbb{X}} \circ c) &&= (r_{\mathcal{A}(\mathbb{X},c)}(T) \circ \epsilon_{T\mathbb{X}}) \circ c,
\end{aligned}$$

where the third step is by naturality of ϵ. Now by commutativity of the above diagram for T we find that $\mathcal{C}(\mathcal{A}(\mathbb{X},c)) \circ \epsilon_{\mathbb{X}} = T(\epsilon_{\mathbb{X}}) \circ c$, which is nothing but stating that $\epsilon_{\mathbb{X}}$ is a coalgebra homomorphism. But then since $\epsilon_{\mathbb{X}}$ is an isomorphism between Stone spaces we may conclude that it is also an isomorphism between the two given coalgebras. QED

The functor $\mathcal{C}$ is not faithful in general; however, when it comes to morphisms having a complex algebra $\mathcal{A}(\mathbb{X},c)$ as their domain, we can prove the following.

Proposition 5.2 *Let $(\mathbb{X},c)$ be a T-coalgebra and Φ be a T-BAO. Furthermore let $v, v' : \mathcal{A}(\mathbb{X},c) \to \Phi$ be morphisms in BAO_T. Then $\mathcal{C}(v) = \mathcal{C}(v')$ implies $v = v'$.*

Proof. Let $(\mathbb{X},c)$, Φ, v and v' be as in the statement of the Proposition, and assume that $\mathcal{C}(v) = \mathcal{C}(v')$. Then it is clear that we have $v_{\mathbb{I}} = v'_{\mathbb{I}}$. With the help of Lemma 5.3 below we therefore get $v = v'$. QED

The following lemma, which forms the heart of the proof of Proposition 5.2, is stated separately because we need it again further on.

Lemma 5.3 *Let $(\mathbb{X}, c)$ be a T-coalgebra and Φ a T-BAO. Furthermore let $v, v' : \mathcal{A}(\mathbb{X}, c) \to \Phi$ be natural transformations such that their components preserve all the Boolean structure, $v_{\mathbb{I}} = v'_{\mathbb{I}}$ and $v_{\mathbb{Q}} = v'_{\mathbb{Q}}$ for all constants $\mathbb{Q} \in \mathsf{Ing}(T)$. Then $v = v'$.*

Proof. Assume that we have two natural transformations $v, v' : \mathcal{A}(\mathbb{X}, c) \to \Phi$ as required in the lemma. In order to prove that $v = v'$, it suffices to show that

$$v_S = v'_S \text{ for all } S \in \mathsf{Ing}(T). \tag{3}$$

We will prove (3) by induction on S. In the base case ($S = \mathbb{I}$ or $S = \mathbb{Q}$ for some constant functor $\mathbb{Q}$), it follows immediately that $v_S = v'_S$.

Unfortunately, for the inductive step of the proof, we only have enough space to give a rough sketch of the proof idea. In each case, in order to show that $v_S(U) = v_{S'}(U)$ for every clopen U of $S\mathbb{X}$, we try and find a clopen subbasis $\mathcal{B}$ such that $v_S(W) = v'_S(W)$ for all subbasic W. For instance, in the case that $S = \mathbb{V}S'$, put

$$\mathcal{B} := \left\{W \mid W \in (\_)^{\mathbb{V}}[\mathsf{Clp}_{S'\mathbb{X}}]\right\} \cup \left\{-W \mid W \in (\_)^{\mathbb{V}}[\mathsf{Clp}_{S'\mathbb{X}}]\right\},$$

and let $W \in \mathcal{B}$. Then one can easily check that we have $v_S(W) = v'_S(W)$ for all $W \in \mathcal{B}$ and by the fact that $\mathcal{B}$ is a clopen subbasis of the Vietoris topology one can use a straightforward argument to show that $v_S = v'_S$. QED

We are now ready to show that the functors $\mathcal{C} : \mathsf{BAO}_T \to \mathsf{Coalg}(T)^{\mathrm{op}}$ and $\mathcal{A} : \mathsf{Coalg}(T)^{\mathrm{op}} \to \mathsf{BAO}_T$ form a so-called dual representation. That is, $\mathcal{C}$ is right adjoint to $\mathcal{A}$ and the unit of the adjunction is an isomorphism. We first define the unit γ and the counit α of the adjunction. Recall that we proved in Theorem 5.1 that ϵ is an isomorphism; for r_Φ see Definition 4.8 and for $i_{\Phi(S)}$ Definition 2.4.

