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Stone Coalgebras

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Abstract

In this paper we argue that the category of Stone spaces forms an interesting base category for coalgebras, in particular, if one considers the Vietoris functor as an analogue to the power set functor. We prove that the so-called descriptive general frames, which play a fundamental role in the semantics of modal logics, can be seen as Stone coalgebras in a natural way. This yields a duality between the category of modal algebras and that of coalgebras over the Vietoris functor. Building on this idea, we introduce the notion of a Vietoris polynomial functor over the category of Stone spaces. For each such functor T we establish a link between the category of T-sorted Boolean algebras with operators and the category of Stone coalgebras over T. Applications include a general theorem providing final coalgebras in the category of T-coalgebras.

Key words: coalgebra, Stone spaces, Vietoris topology, modal logic, descriptive general frames, Kripke polynomial functors

1 Introduction

Technically, every coalgebra is based on a carrier which itself is an object in the so-called base category. Most of the literature on coalgebras either focuses on Set as the base category, or takes a very general perspective, allowing arbitrary base categories, possibly restricted by some constraints. The aim of this paper is to argue that, besides Set, the category Stone of Stone spaces is of relevance as a base category. We have a number of reasons for believing that *Stone coalgebras*, that is, coalgebras based on Stone, are of interest.

To start with, in Section 3 we discuss interesting examples of Stone coalgebras, namely the ones that are associated with the *Vietoris functor* $\mathbb{V}$: $\mathsf{Stone} \to \mathsf{Stone}$. This $\mathbb{V}$ is the functorial extension of the Vietoris construction

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which is a well-known topological analogue of the power set construction: the Vietoris topology of a topology τ is based on the collection of sets that are closed in τ [9]. This construction preserves a number of nice topological properties; in particular, it turns Stone spaces into Stone spaces [16]. As we will see further on, the category $\mathsf{Coalg}(\mathbb{V})$ of coalgebras over this Vietoris functor is of interest because it is isomorphic to the category DGF of descriptive general frames. This category in its turn is dual to that of modal algebras, and hence, unlike Kripke frames, descriptive general frames form a mathematically adequate semantics for modal logics [6].

The connection with modal logic thus forms a second reason as to why Stone coalgebras are of interest. Since coalgebras can be seen as a very general model of state-based dynamics, and modal logic as a logic for dynamic systems, the relation between modal logic and coalgebras is rather tight. Starting with the work of Moss [22], this has been an active research area [25,15,5,24,13,7]. The relation between modal logic and coalgebras can be seen to dualize that between equational logic and algebra [21,20], an important difference being that the relation with Set-based coalgebras seems to work smoothly only for modal languages that allow infinitary formulas. In the case of the Vietoris functor however, it follows from the duality between $\mathsf{Coalg}(\mathbb{V})$ and the category MA of modal algebras, that $\mathsf{Coalg}(\mathbb{V})$ provides a natural semantics for *finitary* modal logics. Section 4 substantiates this by applying the duality of Section 3 to many-sorted coalgebraic modal logic in the style of Jacobs [15].

Let us add two more observations. First, the duality of descriptive general frames and modal algebras shows that the (trivial) duality between the categories $\mathsf{Coalg}(T)$ and $\mathsf{Alg}(T^{\mathrm{op}})$ has non-trivial instances. Second, it might be interesting to note that Stone provides a meaningful example of a base category for coalgebras which is not finitely locally presentable.

Before we turn to the technical details of the paper, we want to emphasize that in our opinion the main value of this paper lies not so much in the technical contributions; in fact, many of the results that we list are known, or could be obtained by standard methods from known results. The interest of this work, we believe, rather lies in the fact that these results can be grouped together in a natural, coalgebraic light.

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2 Preliminaries

The paper presupposes some familiarity with category theory, general topology and the theory of boolean algebras. The main purpose of this section is to fix our notation and terminology.

Definition 2.1 (Coalgebras) Let C be a category and $T : \mathsf{C} \to \mathsf{C}$ an endofunctor. Then a T-coalgebra is a pair $(X, \xi : X \to TX)$ where X denotes an object of C and ξ a morphism of C. A T-coalgebra morphism $h : (X_1, \xi_1) \to (X_2, \xi_2)$ is a C-morphism $h : X_1 \to X_2$ satisfying $\xi_2 \circ h = Th \circ \xi_1$. The category $\mathsf{Coalg}(T)$ has T-coalgebras as its objects and T-coalgebra morphisms as arrows. Dually, we define a T-algebra to be a T^{op}-coalgebra and $\mathsf{Alg}(T) = (\mathsf{Coalg}(T^{\mathrm{op}}))^{\mathrm{op}}$.

Example 2.2 (Kripke frames) A *Kripke frame* is a structure $\mathbb{F} = (X, R)$ such that R is a binary relation on X. It is by now well-known that Kripke frames can be seen as coalgebras for the power set functor $\mathcal{P}$ over Set. The idea here is to replace the binary relation R of a frame $\mathbb{F} = (X, R)$ with the map $R[\_] : X \to \mathcal{P}(X)$ given by

$$R[s] := \{t \in X \mid Rst\}.$$

In fact, Kripke frames (and models) form some of the prime examples of coalgebras — many coalgebraic concepts have been developed as generalizations of notions applying to Kripke structures. This applies for instance to the notion of a *bounded morphism* between Kripke frames; we will use this terminology for $\mathcal{P}$-coalgebra morphisms.

Definition 2.3 (Stone spaces) A topological space $\mathbb{X} = (X, \tau)$ is called a Stone space if τ is a compact Hausdorff topology which is in addition zero-dimensional, that is, it has a basis of clopen sets. The category Stone of Stone spaces has as its objects Stone spaces and as its morphisms the continuous functions between them.

Definition 2.4 (Stone duality) The category of Boolean algebras with homomorphisms is denoted as BA.

The Stone space $(\mathsf{Sp}\,\mathbb{B}, \tau_{\mathbb{B}})$ dual to a Boolean algebra $\mathbb{B}$ is given by the collection $\mathsf{Sp}\,\mathbb{B}$ of ultrafilters of $\mathbb{B}$ and the topology $\tau_{\mathbb{B}}$ generated by basic opens of the form $\{u \in \mathsf{Sp}\,\mathbb{B} \mid b \in u\}$ for any b in $\mathbb{B}$. We let $\mathbb{S}\mathsf{p}$ denote the functor that associates with a Boolean algebra its dual Stone space, and with a Boolean homomorphism its inverse image function.

Conversely, if $\mathbb{X}$ is a Stone space we denote by $\mathsf{Clp}_{\mathbb{X}}$ the set of clopen subsets of X; the functor mapping a Stone space to the Boolean algebra of its clopens, and a continuous morphism to its inverse image function, is denoted as $\mathbb{C}\mathsf{lp}$.

It is well-known that the functors $\mathbb{S}\mathsf{p}$ and $\mathbb{C}\mathsf{lp}$ induce a dual equivalence between the categories Stone and BA.

Definition 2.5 (Vietoris topology) Let $\mathbb{X} = (X, \tau)$ be a topological space. We let $K(\mathbb{X})$ denote the collection of all closed subsets of X. We define the operations $[\ni], \langle\ni\rangle : \mathcal{P}(X) \to \mathcal{P}(K(\mathbb{X}))$ by

$$[\ni]U := \{F \in K(X) \mid F \subseteq U\},$$
$$\langle\ni\rangle U := \{F \in K(X) \mid F \cap U \neq \varnothing\}.$$

Given a subset Q of $\mathcal{P}(X)$, define

$$V_Q \ := \ \{[\ni]U \mid U \in Q\} \cup \{\langle\ni\rangle U \mid U \in Q\}\,.$$

The Vietoris space $\mathbb{V}(\mathbb{X})$ associated with $\mathbb{X}$ is given by the topology $\upsilon_{\mathbb{X}}$ on $K(\mathbb{X})$ which is generated by the subbasis V_τ.