Definition 5.4 $(\boldsymbol{\alpha}, \boldsymbol{\gamma})$ For a T-BAO (Φ, next) and a $S \in \mathsf{Ing}(T)$ we define

$$\alpha_\Phi : \mathcal{AC}(\Phi) \to \Phi$$

via $\alpha_\Phi(S) := j_{\Phi(S)} \circ \mathbb{C}\mathsf{lp}(r_\Phi(S))$, where $j_{\Phi(S)}$ denotes the inverse of the isomorphism $i_{\Phi(S)} : \Phi(S) \to \mathbb{C}\mathsf{lp}\,\mathbb{S}\mathsf{p}\,\Phi(S)$.

For a T-coalgebra $(\mathbb{X}, c)$, we define

$$\gamma_{(\mathbb{X},c)} : (\mathbb{X}, c) \to \mathcal{CA}(\mathbb{X}, c) \quad \text{in } \mathsf{Coalg}(T)^{\mathrm{op}}$$

as the inverse $\gamma_{(\mathbb{X},c)} : \mathcal{CA}(\mathbb{X}, c) \to (\mathbb{X}, c)$ of the morphism $\epsilon_{(\mathbb{X},c)} : (\mathbb{X}, c) \to \mathcal{CA}(\mathbb{X}, c)$ in $\mathsf{Coalg}(T)$. $\lhd$

Intuitively, the next theorem establishes a duality between $\mathsf{Coalg}(T)$ and BAO_T in which every coalgebra $(\mathbb{X}, c)$ can be represented in a canonical way by the algebra $\mathcal{A}(\mathbb{X}, c)$.

Theorem 5.5 *Let T be a VPF. Then $\mathcal{A} : \mathsf{Coalg}(T)^{\mathrm{op}} \to \mathsf{BAO}_T$ is a full embedding and has $\mathcal{C} : \mathsf{BAO}_T \to \mathsf{Coalg}(T)^{\mathrm{op}}$ as a right adjoint with γ and α as unit and counit. That is, $\mathsf{Coalg}(T)^{\mathrm{op}}$ is (isomorphic to) a full coreflective subcategory of BAO_T.*

Before we turn to the proof of this theorem, we first show that α is indeed a morphism of T-BAOs.

Lemma 5.6 *The family of maps $\alpha_\Phi(\_) : \mathcal{AC}\Phi \to \Phi$ is a morphism of T-BAOs.*

Proof. We have to show that $\alpha_\Phi(\_)$ is a natural transformation and that $\alpha_\Phi(\_)$ fulfils an additional naturality condition with respect to the next-operator.

Concerning the first claim we must prove that for all $S \overset{p}{\rightsquigarrow} S'$ in $\mathsf{Ing}(T)$ we have

$$\Phi(p) \circ \alpha_{\Phi(S')} = \alpha_{\Phi(S)} \circ (\_)^p.$$

It suffices to show, by a case distinction, that this equation holds for paths of length at most one. As all of these proofs boil down to a tedious but straightforward unravelling of definitions, we confine ourselves to the case that $p = \mathbb{V}$ and $S = \mathbb{V}S_1$. Take an arbitrary $U \in \mathsf{Clp}_{S_1\,\mathsf{Sp}\,\Phi(\mathbb{I})}$ and let $a \in \Phi(S_1)$ be such that $\mathbb{C}\mathsf{lp}(r_\Phi(S_1))(U) = \hat{a}$. Then

$$\begin{array}{rcl}
\alpha_\Phi(S)((U)^{\mathbb{V}}) & = & (j_{\Phi(S)} \circ \mathbb{C}\mathsf{lp}(r_{\Phi(S)}))((U)^{\mathbb{V}}) \\
& = & (j_{\Phi(S)} \circ r^{-1}_{\Phi(S)})(\{\beta \subseteq U \mid \beta \subseteq S_1\,\mathsf{Sp}\,\Phi(\mathbb{I})\text{ closed}\}) \\
& = & j_{\Phi(S)}\left(\{u \in \mathsf{Sp}\,\Phi(S) \mid r_{\Phi(S)}(u) \subseteq U\}\right) \\
& = & j_{\Phi(S)}\left(\{u \in \mathsf{Sp}\,\Phi(S) \mid \{r_{\Phi(S_1)}(v) \mid \Phi(\mathbb{V})^{-1}(u) \subseteq v\} \subseteq U\}\right) \\
& = & j_{\Phi(S)}\left(\{u \in \mathsf{Sp}\,\Phi(S) \mid \{v \mid \Phi(\mathbb{V})^{-1}(u) \subseteq v\} \subseteq \mathbb{C}\mathsf{lp}(r_{\Phi(S_1)})(U)\}\right) \\
& = & j_{\Phi(S)}\left(\{u \in \mathsf{Sp}\,\Phi(S) \mid \{v \mid \Phi(\mathbb{V})^{-1}(u) \subseteq v\} \subseteq \hat{a}\}\right) \\
& = & j_{\Phi(S)}\left(\{u \in \mathsf{Sp}\,\Phi(S) \mid \Phi(\mathbb{V})^{-1}(u) \subseteq v \Rightarrow a \in v\}\right) \\
& = & j_{\Phi(S)}(\{u \in \mathsf{Sp}\,\Phi(S) \mid \Phi(\mathbb{V})(a) \in u\}) \\
& = & \Phi(\mathbb{V})(a) \\
& = & \Phi(\mathbb{V})\left(j_{\Phi(S_1)} \circ \mathbb{C}\mathsf{lp}(r_{\Phi(S_1)})(U)\right) \\
& = & \left(\Phi(\mathbb{V}) \circ \alpha_\Phi(S_1)\right)(U)
\end{array}$$

and we get $\alpha_\Phi(S) \circ (\_)^{\mathbb{V}} = \Phi(\mathbb{V}) \circ \alpha_\Phi(S_1)$, as required.

Now we turn to the second claim. The 'additional naturality condition with respect to the next-operator' is the following: $\mathsf{next} \circ \alpha_\Phi(T) = \alpha_\Phi(\mathbb{I}) \circ \mathbb{C}\mathsf{lp}(r_\Phi(T) \circ \mathbb{S}\mathsf{p}\,\mathsf{next})$. This is easily shown to hold (the second identity being due the naturality of j).

$$\begin{array}{rcl}
\alpha_\Phi(\mathbb{I}) \circ \mathbb{C}\mathsf{lp}(r_\Phi(T) \circ \mathbb{S}\mathsf{p}(\mathsf{next})) & = & j_{\Phi(\mathbb{I})} \circ \mathbb{C}\mathsf{lp}(\mathbb{S}\mathsf{p}(\mathsf{next})) \circ \mathbb{C}\mathsf{lp}(r_{\Phi(T)}) \\
& = & \mathsf{next} \circ j_{\Phi(T)} \circ \mathbb{C}\mathsf{lp}(r_\Phi(T)) \\
& = & \mathsf{next} \circ \alpha_\Phi(T).
\end{array}$$

QED

Proof of Theorem 5.5. For the adjunction ([23], p. 81), we show that for all $(\mathbb{X}, c) \in \mathsf{Stone}$ and for all $u : \mathcal{C}(\Phi) \to (\mathbb{X}, c)$ there is a unique $v : \mathcal{A}(\mathbb{X}, c) \to \Phi$ such that the following diagram in $\mathsf{Coalg}(T)$ commutes:

$$\begin{tikzcd}
\mathcal{CA}(\mathbb{X},c) \arrow[r, "\gamma_{\mathbb{X}}"] & (\mathbb{X},c) \\
\mathcal{C}\Phi \arrow[u, "\mathcal{C}(v)"] \arrow[ur, "u"'] &
\end{tikzcd}$$

Indeed, defining $v = \alpha_\Phi \circ \mathcal{A}(u)$, we calculate

$$\begin{array}{rcl}
\gamma_{\mathbb{X}} \circ \mathcal{C}(\alpha_\Phi \circ \mathcal{A}(u)) & = & \gamma_{\mathbb{X}} \circ \mathbb{S}\mathsf{p}\,(\alpha_\Phi(\mathbb{I}) \circ \mathcal{A}(u)(\mathbb{I})) \\
& = & \gamma_{\mathbb{X}} \circ \mathbb{S}\mathsf{p}(j_{\Phi(\mathbb{I})} \circ r_\Phi(\mathbb{I}) \circ \mathbb{C}\mathsf{lp}(u)) \\
& = & \gamma_{\mathbb{X}} \circ \mathbb{S}\mathsf{p}(\mathbb{C}\mathsf{lp}(u)) \circ \mathbb{S}\mathsf{p}(j_{\Phi(\mathbb{I})}) \\
& = & u \circ \gamma_{\mathbb{S}\mathsf{p}(\Phi(\mathbb{I}))} \circ \mathbb{S}\mathsf{p}(j_{\Phi(\mathbb{I})}) \\
& = & u
\end{array}$$