Modal logicians will recognize the above notation as indicating that $[\ni]$ and $\langle\ni\rangle$ are the 'box' and the 'diamond' associated with the converse membership relation $\ni \subseteq K(\mathbb{X}) \times X$.

In case the original topology is compact, then we might as well have generated the Vietoris topology in other ways. This has nice consequences for the case that the original topology is a Stone space.

Fact 2.6 *Let $\mathbb{X} = (X, \tau)$ be a compact topological space and let β be a basis of τ that is closed under finite unions. Then the set V_β forms a subbasis for $\upsilon_{\mathbb{X}}$. In particular, if $\mathbb{X}$ is a Stone space, then the set $V_{\mathsf{Clp}_{\mathbb{X}}}$ is a subbasis for the Vietoris topology.*

The last basic fact gathered here states that the Vietoris construction preserves various nice topological properties.

Fact 2.7 *Let $\mathbb{X} = (X, \tau)$ be a topological space. If τ is compact Hausdorff, then so is its Vietoris topology. If $\mathbb{X}$ is in addition zero-dimensional, then so is $\mathbb{V}(\mathbb{X})$. Hence, the Vietoris space of a Stone space is a Stone space.*

3 Descriptive general frames as Stone coalgebras

In this section we discuss what are probably the prime examples of Stone coalgebras, namely those for the Vietoris functor $\mathbb{V}$ (to be defined below). As we will see, the importance of these structures lies in the fact that the category $\mathsf{Coalg}(\mathbb{V})$ is isomorphic to the category of so-called *descriptive general frames.* We hasten to remark that when it comes down to the technicalities, this section contains little news; most of the results in this section can be obtained by exposing existing material from Esakia [10], Goldblatt [12], Johnstone [16], and Sambin and Vaccaro [27] in a new, coalgebraic framework.

General frames, and in particular, descriptive general frames, play a crucial role in the theory of modal logic. Together with their duals, the modal algebras, they provide an important class of structures interpreting modal languages. From a mathematical perspective they rank perhaps even higher than Kripke frames, since the Kripke semantics suffers from a fundamental incompleteness result: not every modal logic (in the technical sense of the word) is complete with respect to the class of Kripke frames on which it is valid (see e.g. [6], chapter 4). Putting it differently, Kripke frames provide too poor a tool to make the required distinctions between modal logics. The algebraic semantics for modal logic does not suffer from this shortcoming: every modal logic is determined by the class of modal algebras on which it is valid.

Definition 3.1 (Modal algebras) Let $\mathbb{B}$ and $\mathbb{B}'$ be boolean algebras; an operation $g : B \to B'$ on their carriers is said to *preserve finite meets* if $g(\top) = \top'$ and $g(b_1 \wedge b_2) = g(b_1) \wedge' g(b_2)$. A *modal algebra* is a structure $\mathbb{A} = (A, \wedge, -, \bot, \top, g)$ such that the reduct $(A, \wedge, -, \bot, \top)$ of $\mathbb{A}$ is a Boolean algebra, and $g : A \to A$ preserves finite meets. The category of modal algebras (with homomorphisms) is denoted by MA.

The intended meaning of g is to provide an interpretation of the modal operator $\Box$. Thinking of $a \in A$ as the interpretation of a modal formula φ, $g(a)$ provides the interpretation of $\Box\varphi$.

Example 3.2 (i) If (X, R) is a Kripke frame then $(\mathcal{P}X, \cap, -, \emptyset, X, [R])$ is a modal algebra where $[R](a) = \{x \in X \mid x\ R\ y \ \Rightarrow\ y \in a\}$.

(ii) Let Prop be a set of propositional variables and $\mathcal{L}(\mathsf{Prop})$ be the set of modal formulae over Prop quotiented by $\varphi \equiv \psi \ \Leftrightarrow\ \vdash_{\mathbf{K}} \varphi \leftrightarrow \psi$ where $\vdash_{\mathbf{K}}$ denotes derivability in the basic modal logic $\mathbf{K}$ (see eg [6]). Then $\mathcal{L}(\mathsf{Prop})$—equipped with the obvious operations—is a modal algebra. In fact, $\mathcal{L}(\mathsf{Prop})$ is the modal algebra free over Prop and is called the Lindenbaum-Tarski algebra (over Prop).

Remark 3.3 Although not needed in the following, we indicate how modal formulae are evaluated in modal algebras. Let φ be a modal formula taking propositional variables from Prop and let $\mathbb{A} = (A, \wedge, -, \bot, \top, g)$ be a modal algebra. Employing the freeness of the modal algebra $\mathcal{L}(\mathsf{Prop})$ we can identify valuations of variables $v : \mathsf{Prop} \to A$ with algebra morphisms $\mathcal{L}(\mathsf{Prop}) \to \mathbb{A}$ and define $\mathbb{A} \models \varphi$ if $v([\varphi]_\equiv) = \top$ for all morphisms $v : \mathcal{L}(\mathsf{Prop}) \to \mathbb{A}$.

However, modal algebras are fairly abstract in nature and many modal logicians prefer the intuitive, geometric appeal of Kripke frames. *General frames*, unifying the algebraic and the Kripke semantics in one structure, provide a nice compromise.

Definition 3.4 (General frames) Formally, a general frame is a structure $\mathbb{G} = (G, R, A)$ such that (G, R) is a Kripke frame and A is a collection of so-called *admissible* subsets of G that is closed under the boolean operations and under the operation $\langle R \rangle : \mathcal{P}(G) \to \mathcal{P}(G)$ given by:

$$\langle R \rangle X \ := \ \{y \in G \mid Ryx \text{ for some } x \in X\}.$$

A general frame $\mathbb{G} = (G, R, A)$ is called *differentiated* if for all distinct $s_1, s_2 \in G$ there is a 'witness' $a \in A$ such that $s_1 \in a$ while $s_2 \notin a$; *tight* if whenever t is not an R-successor of s, then there is a 'witness' $a \in A$ such that $t \in a$ while $s \notin \langle R \rangle a$; and *compact* if $\bigcap A_0 \neq \varnothing$ for every subset A_0 of A which has the finite intersection property. A general frame is *descriptive* if it is differentiated, tight and compact.

Example 3.5 (i) Any Kripke frame (X, R) can be considered as a general frame $(X, R, \mathcal{P}X)$.

(ii) If $\mathbb{A} = (A, \wedge, -, \bot, \top, g)$ is a modal algebra then $(\mathsf{Sp}\,\mathbb{A}, R, \hat{A})$ where $R = \{(u, v) \mid a \in u \;\Rightarrow\; g(a) \in v\}$ and $\hat{A} = \{\{u \in \mathsf{Sp}\,\mathbb{A} \mid a \in u\} \mid a \in A\}$ is a descriptive general frame.

(iii) If $\mathbb{G} = (G, R, A)$ is a general frame then $(A, \cap, -, \emptyset, G, [R])$ is a modal algebra.

The following remark explains the terminology of 'admissible' subsets.

Remark 3.6 Let $\mathbb{G} = (X, R, A)$ be a general frame and consider a modal formula φ taking its propositional variables from the set Prop. Note that, given a function $v : X \to \prod_{\mathsf{Prop}} 2$, where $2 = \{0, 1\}$ is the set of truth values, (X, R, v) is a Kripke model. v is called a *valuation for* $\mathbb{G}$ if the extensions of all propositions are admissible, that is, if $\{x \in X \mid v(x)_p = 1\} \in A$ for all $p \in \mathsf{Prop}$. The validity of a modal formula in general frame is then defined as $\mathbb{G} \models \varphi$ if $(X, R, v) \models \varphi$ for all valuations v for $\mathbb{G}$.

Since Kripke frames (and models) form some of the prime examples of coalgebras, the question naturally arises whether (descriptive) general frames can be seen as coalgebras as well. In this and the following section we will answer this question in the positive.

Two crucial observations connect descriptive general frames with coalgebras. First, the admissible sets of a descriptive frame form a basis for a topology. This topology is compact, Hausdorff, and zero-dimensional because descriptive general frames are compact, differentiated and the admissible sets are closed under boolean operations. It follows that descriptive general frames give rise to a Stone space with the admissible sets appearing as the collection of clopens.

Second, the tightness condition of descriptive general frames can be reformulated as the requirement that the relation is *point-closed*; that is, the successor set of any point is closed in the Stone topology. This suggests that if we are looking for a coalgebraic counterpart of a descriptive general frame $\mathbb{G} = (G, R, A)$, it should be of the form

$$(G, \tau) \xrightarrow{R[\_]} (K(G), \tau_?)$$

where $K(G)$ is the collection of closed sets in the Stone topology τ on G and $\tau_?$ is some suitable topology on $K(G)$, which turns $K(G)$ again into a Stone space. A good candidate is the Vietoris topology: it is based on the closed sets of τ and it yields a Stone space if we started from one. Moreover, as we will see, choosing the Vietoris topology for $\tau_?$, continuity of the map $R[\_]$ corresponds to the admissible sets being closed under $\langle R \rangle$.

Turning these intuitions into a more precise statement, we will prove that the category of descriptive general frames and the category $\mathsf{Coalg}(\mathbb{V})$ of coalgebras for the Vietoris functor are in fact *isomorphic*. Before we can go into the details of this, there are two obvious tasks waiting: first, we have to define the morphisms that make the descriptive general frames into a category, and second, we have to show that the Vietoris construction, which until now has just been defined for objects, can be turned into a functor.

Definition 3.7 (General frame morphisms) A morphism $\theta : (G, R, A) \to (G', R', A')$ is a function from W to W' such that (i) $\theta : (W, R) \to (W', R')$ is a bounded morphism (see Example 2.2) and (ii) $\theta^{-1}(a') \in A$ for all $a' \in A'$.

We let GF (DGF) denote the category with general frames (descriptive general frames, respectively) as its objects, and the general frame morphisms as the morphisms.

In the future we will need the fact that there is a dual equivalence [4] between the categories of modal algebras and descriptive general frames:

$$\mathsf{MA} \simeq \mathsf{DGF}^{\mathrm{op}}.$$

We will now see how the Vietoris construction can be upgraded to a proper endofunctor on the category of Stone spaces. For that purpose, we need to show how continuous maps between Stone spaces can be lifted to continuous maps between their Vietoris spaces; as a first step, we need the fact that whenever $f : \mathbb{X} \to \mathbb{X}'$ is a continuous map between compact Hausdorff spaces, then the image map $f[\_]$ is of the right type, that is, sends closed sets to closed sets. Fortunately, this is standard topology.

Fact 3.8 *Let $f : \mathbb{X} \to \mathbb{X}'$ be a continuous map between compact Hausdorff spaces. Then the function $\mathbb{V}(f)$ given by*

$$\mathbb{V}(f)(F) \;:=\; f[F] \;(= \{f(x) \mid x \in X\})$$

maps closed sets in $\mathbb{X}$ to closed sets in $\mathbb{X}'$.

Moreover, $\mathbb{V}$ is functorial:

Lemma 3.9 *Let $f : \mathbb{X} \to \mathbb{X}'$ be a continuous map between compact Hausdorff spaces. Then the function $\mathbb{V}(f)$ is a continuous map from $\mathbb{V}(\mathbb{X})$ to $\mathbb{V}(\mathbb{X}')$, and satisfies the functorial laws: $\mathbb{V}(id_{\mathbb{X}}) = \mathbb{V}(id_{\mathbb{V}(\mathbb{X})})$, and $\mathbb{V}(f \circ g) = \mathbb{V}(f) \circ \mathbb{V}(g)$.*

Proof. Assume that f is a continuous map between the Stone spaces $\mathbb{X} = (X, \tau)$ and $\mathbb{X}' = (X', \tau')$. In order to show that $\mathbb{V}(f)$ is a continuous map from $\mathbb{V}(\mathbb{X})$ to $\mathbb{V}(\mathbb{X}')$, we show that the pre-images of subbasic elements of the Vietoris topology $\upsilon_{\mathbb{X}'}$ are open in the Vietoris topology $\upsilon_{\mathbb{X}}$.

Let U' be an arbitrary element of $V_{\mathbb{X}'}$; there are two cases to consider. To start with, if U' is of the form $[\ni]Q'$ for some $Q' \in \tau'$, then we see that $\mathbb{V}(f)^{-1}(U') = \{F \in K(\mathbb{X}) \mid \mathbb{V}(F) \in [\ni]Q'\} = \{F \in K(\mathbb{X}) \mid f[F] \subseteq Q'\} = \{F \in K(\mathbb{X}) \mid F \subseteq f^{-1}(Q')\} = [\ni]f^{-1}(Q')$. And second, if U' is of the form $\langle\ni\rangle Q'$ for some $Q' \in \tau'$, then we have $\mathbb{V}(f)^{-1}(U') = \{F \in K(\mathbb{X}) \mid \mathbb{V}(F) \in \langle\ni$

[4] On objects the equivalence is given by Example 3.5, (ii) and (iii).

$\rangle Q'\} = \{F \in K(\mathbb{X}) \mid f[F] \cap Q' \neq \varnothing\} = \{F \in K(\mathbb{X}) \mid F \cap f^{-1}(Q') \neq \varnothing\} = \langle\ni\rangle f^{-1}(Q')$. In both cases we find that $\mathbb{V}(f)^{-1}(U')$ is (basic) open, as required.

We leave it to the reader to verify that $\mathbb{V}$ satisfies the functorial laws. □

Definition 3.10 (Vietoris functor) The *Vietoris functor* on the category of Stone spaces is given on objects as in Definition 2.5 and on morphisms as in Lemma 3.8, i.e., for $(X, \tau) \in \mathsf{Stone}$

$$\begin{array}{rcl} (X,\tau) & \mapsto & (K(\mathbb{X}), \tau_V) \\ (f : (X,\tau) \to (Y,\sigma)) & \mapsto & \mathbb{V}(f) \end{array}$$

where $\mathbb{V}(f)[F] := f[F]$ for all closed $F \subseteq X$.

We now turn to the isomorphism between the categories DGF and $\mathsf{Coalg}(\mathbb{V})$. The following rather technical lemma allows us to define the required functors relating the two categories.

Lemma 3.11 *Let X, τ and A be such that τ is a Stone topology on X and A is the collection of clopens of τ, and likewise for X', τ' and A'. Furthermore, suppose that $R \subseteq X^2$ and $\gamma : X \to K(\mathbb{X})$ satisfy*

$$Rxy \textit{ iff } y \in \gamma(x) \tag{1}$$

for all $x, y \in X$; and similarly for $R' \subseteq X'^2$ and $\gamma' : X' \to K(\mathbb{X}')$.

Then $\theta : X \to X'$ is $\mathbb{V}$-coalgebra homomorphism between $((X,\tau),\gamma)$ and $((X',\tau'),\gamma')$ if and only if it is a general frame morphism between (X, R, A) and (X', R', A').