The last two steps use that $\mathbb{S}\mathsf{p}$ and $\mathbb{C}\mathsf{lp}$ are adjoint with (co)units j and γ, see Definitions 2.4 and 5.4. Uniqueness of v is Proposition 5.2. To conclude the proof, recall that a left-adjoint is full and faithful iff the unit is iso ([23], p. 88). Hence $\mathcal{A}$ is full and faithful by Theorem 5.1. It remains to observe that $\mathcal{A}$ is injective on objects. QED

We now turn to a characterisation of the largest subcategory of BAO_T on which the adjunction from Theorem 5.5 restricts to a dual equivalence. The reader might have noticed already that our adjunction is not a dual equivalence since the definition of T-BAOs does not force a T-BAO Φ to respect T-structure. For example, if $S_1 \times S_2$ is an ingredient of T then it may well be that $\Phi(S_1 \times S_2) \neq \Phi(S_1) + \Phi(S_2)$.

Definition 5.7 Let S be a functor $\mathsf{Stone} \to \mathsf{Stone}$. Then

$$\hat{S} := \mathbb{C}\mathsf{lp} \circ S \circ \mathbb{S}\mathsf{p}.$$

defines a corresponding functor $\hat{S}$ on the category BA. $\lhd$

The following definition introduces *exact* T-BAOs, that is, those T-BAOs which do respect T-structure.

Definition 5.8 (exact T-BAO) A T-BAO Φ is called *exact* if there is a family of isomorphisms

$$\tau_S : \hat{S}(\Phi(\mathbb{I})) \to \Phi(S)$$

with the following properties:

- $\tau : \hat{(\_)}(\Phi(\mathbb{I})) \to \Phi$ is a natural transformation, where $\hat{(\_)}(\Phi(\mathbb{I})) : \mathsf{Ing}(T)^{\mathrm{op}} \to \mathsf{BA}_\wedge$ is defined on objects as in Definition 5.7 and on paths $p : S_1 \overset{p}{\rightsquigarrow} S_2$ as $(\_)^p$ in Definition 4.6 (with $\mathbb{X}$ here being $\Phi(\mathbb{I})$).
- $\tau_{\mathbb{I}} = j_{\Phi(\mathbb{I})}$, where again $j_{\Phi(\mathbb{I})}$ denotes the inverse of the isomorphism $i_{\Phi(\mathbb{I})} : \Phi(\mathbb{I}) \to \mathbb{C}\mathsf{lp}\,\mathbb{S}\mathsf{p}\,\Phi(\mathbb{I})$
- $\tau_{\mathbb{Q}} = \mathsf{id}_{\mathsf{Clp}_{\mathbb{Q}}}$ for every constant $\mathbb{Q} \in \mathsf{Ing}(T)$.

BAO^e_T is the full subcategory of BAO_T consisting of the exact T-BAOs. $\lhd$

We will now see that exact T-BAOs are precisely those T-BAOs Φ for which the component α_Φ of the counit of the adjunction is an isomorphism.

Theorem 5.9 *Let T be a VPF. The category BAO^e_T is the largest subcategory of BAO_T on which the adjunction of Theorem 5.5 restricts to a dual equivalence to $\mathsf{Coalg}(T)$.*

Proof. Let $\mathcal{B}$ be the largest subcategory of BAO_T on which the adjunction $\mathcal{A} \dashv \mathcal{C}$ restricts to an equivalence. Then for any $\Phi \in \mathcal{B}$ the map $\alpha_\Phi : \mathcal{AC}\Phi \to \Phi$ consists of a family of isomorphisms going from $\mathcal{AC}\Phi(S) = \hat{S}(\Phi)(\mathbb{I})$ to $\Phi(S)$. Therefore we can define a family of isomorphisms $\tau_S : \hat{S}(\Phi)(\mathbb{I}) \to \Phi(S)$ by letting $\tau = \alpha_\Phi$. It is straightforward to check that this family satisfies the conditions in Definition 5.8. Hence $\Phi \in \mathsf{BAO}^e_T$.