Proof. Both directions of the proof are straightforward. We only show the direction from left to right, leaving the other direction to the reader. Suppose that θ is a coalgebra morphism. Then θ is a continuous map from (X, τ) to (X', τ'), so the θ-inverse of a clopen set in τ' is clopen in τ. This shows that $\theta^{-1}(a') \in A$ for all $a' \in A'$.

In order to show that θ is a bounded morphism, first let Rxy. This implies that $y \in \gamma(x)$. Because θ is a coalgebra morphism we have

$$\theta[\gamma(x)] = \gamma'(\theta(x)),$$

so we get $\theta(y) \in \gamma'(\theta(x))$, i.e. $R'\theta(x)\theta(y)$. Now suppose that $R'\theta(x)y'$. Then $y' \in \gamma'(\theta(x))$ so by the above equation $y' \in \theta[\gamma(x)]$; that is, there is a $y \in X$ such that Rxy and $\theta(y) = y'$. □

Lemma 3.11, together with our earlier observation on the connection between the admissible sets of a descriptive general frame and the clopens of the Stone space induced by taking these admissible sets as a basis, ensures that the following definition is correct. That is, if the reader is willing to check for himself that the maps defined below are indeed functors.

Definition 3.12 We define the functor $\mathbb{C} : \mathsf{DGF} \to \mathsf{Coalg}(\mathbb{V})$ as follows:

$$(G, R, A) \quad \mapsto \quad (G, \sigma_A) \xrightarrow{R[\_]} \mathbb{V}(G, \sigma_A)$$

Here σ_A denotes the Stone topology generated by taking A as a basis. Conversely, there is a functor $\mathbb{D} : \mathsf{Coalg}(\mathbb{V}) \to \mathsf{DGF}$ given by:

$$((X, \tau), \gamma) \quad \mapsto \quad (X, R_\gamma, \mathsf{Clp}_{(X,\tau)})$$

where R_γ is defined by $R_\gamma s_1 s_2$ iff $s_2 \in \gamma(s_1)$. On morphisms both functors act as the identity with respect to the underlying Set-functions.

The following theorem now easily follows from spelling out the respective definitions.

Theorem 3.13 *The functors $\mathbb{C}$ and $\mathbb{D}$ form an isomorphism between the categories DGF and $\mathsf{Coalg}(\mathbb{V})$.*

Remark 3.14 (Valuations of Propositional Variables)
For a set-coalgebra (X, ξ), a valuation of propositional variables $p \in \mathsf{Prop}$ is a function $X \to \prod_{\mathsf{Prop}} 2$ where 2 is the two-element set of truth-values. For a Stone-coalgebra $(\mathbb{X}, \xi)$, a valuation is a continuous map $v : \mathbb{X} \to \prod_{\mathsf{Prop}} 2$ where 2 is taken with the discrete topology. The continuity of v is equivalent to the statement that the propositional variables take their values in admissible sets. Indeed, writing $\pi_p : \prod_{\mathsf{Prop}} 2 \to 2$ ($p \in \mathsf{Prop}$) for the projections, continuity of v is equivalent to $v^{-1}(\pi_p^{-1}(\{1\}))$ clopen for all $p \in \mathsf{Prop}$. Observing that $v^{-1}(\pi_p^{-1}(\{1\})) = \{x \in X \mid v(x)_p = 1\}$ is the extension of p the claim now follows from the fact that the clopens coincide with the admissible sets.

Remark 3.15 (General Frames as Coalgebras) Stone spaces provide a convenient framework to study descriptive general frames since the admissible sets can be recovered from the topology. Making a generalization to arbitrary general frames, we can still work in a coalgebraic framework, but we have to make two adjustments.

First, we work directly with admissible sets instead of with topologies: the category RBA (referential or represented Boolean algebras) has objects (X, A) where X is a set and A a set of subsets of X closed under boolean operations. It has morphisms $f : (X, A) \to (Y, B)$ where f is a function $X \to Y$ such that $f^{-1}(b) \in A$ for all $b \in B$.

And second, in the absence of tightness, the relation of the general frame will no longer be point-closed. Hence, its coalgebraic version has the full power set as its codomain. For $\mathbb{X} = (X, A) \in \mathsf{RBA}$ let $\mathbb{W}(\mathbb{X}) = (\mathcal{P}(X), v_{\mathbb{X}})$ where $v_{\mathbb{X}}$ is the Boolean algebra generated by $\{\{F \in \mathcal{P}X \mid F \cap a \neq \varnothing\} \mid a \in A\}$. On morphisms let $\mathbb{W}(f) = \mathcal{P}(f)$. This clearly defines an endofunctor on the category RBA, and the induced category $\mathsf{Coalg}(\mathbb{W})$ is the coalgebraic version of general frames:

$$\text{There is an isomorphism between } \mathsf{GF} \text{ and } \mathsf{Coalg}(\mathbb{W}) \tag{2}$$

The crucial observation in the proof of (2) is that, for $\mathbb{X} = (X, A) \in \mathsf{RBA}$ and R a relation on X, we have that A is closed under $\langle R\rangle$ iff $R[\_] : X \to \mathcal{P}X$ is a RBA-morphism $\mathbb{X} \to \mathbb{W}(\mathbb{X})$. This follows from the fact that $\langle R\rangle a = (R[\_])^{-1}(\{F \in \mathcal{P}X \mid F \cap a \neq \varnothing\})$. Further details are left to the reader.

Finally, to finish off this section, let us note two corollaries of Theorem 3.13.

Using $\mathsf{MA} \simeq \mathsf{DGF}^{\mathrm{op}}$ and $(\mathsf{Coalg}(\mathbb{V}))^{\mathrm{op}} = \mathsf{Alg}(\mathbb{V}^{\mathrm{op}})$, it follows $\mathsf{MA} \simeq \mathsf{Alg}(\mathbb{V}^{\mathrm{op}})$. With $\mathsf{Stone}^{\mathrm{op}} \simeq \mathsf{BA}$ we obtain the following.

Corollary 3.16 *There is a functor $H : \mathsf{BA} \to \mathsf{BA}$ such that the category of modal algebras MA is equivalent to the category $\mathsf{Alg}(H)$ of algebras for the functor H.*

Proof. With the help of the contravariant functors $\mathbb{C}\mathsf{lp} : \mathsf{Stone} \to \mathsf{BA}$, $\mathbb{S}\mathsf{p} : \mathsf{BA} \to \mathsf{Stone}$, we let $H = \mathbb{C}\mathsf{lp}\mathbb{V}\,\mathbb{S}\mathsf{p}$. The claim now follows from the observation that $\mathsf{Alg}(H)$ is dual to $\mathsf{Coalg}(\mathbb{V})$: An algebra $HA \xrightarrow{\alpha} A$ corresponds to the coalgebra $\mathbb{S}\mathsf{p}\,A \xrightarrow{\mathbb{S}\mathsf{p}\,\alpha} \mathbb{S}\mathsf{p}\,HA \cong \mathbb{V}\,\mathbb{S}\mathsf{p}\,A$ and a coalgebra $\mathbb{X} \xrightarrow{\xi} \mathbb{V}\mathbb{X}$ corresponds to the algebra $H\mathbb{C}\mathsf{lp}\mathbb{X} \cong \mathbb{C}\mathsf{lp}\mathbb{V}\mathbb{X} \xrightarrow{\mathbb{C}\mathsf{lp}\xi} \mathbb{C}\mathsf{lp}\mathbb{X}$. □

An explicit description of H not involving the Vietoris functor is given by the following proposition.