Now let $\Phi \in \mathsf{BAO}^e_T$. We have to show that the counit α_Φ is an isomorphism. As $\Phi \in \mathsf{BAO}^e_T$ there is a family of isomorphisms

$$\tau_S : (\mathcal{AC}\Phi)(S) \to \Phi(S).$$

which is natural in S and for which we have $\tau_{\mathbb{I}} = j_{\Phi(\mathbb{I})} = \alpha_\Phi(\mathbb{I})$ and $\tau_{\mathbb{Q}} = \mathsf{id}_{\mathsf{Clp}_{\mathbb{Q}}} = \alpha_\Phi(\mathbb{Q})$ for all constants $\mathbb{Q} \in \mathsf{Ing}(T)$. Using Lemma 5.3 one can therefore show that $\tau_S = \alpha_S$ for all $S \in \mathsf{Ing}(T)$. But this means in particular that α_Φ is an isomorphism. QED

We now show that the final object in $\mathsf{Coalg}(T)$ is obtained as the dual of the initial object in BAO_T. This is a direct consequence of Theorem 5.5 and a special case of the more general fact that the right adjoint $\mathcal{C}$ preserves colimits of diagrams that take values in the $\mathcal{A}$-image of $\mathsf{Coalg}(T)^{\mathrm{op}}$.

Theorem 5.10 *Let T be a VPF and $\mathcal{L}_T$ be the initial object in BAO_T. Then $\mathcal{C}\mathcal{L}_T$ is final in $\mathsf{Coalg}(T)$.*

Proof. We prove the theorem by showing that $\alpha_{\mathcal{L}_T}$ is an isomorphism, i.e. $\mathcal{L}_T \in \mathsf{BAO}^e_T$. Finality of $\mathcal{C}\mathcal{L}_T$ follows then immediately from the duality between $\mathsf{Coalg}(T)$ and BAO^e_T.

Since $\mathcal{L}_T$ is initial there is a morphism $m : \mathcal{L}_T \to \mathcal{A}\mathcal{C}\mathcal{L}_T$. Since $id_{\mathcal{L}_T}$ is the unique morphism $\mathcal{L}_T \to \mathcal{L}_T$ it follows that $\alpha_{\mathcal{L}_T} \circ m = id_{\mathcal{L}_T}$. We want to show that $m \circ \alpha_{\mathcal{L}_T} : \mathcal{A}\mathcal{C}\mathcal{L}_T \to \mathcal{A}\mathcal{C}\mathcal{L}_T$ is in fact the identity on $\mathcal{A}\mathcal{C}\mathcal{L}_T$. Since $\mathcal{A}$ is full (cf. Theorem 5.5) there is $f : \mathcal{C}\mathcal{L}_T \to \mathcal{C}\mathcal{L}_T$ in $\mathsf{Coalg}(T)$ such that $\mathcal{A}(f) = m \circ \alpha_{\mathcal{L}_T}$. We obtain $\alpha_{\mathcal{L}_T} \circ \mathcal{A}(f) = \alpha_{\mathcal{L}_T} \circ m \circ \alpha_{\mathcal{L}_T} = \alpha_{\mathcal{L}_T} = \alpha_{\mathcal{L}_T} \circ \mathcal{A}(id_{\mathcal{C}\mathcal{L}_T})$ and the universal property of the coreflection tells us that $f = id_{\mathcal{C}\mathcal{L}_T}$, hence, $id_{\mathcal{A}\mathcal{C}\mathcal{L}_T} = m \circ \alpha_{\mathcal{L}_T}$ and $\alpha_{\mathcal{L}_T}$ is iso. QED

Theorem 5.10 indicates that we can now use the standard Stone duality approach to prove completeness for multi-sorted coalgebraic logic ([28, 17]). Lacking the space to go into any kind of detail, let us just mention that the initial algebra $\mathcal{L}_T$ is in fact the *Lindenbaum-Tarski* algebra of the multi-sorted modal logic of Jacobs. From the properties of such algebras and Theorem 5.10 it follows immediately that any non-theorem of this logic can in fact be refuted on the final coalgebra of $\mathsf{Coalg}(T)$.

Remark 5.11 In [17], Jacobs states a similar final coalgebra theorem for set-based Kripke polynomial functors. Unfortunately, there is a defect in his proof. The problem involves his functor $\mathcal{C} : \mathsf{BAO}_{T_J} \to \mathsf{Coalg}(T_J)^{\mathrm{op}}$. Note that Jacobs's functor T_J is the set-based analogue of our T. (To obtain Jacobs's T_J from ours, simply replace all occurrences of the Vietoris functor with the power set functor, and interpret all polynomial functors occurring in T in the standard way.) Thus Jacobs studies the relation between T_J-BAOs and *set-based* T_J-coalgebras. However, as mentioned already, on the algebraic side, we may identify T_J-BAOs with T-BAOs. Thus we may compare Jacobs's way of relating BAO_T with the Set-based $\mathsf{Coalg}(T_J)$ to our way of relating BAO_T to the Stone-based $\mathsf{Coalg}(T)$. Jacobs assigns a modal logic to each Kripke polynomial functor, and he proves that the coalgebras for these functors form a sound and complete semantics for these logics. In order to obtain the final coalgebra for a so-called *finite* KPF T, that is, a KPF which may only contain the finite-power set functor, he maps the Lindenbaum-Tarski algebra $\mathcal{L}_T$ to its corresponding coalgebra $\mathcal{C}(\mathcal{L}_T)$, using the above-mentioned functor $\mathcal{C}$. His construction works for functors $\mathcal{C}$ mapping T-BAOs, for a finite KPF T, to a T-coalgebra. This is however only the case for functors T not containing the finite power set functor $\mathcal{P}_\omega$ (Jacobs's construction maps a $(\mathcal{P}_\omega\mathbb{I})$-BAO to a $\mathcal{P}$-coalgebra and not to a $\mathcal{P}_\omega$-coalgebra). This means that Jacobs's construction of final objects in $\mathsf{Coalg}(T)$ works only for Kripke polynomial functors that do not contain the power set functor or its finitary version. Moving from the category of sets to Stone enables us to repair this defect.

6 Conclusions

What we have done so far can be viewed from various distinct perspectives. Here we summarise some of these, indicating possible future research directions.

Stone Coalgebras and Modal Logic Research on the relation between coalgebras and modal logic started with Moss [25] although earlier work, e.g. by Rutten [29] already showed that Kripke frames and models are instances of coalgebras. Kurz [20, 19] showed that modal logic for coalgebras dualises equational logic for algebras, the idea being that equations describe quotients of free algebras and modal formulae describe subsets of final (or cofree) coalgebras. Another account of the duality has been given in [22] where it was shown that modalities dualise algebraic operations. But whereas, usually, any quotient of a free algebra can be defined by a set of ordinary equations, one needs *infinitary* modal formulae to define all subsets of a final coalgebra. As a consequence, while we have a satisfactory description of the coalgebraic semantics of infinitary modal logics, we do not completely understand the relationship between coalgebras and finitary modal logic. The results in this paper show that Stone coalgebras provide a natural and adequate semantics for finitary modal logics, but there is ample room for clarification here.

Another approach to a coalgebraic semantics for finitary modal logics was given in [21]. There, the idea is to modify coalgebra morphisms in such a way that they capture not bisimulation but only bisimulation up to rank ω. Since finitary modal logics capture precisely bisimulation up to rank ω, the resulting category Beh_ω provides a convenient framework to study the coalgebraic semantics of finitary modal logic. So an important next step is to understand the relation between both approaches.

Stone Coalgebras as Systems We investigated coalgebras over Stone spaces as models for modal logic. But what is the significance of Stone-coalgebras from the point of view of systems (that is, coalgebras over Set, cf. [30])? What is the relationship between Set-coalgebras and Stone-coalgebras? An interesting observation is here that their notions of behavioural equivalence coincide. Recall that two elements of two coalgebras are behaviourally equivalent iff they can be identified by some coalgebra morphisms. Since Stone-coalgebra morphisms have to be continuous, we expect that less states are identified under Stone-behavioural equivalence than under Set-behavioural equivalence. But the following holds.

Consider a Vietoris polynomial functor $T : \mathsf{Stone} \to \mathsf{Stone}$ and its corresponding (Kripke polynomial) functor $\breve{T} : \mathsf{Set} \to \mathsf{Set}$. There is an obvious functor $F : \mathsf{Coalg}(T) \to \mathsf{Coalg}(\breve{T})$. Now let $(\mathbb{X}_1, c_1)$, $(\mathbb{X}_2, c_2)$ be two T-coalgebras and x_1, x_2 be two elements in $\mathbb{X}_1$, $\mathbb{X}_2$, respectively. Then $((\mathbb{X}_1, c_1), x_1)$ and $((\mathbb{X}_2, c_2), x_2)$ are behaviourally equivalent iff $(F(\mathbb{X}_1, c_1), x_1)$ and $(F(\mathbb{X}_2, c_2), x_2)$ are behavioural equivalent. — Proof: 'only if' is immediate. The converse follows from the fact that the final T-coalgebra appears as the ω-limit of the terminal sequence (see [33]) of $\breve{T}$.