Proposition 3.17 *Let $H : \mathsf{BA} \to \mathsf{BA}$ be the functor that assigns to a Boolean algebra the free Boolean algebra over its underlying meet-semilattice. Then $\mathsf{Alg}(H)$ is isomorphic to the category of modal algebras MA.*

Proof. We use the well-known fact that MA is isomorphic to the category MPF which is defined as follows. An object of MPF is an endofunction $A \xrightarrow{m} A$ on a Boolean algebra A that preserves finite meets (i.e. binary meets and the top-element). A morphism $f : (A \xrightarrow{m} A) \longrightarrow (A' \xrightarrow{m'} A')$ is a Boolean algebra morphism $f : A \to A'$ such that $m' \circ f = f \circ m$. We also write $\mathsf{BA}_\wedge$ for the category with Boolean algebras as objects and finite meet preserving functions as morphisms.

To prove that $\mathsf{Alg}(H)$ and MPF are isomorphic categories, we first show that $\mathsf{BA}(HA, A) \cong \mathsf{BA}_\wedge(A, A)$, or to be slightly more general and precise, $\mathsf{BA}(HA, B) \cong \mathsf{BA}_\wedge(IA, IB)$ where $I : \mathsf{BA} \hookrightarrow \mathsf{BA}_\wedge$. (Here we denote, for a category C and objects A, B in C, the set of morphisms between A and B by $\mathsf{C}(A, B)$.) Indeed, consider the forgetful functors $U : \mathsf{BA} \to \mathsf{SL}$, $V : \mathsf{BA}_\wedge \to \mathsf{SL}$ to the category SL of meet-semilattices with top element and the left adjoint F of U. Using our assumption $H = FU$, we calculate $\mathsf{BA}(HA, B) = \mathsf{BA}(FUA, B) \cong \mathsf{SL}(UA, UB) \cong \mathsf{SL}(VIA, VIB) \cong \mathsf{BA}_\wedge(IA, IB)$. The isomorphisms $\varphi_A : \mathsf{BA}(HA, A) \to \mathsf{BA}_\wedge(A, A)$, $A \in \mathsf{BA}$, give us an isomorphism φ between the objects of $\mathsf{Alg}(H)$ and MPF. On morphisms, we define φ to be the identity. This is well-defined because the isomorphisms $\mathsf{BA}(HA, B) \cong \mathsf{BA}_\wedge(IA, IB)$ are natural in A and B. □

As another corollary to the duality we obtain that $\mathsf{Coalg}(\mathbb{V})$ has cofree coalgebras.

Corollary 3.18 *The forgetful functor* $\mathsf{Coalg}(\mathbb{V}) \to \mathsf{Stone}$ *has a right adjoint.*

Proof. Consider the forgetful functors $R : \mathsf{MA} \to \mathsf{BA}$, $U : \mathsf{MA} \to \mathsf{Set}$, $V : \mathsf{BA} \to \mathsf{Set}$. Since U and V are monadic, R has a left adjoint. Hence, by duality, $\mathsf{Coalg}(\mathbb{V}) \to \mathsf{Stone}$ has a right adjoint. $\Box$

4 Vietoris Polynomial Functors

In this section we introduce the notion of a Vietoris polynomial functor (short: VPF) as a natural analogue for the category Stone of what the so-called Kripke polynomial functors [25,15] are for Set. This section can be therefore seen as a first application of the observation that coalgebras over Stone can be used as semantics for (coalgebraic) modal logics.

Although we have kept this section self-contained, most of its content builds on the work by Jacobs in [15]. Note however that we not only translate the entire setting to the category of Stone spaces but also repair a defect of the original construction (see Remark 4.16 for more details).

4.1 Polynomial functors

Definition 4.1 (Vietoris polynomial functors) The collection of *Vietoris polynomial functors*, in brief: VPFs, over Stone is inductively defined as follows:

$$T \;::=\; \mathbb{I} \mid \mathbb{Q} \mid T_1 + T_2 \mid T_1 \times T_2 \mid T^D \mid \mathbb{V}T.$$

Here $\mathbb{I}$ is the identity functor on the category Stone; $\mathbb{Q}$ denotes a finite Stone space (that is, the functor $\mathbb{Q}$ is a constant functor); '+' and '$\times$' denote disjoint union and binary product, respectively; and, for an arbitrary set D, T^D denotes the functor sending a Stone space $\mathbb{X}$ to the D-fold product [5] $(T(\mathbb{X})^D)$.

Associated with this we inductively define the notion of a *path*:

$$p \;::=\; <>\mid \pi_1 \cdot p \mid \pi_2 \cdot p \mid \kappa_1 \cdot p \mid \kappa_2 \cdot p \mid [\mathrm{ev}(d)] \cdot p \mid \mathbb{V} \cdot p.$$

By induction on the complexity of paths we now define when two VPFs T_1

[5] We leave it as an exercise for the reader to verify that the class of Stone spaces is closed under taking topological products.

and T_2 are related by a path p, notation: $T_1 \overset{p}{\leadsto} T_2$:

$$
\begin{array}{rcll}
T & \overset{<>}{\leadsto} & T & \\
T_1 \times T_2 & \overset{\pi_i \cdot p}{\leadsto} & T' & \text{if} \quad T_i \overset{p}{\leadsto} T' \\
T_1 + T_2 & \overset{\kappa_i \cdot p}{\leadsto} & T' & \text{if} \quad T_i \overset{p}{\leadsto} T' \\
T^D & \overset{[\mathrm{ev}(d)] \cdot p}{\leadsto} & T' & \text{if} \quad T \overset{p}{\leadsto} T' \text{ and } d \in D \\
\mathbb{V}T & \overset{\mathbb{V} \cdot p}{\leadsto} & T' & \text{if} \quad T \overset{p}{\leadsto} T'.
\end{array}
$$

Finally, for a VPF T we define $\mathsf{Ing}(T)$ to be the category with the set $\mathrm{Ing}(T) := \{S \mid \exists p.T \overset{p}{\leadsto} S\}$ as the set of objects and the paths as morphisms between them.

Remark 4.2 The fact that VPFs are defined on Stone spaces makes it possible to include infinite constants $\mathbb{Q}$ in our discussion. Dually to the Boolean product construction for Boolean algebras one can also define an infinite sum of Stone spaces, cf. [11]. Because of space limitations we confine ourselves to the standard case, in which only finite constants and finite sums are allowed.

4.2 Algebras

It follows from the general definition of coalgebras, what the definition of a T-coalgebra is, for an arbitrary VPF T. Dually, we will make good use of a kind of algebras for T; the definition of a so-called T-BAO may look slightly involved, but it is based on a simple generalization of the concept of a modal algebra. The generalization is that instead of dealing with one single Boolean algebra, we will be working with a *family* $(\Phi(S))_{S \in \mathrm{Ing}(T)}$ of Boolean algebras, linked by finite-meet preserving operations. We let $\mathsf{BA}_\wedge$ denote the category of Boolean algebras with finite-meet preserving operations.

Definition 4.3 (T-BAO) Let T be a VPF. A *T-sorted Boolean algebra with operators*, T-BAO, consists of

- a functor $\Phi : \mathsf{Ing}(T)^{\mathrm{op}} \longrightarrow \mathsf{BA}_\wedge$, together with
- (in the case that $\mathbb{I} \in \mathsf{Ing}(T)$) an additional map $\mathsf{next} : \Phi(T) \to \Phi(\mathbb{I})$ which preserves all Boolean operations.