Generalising Stone Coalgebras Coalgebras over Stone spaces can be generalised in different ways. We have seen that replacing the topologies by represented Boolean algebras leads to general frames. But it will also be of interest to consider other topological spaces as base categories.

From the point of view of modal logic, it will be interesting to investigate the Vietoris functors on other base categories. For example, [26] shows that the Vietoris functor can be defined on Priestley spaces, leading to an adequate semantics for positive modal logic.

From the point of view of the theory of coalgebras, the value of the move from Set to Stone as a base category can be explained as follows. For a functor on Set the notion of behavioural equivalence is, in general, characterised by the whole terminal sequence running through all ordinals. But often, one is interested only in finitary approximations. In the examples considered in this paper, the move from a functor on Set to its version on Stone has the consequence that the final coalgebra is the limit of the finitary approximants of the terminal sequence (and, therefore, behavioural equivalence is completely characterised by the finitary approximants of the terminal sequence). We expect that this idea of topologising a functor T in order to tailor the behaviour of T-coalgebras to meet a specific notion of observable behaviour will have further applications to universal coalgebra.

Compared to work in domain theory, the perspective taken in universal coalgebra differs in the following way. First, the models of interest are not only the final but all T-coalgebras (and all T-BAOs). Second, we are not necessarily interested in modelling function space but typically focus on polynomial functors. This allows to work in simpler categories than usually considered in domain theory. An interesting advantage is that, for set coalgebras, there are no problems in combining non-deterministic and probabilistic type constructors (for a recent overview see [7]). It would be interesting to see how probability can be treated in a topological setting.

Coalgebras and Duality Theory Whereas many, or most, common dualities are induced by a schizophrenic object (see [18], Section VI.4.1), the duality of modal algebras and descriptive general frames is not. To see why this is so, write $K : \mathsf{MA} \to \mathsf{DGF}$, $L : \mathsf{DGF} \to \mathsf{MA}$ for the contravariant functors witnessing the duality and suppose, for contradiction that there is a schizophrenic object S. That is, assume that $\mathsf{MA}(\mathbb{A}, S) \cong UK(\mathbb{A})$ where U denotes the forgetful functor $\mathsf{DGF} \to \mathsf{Set}$. Then $\mathsf{Set}(1, U\mathbb{G}) \cong U\mathbb{G} \cong UKL\mathbb{G} \cong \mathsf{MA}(L\mathbb{G}, S) \cong \mathsf{DGF}(KS, KL\mathbb{G}) \cong \mathsf{DGF}(KS, \mathbb{G})$, showing that KS is a free object over one generator in DGF. But since DGF-morphisms are also bisimulations it is not hard to see that such an object cannot exists.

On the other hand, this duality is an instance of the duality $\mathsf{Alg}(T^{\mathrm{op}}) \cong \mathsf{Coalg}(T)^{\mathrm{op}}$ of algebras and coalgebras, with the Vietoris functor $\mathbb{V}$ as the functor T. It seems therefore of interest to explore which dualities are instances of the algebra/coalgebra duality. As a first step in this direction, [26] shows that the duality between positive modal algebras and K^+-spaces can be described in a similar way as in Section 3 (although the technical details are substantially more complicated).

References

[1] S. Abramsky. A Cook's Tour of the Finitary Non-Well-Founded Sets. Invited Lecture at BCTCS 1988.

[2] S. Abramsky. A domain equation for bisimulation. *Information and Computation*, 92(2):161–218, 1991.

[3] S. Abramsky. Domain theory in logical form. *Annals of Pure and Applied Logic*, 51:1–77, 1991.

[4] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science Volume 3*, pages 1–168. Oxford University Press, 1994.

[5] F. Alessi, P. Baldan, and F. Honsell. Partializing Stone Spaces Using SFP Domains (extended abstract). In M. Bidoit et al, editor, *CAAP '97*, volume 1158 of *LNCS*, pages 478–489. Springer, 1997.

[6] A. Baltag. A logic for coalgebraic simulation. In *Coalgebraic Methods in Computer Science (CMCS'00)*, volume 33 of *Electronic Notes in Theoretical Computer Science*, pages 41–60, 2000.

[7] F. Bartels, A. Sokolova, and E. de Vink. A hierarchy of probabilistic system types. In H.P. Gumm, editor, *Coalgebraic Methods in Computer Science (CMCS'03)*, volume 86.1 of *ENTCS*. Elsevier, 2003.

[8] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, 2001.

[9] C. Cîrstea. On specification logics for algebra-coalgebra structures: Reconciling reachability and observability. In M. Nielsen, editor, *Proceedings of FoSSaCS 2002*, volume 2303 of *LNCS*, pages 82–97. Springer, 2002.

[10] B. Davey and J. Galati. A coalgebraic view of Heyting duality. *Studia Logica*. To appear.

[11] J. de Bakker and E. de Vink. *Control Flow Semantics*. MIT Press, 1996.

[12] R. Engelking. *General Topology*. Heldermann Verlag, 1989.

[13] L.L. Esakia. Topological Kripke models. *Soviet Mathematics Doklady*, 15:147–151, 1974.

[14] M. Gehrke. The order structure of stone spaces and the t_d-separation axiom. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 37:5–15, 1991.

[15] R. Goldblatt. A calculus of terms for coalgebras of polynomial functors. In A. Corradini, M. Lenisa, and U. Montanari, editors, *Coalgebraic Methods in Computer Science (CMCS'01)*, volume 44 of *Electronic Notes in Theoretical Computer Science*. Elsevier, 2001.

[16] R.I. Goldblatt. Metamathematics of modal logic I. *Reports on Mathematical Logic*, 6:41–78, 1976.

[17] B. Jacobs. Many-sorted coalgebraic modal logic: a model-theoretic study. *Theoretical Informatics and Applications*, 35(1):31–59, 2001.

[18] P. Johnstone. *Stone Spaces*. Cambridge University Press, 1982.

[19] A. Kurz. *Logics for Coalgebras and Applications to Computer Science*. PhD thesis, Ludwig-Maximilians-Universität München, 2000.

[20] A. Kurz. A co-variety-theorem for modal logic. In M. Zakharyaschev et alii, editor, *Advances in Modal Logic 2*. CSLI, Stanford University, 2001.

[21] A. Kurz and D. Pattinson. Coalgebraic modal logic of finite rank. *Mathematical Structures in Computer Science*. Forthcoming.

[22] A. Kurz and J. Rosický. Modal predicates and coequations. In L. Moss, editor, *Proceedings of CMCS 2002*, volume 65 of *Electronic Notes in Theoretical Computer Science*. Elsevier, 2002.

[23] S. Mac Lane. *Category Theory for the Working Mathematician*. Springer, 1971.

[24] E. Michael. Topologies on spaces of subsets. *Transactions of the American Mathematical Society*, 71:152–182, 1951.

[25] L. Moss. Coalgebraic logic. *Annals of Pure and Applied Logic*, 96:277–317, 1999.

[26] A. Palmigiano. Coalgebraic semantics for positive modal logic. In H.-P. Gumm, editor, *Coalgebraic Methods in Computer Science (CMCS'03)*, volume 82 of *Electronic Notes in Theoretical Computer Science*, 2003.

[27] D. Pattinson. Semantical principles in the modal logic of coalgebras. In *Proceedings 18th International Symposium on Theoretical Aspects of Computer Science (STACS 2001)*, volume 2010 of *LNCS*, Berlin, 2001. Springer.

[28] M. Rößiger. Coalgebras and modal logic. In H. Reichel, editor, *Electronic Notes in Theoretical Computer Science*, volume 33. Elsevier, 2000.

[29] J.J.M.M. Rutten. A calculus of transition systems (towards universal coalgebra). In A. Ponse, M. de Rijke, and Y. Venema, editors, *Modal Logic and Process Algebra*, volume 53 of *CSLI Lecture Notes*. CSLI, Stanford University, 1995.

[30] J.J.M.M. Rutten. Universal coalgebra: A theory of systems. *Theoretical Computer Science*, 249:3–80, 2000.

[31] G. Sambin and V. Vaccaro. Topology and duality in modal logic. *Annals of Pure and Applied Logic*, 37:249–296, 1988.

[32] D. Turi and J. Rutten. On the foundations of final coalgebra semantics: non-well-founded sets, partial orders, metric spaces. *Mathematical Structures in Computer Science*, 8, 1998.

[33] J. Worrell. On the final sequence of a finitary set functor. *Theoretical Computer Science*. Forthcoming.