This functor is supposed to satisfy the following conditions:

(i) $\Phi(\mathbb{Q}) = \mathsf{Clp}_\mathbb{Q}$

(ii) the functions $\Phi(\pi_i)$ and $\Phi([\mathrm{ev}(d)])$ are Boolean homomorphisms

(iii) the functions $\Phi(\kappa_i)$ induced by the injection paths satisfy

$$\begin{aligned} -\Phi(\kappa_1)(\bot) \vee -\Phi(\kappa_2)(\bot) &= \top \\ -\Phi(\kappa_1)(\bot) \wedge -\Phi(\kappa_2)(\bot) &= \bot \\ -\Phi(\kappa_i)(\bot) \wedge \Phi(\kappa_i)(-\alpha) &\leq -\Phi(\kappa_i)(\alpha) \end{aligned}$$

Example 4.4 Let $\mathbb{A} = (A, \wedge, -, \bot, \top, g)$ be a modal algebra, cf. Definition 3.1. Then $\mathrm{Ing}(T) = \{\mathbb{I}, \mathbb{V}\mathbb{I}\}$ and we have $\mathbb{V}\mathbb{I} \overset{\mathbb{V}}{\rightsquigarrow} \mathbb{I}$. If we define $\Phi(\mathbb{I}) := A$, $\Phi(\mathbb{V}\mathbb{I}) := A$, $\Phi(\mathbb{V}) := g$, and take $\mathsf{next} : \Phi(\mathbb{V}\mathbb{I}) \to \Phi(\mathbb{I})$ to be the identity map, we get a $\mathbb{V}\mathbb{I}$-BAO (Φ, next) that corresponds to the original modal algebra.

Definition 4.5 **(BAO_T)** Let T be a Vietoris polynomial functor; a *morphism* from one T-BAO (Φ, next) to another (Φ', next') is a natural transformation $t : \Phi \to \Phi'$ such that for each ingredient S of T the component $t_S : \Phi(S) \to \Phi'(S)$ preserves the Boolean structure, such that $t_{\mathbb{I}}$ and t_T satisfy the following naturality condition with respect to next and next':

$$\mathsf{next}' \circ t_T = t_{\mathbb{I}} \circ \mathsf{next}$$

and such that $t_{\mathbb{Q}} = \mathsf{id}_{\mathsf{Clp}_{\mathbb{Q}}}$ for all constants $\mathbb{Q} \in \mathsf{Ing}(T)$. This yields the category BAO_T.

4.3 From coalgebras to algebras and back

It is not difficult to transform a T-coalgebra into a T-BAO; basically, we are dealing with a sorted version of Stone duality (see Definition 2.4 for terminology and notation), together with a path-indexed predicate lifting.

Lemma and Definition 4.6 Let T be a VPF and let $\mathbb{X}$ be a Stone space. Then the following definition on the complexity of paths

$$\begin{aligned} \alpha^{<>} &:= \alpha \\ \alpha^{\pi_1 \cdot p} &:= \pi_1^{-1}(\alpha^p) \\ \alpha^{\pi_2 \cdot p} &:= \pi_2^{-1}(\alpha^p) \\ \alpha^{\kappa_1 \cdot p} &:= \kappa_1(\alpha^p) \cup \kappa_2 S_2(X) && \text{for } T_2 = S_1 + S_2 \\ \alpha^{\kappa_2 \cdot p} &:= \kappa_1 S_1(X) \cup \kappa_2(\alpha^p) && \text{for } T_2 = S_1 + S_2 \\ \alpha^{[\mathrm{ev}(d)] \cdot p} &:= \pi_d^{-1}(\alpha^p) \\ \alpha^{\mathbb{V} \cdot p} &:= \{\beta \mid \beta \subseteq \alpha^p \text{ and } \beta \text{ closed }\} \ (= [\ni]\alpha) \end{aligned}$$

provides, for any two functors $T_1, T_2 \in \mathsf{Ing}(T)$ such that $T_1 \overset{p}{\rightsquigarrow} T_2$, a so-called *predicate lifting* $(\_)^p : \mathsf{Clp}_{T_2\mathbb{X}} \to \mathsf{Clp}_{T_1\mathbb{X}}$.

Lemma 4.7 *For each Vietoris polynomial functor T, each T-coalgebra $(\mathbb{X},\xi)$ gives rise to a T-BAO, namely, the 'complex algebra' functor* $\Gamma(\mathbb{X},\xi) : \mathsf{Ing}(T)^{\mathrm{op}} \to \mathsf{BA}_\wedge$ *given by*

$$\begin{aligned} S &\mapsto \mathbb{C}\mathsf{lp}S(\mathbb{X}) \\ \left(S_1 \overset{p}{\rightsquigarrow} S_2\right) &\mapsto ((\_)^p : \mathbb{C}\mathsf{lp}S_2(\mathbb{X}) \to \mathbb{C}\mathsf{lp}S_1(\mathbb{X}))\,, \end{aligned}$$

accompanied by the map $\mathsf{next} : \mathbb{C}\mathsf{lp}(T\mathbb{X}) \to \mathbb{C}\mathsf{lp}(\mathbb{X})$ *given by* $\mathsf{next} := \xi^{-1}$ *in the case that* $\mathbb{I} \in \mathrm{Ing}(T)$.

Proof. To start with, we need to show that $\Gamma(\mathbb{X},\xi)$ is a functor from $\mathsf{Ing}(T)^{\mathrm{op}}$ to $\mathsf{BA}_\wedge$. To that aim, we prove that the predicate lifting $(\_)^p : \mathsf{Clp}_{T_1\mathbb{X}} \to \mathsf{Clp}_{T_2\mathbb{X}}$ constitutes a $\mathsf{BA}_\wedge$-morphism between $\mathbb{C}\mathsf{lp}T_1\mathbb{X}$ and $\mathbb{C}\mathsf{lp}T_2\mathbb{X}$; and that it satisfies the functorial laws.

Finally, we have to show that the functor $\Gamma(\mathbb{X},\xi) : \mathsf{Ing}(T)^{\mathrm{op}} \longrightarrow \mathsf{BA}_\wedge$, together with the map $\mathsf{next} := \xi^{-1}$ in case $\mathbb{I} \in \mathrm{Ing}(T)$, meets the requirements listed in Definition 4.3. All of these results can be proved in a fairly straightforward way. $\square$

Conversely, with each T-BAO Φ we may associate a T-coalgebra $\Sigma(\Phi)$. Assume that T has the identity functor as an ingredient; given our results in the previous section, and the well-known Stone duality, it seems fairly obvious that we should take the dual Stone space $\mathbb{S}\mathsf{p}\,\Phi(\mathbb{I})$ as the carrier of this dual coalgebra. However, how to obtain T-coalgebra structure on this? Applying duality theory to the Boolean algebras obtained from Φ only seems to provide information on the spaces $\mathbb{S}\mathsf{p}\,\Phi(S)$, whereas we need to work with $S(\mathbb{S}\mathsf{p}(\Phi(\mathbb{I})))$ in order to correctly define a T-coalgebra. Fortunately, in the next lemma and definition we show that there exists a map r which produces the S-structure. The definition of r is taken from [15]; what we have to show is that it works also in our new topological setting.

Lemma and Definition 4.8 (r_Φ) Let T be a VPF with $\mathbb{I} \in \mathrm{Ing}(T)$ and let (Φ, next) be a T-BAO. Then the following definition by induction on the structure of ingredient functors of T:

$$\begin{aligned} r_\Phi(\mathbb{I})(U) &:= U \\ r_\Phi(\mathbb{Q})(U) &:= a \quad \text{if } \bigcap U = \{a\} \\ r_\Phi(S_1 \times S_2)(U) &:= \left\langle r_\Phi(S_1)(\Phi(\pi_1)^{-1}(U)), r_\Phi(S_2)(\Phi(\pi_2)^{-1}(U))\right\rangle \\ r_\Phi(S_1 + S_2)(U) &:= \begin{cases} \kappa_1 r_\Phi(S_1)(\Phi(\kappa_1)^{-1}(U)) \text{ if } -\Phi(\kappa_1)(\bot) \in U \\ \kappa_2 r_\Phi(S_2)(\Phi(\kappa_2)^{-1}(U)) \text{ if } -\Phi(\kappa_2)(\bot) \in U \end{cases} \\ r_\Phi(S^D)(U) &:= \lambda d \in D. r_\Phi(S)(\Phi(\mathrm{ev}(d))^{-1}(U)) \\ r_\Phi(\mathbb{V}S)(U) &:= \left\{ r_\Phi(S)(V) \mid V \in \mathsf{Sp}\,\Phi(S) \text{ and } \Phi(\mathbb{V})^{-1}(U) \subseteq V \right\} \end{aligned}$$

defines, for every $S \in \mathrm{Ing}(T)$ a continuous map:

$$r_\Phi(S) : \mathbb{S}\mathsf{p}(\Phi(S)) \longrightarrow S(\mathbb{S}\mathsf{p}(\Phi(\mathbb{I})))$$

Furthermore, the inverse image map next^{-1} is a continuous map

$$\mathsf{next}^{-1} : \mathbb{S}\mathsf{p}(\Phi(\mathbb{I})) \longrightarrow T(\mathbb{S}\mathsf{p}(\Phi(\mathbb{I})))$$

Proof. It can be proved, by a simultaneous induction on the structure of S, that $r_\Phi(S)$ maps ultrafilters of $\Phi(S)$ to elements of the (underlying set of) $(\mathsf{Sp}\,\Phi(\mathbb{I}))$, and that this in fact provides a continuous map between the respective Stone spaces. For lack of space we cannot go into details here.

The claim on the map next^{-1} is a simple consequence of Stone duality. □

The above lemma allows us to define a T-coalgebra for a given T-BAO.

Definition 4.9 Let T be a VPF with $\mathbb{I} \in \mathrm{Ing}(T)$ and let (Φ, next) be a T-BAO. We define the coalgebra $\Sigma(\Phi, \mathsf{next})$ as the structure $(\mathbb{S}\mathsf{p}(\Phi(\mathbb{I})), r_\Phi(T) \circ \mathsf{next}^{-1})$.

4.4 Relating the categories

The maps Γ and Σ that allow us to move from a given T-BAO to a T-coalgebra and vice versa can be extended to functors.

Fix a Vietoris polynomial functor T, and let $f : (\mathbb{X}, \xi) \to (\mathbb{X}', \xi')$ be a $\mathsf{Coalg}(T)$-morphism. Then we define $\Gamma(f) : \Gamma(\mathbb{X}', \xi') \to \Gamma(\mathbb{X}, \xi)$ as follows. For each $S \in \mathrm{Ing}(T)$ let $\Gamma(f)(S) := \mathbb{C}\mathsf{lp}(S(f))$. Naturality of $\Gamma(f)$ can be proven by induction on paths and the additional condition in Definition 4.5 concerning the next functions is fulfilled because f is a T-coalgebra homomorphism.

Conversely, given a BAO_T-morphism $t : (\Phi, \mathsf{next}) \to (\Phi', \mathsf{next}')$, define the map $\Sigma(t) : \mathsf{Sp}(\Phi'(\mathbb{I})) \to \mathsf{Sp}(\Phi(\mathbb{I}))$ to be the inverse image map of $t_\mathbb{I} : \Phi(\mathbb{I}) \to \Phi'(\mathbb{I})$. We leave it to the reader to verify that $\Sigma(t)$ is in fact a $\mathsf{Coalg}(T)$ morphism between $\Sigma(\Phi, \mathsf{next})$ and $\Sigma(\Phi', \mathsf{next}')$ (cf. the proof of Proposition 5.3 in [15]).

Lemma 4.10 *If we extend Γ and Σ as described above we obtain functors*

$$\Gamma : \mathsf{Coalg}(T)^{\mathrm{op}} \to \mathsf{BAO}_T \quad \textit{and} \quad \Sigma : \mathsf{BAO}_T \to \mathsf{Coalg}(T)^{\mathrm{op}}.$$

Proof. We already provided the arguments why Γ and Σ are well-defined. That they preserve the composition of morphisms and identities is obvious.□

We are now ready to prove the following representation theorem, stating that every T-coalgebra is isomorphic to the Σ-image of some T-MBAO.

Theorem 4.11 *Let T be a Vietoris polynomial functor, and let $(\mathbb{X}, \xi)$ be a T-coalgebra. Then the map $\epsilon_\mathbb{X} : \mathbb{X} \to \mathsf{Sp}(\mathsf{Clp}_\mathbb{X})$ defined by $\epsilon_\mathbb{X}(x) := \{C \in \mathsf{Clp}_\mathbb{X} \mid x \in C\}$ is a $\mathsf{Coalg}(T)$-isomorphism witnessing that*

$$(\mathbb{X}, \xi) \cong \Sigma(\Gamma((\mathbb{X}, \xi))).$$

Proof. Following the proof idea of [15] we first prove that for each sort $S \in \mathsf{Ing}(T)$ the following equation holds:

$$r_{\Gamma(\mathbb{X},\xi)}(S) \circ \epsilon_{S\mathbb{X}} = S(\epsilon_{\mathbb{X}}).$$

This claim is proved by induction on the structure of S, analogous to Lemma 5.6 of [15], the main difference being that compactness is taking over the role of finiteness. (This means that there is no need to restrict ourselves to what Jacobs calls *finite* KPFs, i.e. polynomial functors which only contain the finite powerset functor.)

The proof that $\epsilon_{\mathbb{X}}$ is a coalgebra morphism now works exactly as in [15]. The fact that it is an isomorphism is then an immediate consequence of Stone duality. $\square$

4.5 The final coalgebra

Finally, although the functor pair Σ and Γ do not constitute a duality between the categories $\mathsf{Coalg}(T)$ and BAO_T, the link that they do establish is useful enough. As an example application we will show that the category $\mathsf{Coalg}(T)$ has a final object, obtained as the Γ-image of the initial object of BAO_T. As initial object of BAO_T we can take the so-called Lindenbaum-Tarski algebra $\mathcal{L}_T$ of the multi-sorted modal logic MSML_T (as defined in Definitions 3.1, 3.2 and Example 4.4 of [15]). For, let (Φ, next) be a T-MBAO. Then we get for any $S \in \mathrm{Ing}(T)$ an interpretation function

$$[\![\_]\!]_S : \mathsf{Form}_S \to \Phi(S)$$

as in Definition 4.2 of [15]. Here Form_S denotes the set of all formulas of sort S.

Lemma 4.12 ([15], Proposition 4.8) *The Lindenbaum T-BAO $\mathcal{L}_T$ is an initial object in the category BAO_T: for an arbitrary T-BAO Φ there is a unique homomorphism $[\![\_]\!] : \mathcal{L}_T \to \Phi$. The components of $[\![\_]\!]$ are the above mentioned interpretation functions.*

To be able to prove the final coalgebra theorem one needs the following results that can be found in [15]. The proofs from [15] can be transferred into our setting without problems.

Lemma 4.13 ([15], Lemma 5.4) *Let Φ be a T-BAO and $\varphi \in \mathsf{Form}_S$ for some $S \in \mathsf{Ing}(T)$. Then for an ultrafilter $U \in \mathsf{Sp}\,\Phi(S)$:*

$$[\![\varphi]\!]^{\Phi}_S \in U \quad \textit{iff} \quad r_\Phi(S)(U) \in [\![\varphi]\!]^{\Sigma(\Phi)}_S.$$

Lemma 4.14 ([15], Corollary 3.8) *Let T be a VPF and $f : (\mathbb{X}, \xi) \to (\mathbb{Y}, d)$ a morphism in $\mathsf{Coalg}(T)$. Then for each sort $S \in \mathsf{Ing}(T)$ and formula $\varphi \in \mathsf{Form}_S$ we get*

$$S(f)^{-1}\left([\![\varphi]\!]^{\Gamma(\mathbb{Y},d)}_S\right) = [\![\varphi]\!]^{\Gamma(\mathbb{X},\xi)}_S.$$

With the help of these lemmas one can now prove the final coalgebra theorem.

Theorem 4.15 ([15], Theorem 5.8) *For any Vietoris polynomial functor T, the coalgebra $\Sigma(\mathcal{L}_T)$ is final in the category $\mathsf{Coalg}(T)$. For a T-coalgebra $(\mathbb{X}, \xi)$ the unique homomorphism $! : \mathbb{X} \to \mathsf{Sp}\,\mathcal{L}_T(\mathbb{I})$ is given by $! = \Sigma([\![\_]\!]) \circ \epsilon_{\mathbb{X}}$.*

Proof. The proof works exactly as in [15] using Lemma 4.12 and Lemma 4.13. □

Remark 4.16 The details of the construction can be found in [15]. Note however, that there is a defect in Jacobs' proof. The problem involves the functor $\mathcal{C} : \mathsf{BAO}_T \to \mathsf{Coalg}(T)$.

In [15] Jacobs assigns a modal logic to each Kripke polynomial functor, and he proves that the coalgebras for these functors form a sound and complete semantics for these logics. In order to obtain the final coalgebra for a so-called *finite* KPF T, that is, a KPF which may only contain the finite-power set functor, he maps the Lindenbaum-Tarski algebra $\mathcal{L}_T$ to its corresponding coalgebra $\mathcal{C}(\mathcal{L}_T)$, using the above-mentioned functor $\mathcal{C}$. His construction works, if the functor $\mathcal{C}$ maps a T-BAO for a finite KPF T to a T-coalgebra. This is however only the case for functors T not containing the (finite) power set functor.

Therefore Jacobs' construction of final objects in $\mathsf{Coalg}(S)$ works only for Kripke polynomial functors that do not contain the power set functor or its finitary version. Moving from the category of sets to Stone enables us to repair this defect, using the compactness of the topology on every occasion that the finiteness of a KPF was used before. Thus in our case, the construction works for any Vietoris polynomial functor.

5 Conclusions

What we have done so far can be viewed from different perspectives. We summarise some of them, indicating possible future research directions.

Stone Coalgebras and Modal Logic

Research on the relation between coalgebras and modal logic started with Moss [22]. In [21,20] it was shown that modal logic for coalgebras dualise equational logic for algebras, the idea being that equations describe quotients of free algebras and modal formulae describe subsets of final (or cofree) coalgebras. [6] But whereas, usually, any quotient of a free algebra can be defined by a set of ordinary equations, one needs *infinitary* modal formulae to define all subsets of a final coalgebra. As a consequence, while we have a satisfactory description of the coalgebraic semantics of infinitary modal logics, we do not completely understand the relationship between coalgebras and finitary modal

[6] Another account of the duality has been given in [19] where it was shown that modalities dualise algebraic operations. Related work on dualising equational logic include [14,4,2].

logic. The results in this paper show that Stone coalgebras provide a natural and adequate semantics for finitary modal logics, but there is ample room for clarification here.

Another approach to a coalgebraic semantics for finitary modal logics was given in [18,17]. There, the idea is to modify coalgebra morphisms in such a way that they capture not bisimulation but only bisimulation up to rank ω. Since finitary modal logics capture precisely bisimulation up to rank ω, the resulting category Beh_ω provides a convenient framework to study the coalgebraic semantics of finitary modal logic. So an important next step is to understand the relation of both approaches.

Stone Coalgebras as Systems

We investigated coalgebras over Stone spaces as models for modal logic. But what is the significance of Stone-coalgebras from the point of view of systems? Here, following [26], as systems we consider coalgebras over Set. Compared to these, the addition of (Stone) topological structure basically means two things. First, morphisms have to be continuous, i.e., the topologies allow for more specific notions of behaviour [7]. Second, the carriers have to be compact. This is quite a severe restriction and many interesting transition systems are not compact. So we would like to understand which set-coalgebras are Stone-coalgebras and how Stone-behavioural equivalence relates to Set-behavioural equivalence.

Generalising Stone Coalgebras

Coalgebras over Stone spaces can be generalised in different ways. We have seen that replacing the topologies by Boolean algebras of sets leads to general frames. But it will also be of interest to consider other topological spaces as base categories. Here are two examples.

First, can we find useful examples of coalgebras over topological spaces, if we drop the compactness condition? For instance, can the topologies be used to restrain the behaviour in order to guarantee fairness and liveness properties?

Second, there is a close relationship between Stone spaces and complete ultrametric spaces. [8] Now complete ultrametric spaces are used in the semantics of programming languages (see e.g. [8]), but they also form a base category for coalgebras in [29]; this shows a clear need for further investigations. Moreover, using the results of [3] on how to partialise Stone spaces with a countable base using SFP-domains, it should be possible to establish a precise relation between modal logics for Stone-coalgebras and the logics for domains of [1].

[7] Recall that the notion of bisimulation or behavioural equivalence is defined in terms of the morphisms of the category. Requiring the morphisms to be continuous means that less states are identified under behavioural equivalence.

[8] A topological space is a Stone space with a countable base iff it is a complete totally bounded ultrametric space, see [28], Corollary 6.4.8.

Coalgebras and Duality Theory

Whereas many, or most, common dualities are induced by a schizophrenic object (see [16], Section VI.4.1), the duality of modal algebras and descriptive general frames is not. For a contradiction, write $K : \mathsf{MA} \to \mathsf{DGF}$, $L : \mathsf{DGF} \to \mathsf{MA}$ for the contravariant functors witnessing the duality and suppose that there is a schizophrenic object S, that is, $\mathsf{MA}(\mathbb{A}, S) = UK(\mathbb{A})$ where U denotes the forgetful functor $\mathsf{DGF} \to \mathsf{Set}$. Then $\mathsf{Set}(1, U\mathbb{G}) \cong U\mathbb{G} \cong UKL\mathbb{G} \cong \mathsf{MA}(L\mathbb{G}, S) \cong \mathsf{DGF}(KS, KL\mathbb{G}) \cong \mathsf{DGF}(KS, \mathbb{G})$, showing that KS is a free object over one generator in DGF. But using that the graph of a DGF-morphisms is a bisimulation, it is not hard to see that such an object cannot exist.

On the other hand, the duality $\mathsf{MA} \simeq \mathsf{DGF}^{\mathrm{op}}$ is an instance of the duality $\mathsf{Alg}(T^{\mathrm{op}}) \cong \mathsf{Coalg}(T)^{\mathrm{op}}$ of algebras and coalgebras, with the Vietoris functor $\mathbb{V}$ as the functor T. It seems therefore of interest to explore which dualities are instances of the algebra/coalgebra duality. As a first step in this direction, [23] shows that the duality between positive modal algebras and K^+-spaces can be described in a similar way as here (although the technical details are substantially more complicated).

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