Notions of Behaviour and Reachable-Part and their Institutions

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Abstract. Notions of observability and reachability induce a relation of indistinguishability on the models of specifications. We show how to obtain, in a systematic way, from a given institution, an institution that respects this indistinguishability relation. Moreover, observability and reachability are treated in formally dual way.

Introduction

This paper is concerned with combining notions of observability/reachability with institutions. Notions of observability and reachability play a major role in specification and verification. From the point of view of the user or specifier, such notions introduce an equivalence relation of indistinguishability on the models (implementations) of specifications: the user/specifier wants to reason only up to this equivalence.

Institutions [10] are logics which allow to reason about models for different signatures which is essential to achieve modularity. The central question of the paper is, therefore:

> Given a notion of observability or reachability and our favourite institution, is there a systematic way to build an institution respecting the corresponding relation of indistinguishability?

The aim of this work is twofold. First, to provide one possible answer to the question raised above. Second, following the ideas in [6], to show that observability and reachability can be treated in a formally dual way.

We proceed in three steps and establish four conditions which are sufficient for a positive answer to our question, leading to the notion of black-box institution. Section 1 formalises observability and reachability by taking a black-box point of view. In case of observability, we assume that for each model M, there is a black-box view BM, also called the behaviour of M, together with a quotient $M \to BM$. Dually, in case of reachability, we assume that for each model M, there is a black-box view RM, also called the reachable-part of M, together with an embedding $RM \to M$. The corresponding definitions introduce Conditions (1) and (2).

Section 2 extends the considerations from categories to indexed categories, that is, functors $\mathit{Mod} : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$. For this, we have to add Condition (3) stating that taking behaviours/reachable-parts commutes with reindexing (taking reducts).

Section 3 shows that we can extend our favourite 'standard' logic to a logic respecting the appropriate indistinguishability relation by requiring Condition (4) which states that a model satisfies a formula if the black-box view standardly satisfies the formula. This leads to the notion of black-box institutions, answering the question above in a positive way.

Section 4 shows that our framework applies indeed to our motivating examples of observational logic [13], constructor-based logic [5] and COL [4].

We conclude with a discussion of related work. The technical notions are recalled in an appendix reviewing categorical duality, indexed categories and institutions, and fibred categories.

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1 Notions of Behaviour and Reachable-Part

Our first task is to formalise observability and reachability in a way that is suitable for the concerns of the paper. To establish a formal duality we use the language of category theory which is briefly reviewed in appendix A.

1.1 Observability

Many notions of behaviour found in the literature can be described abstractly by two simple properties.

Definition B 1 *Let $\mathcal{C}$ be a category. Given an operation B on the objects of $\mathcal{C}$ and a family of arrows $\eta = (\eta_M : M \to BM)_{M \in \mathcal{C}}$, we call (B, η) a* ***notion of behaviour*** *for $\mathcal{C}$ iff*

$$\text{all } \eta_M \text{ are epi} \tag{1}$$

$$\forall M, N \in \mathcal{C} \text{ there is } (\cdot)^\sharp : \mathcal{C}(M, BN) \to \mathcal{C}(BM, BN) \text{ such that } f^\sharp \circ \eta_M = f \tag{2}$$

Note that the operation $(\cdot)^\sharp$ is uniquely determined due to (1). In case that $\mathcal{C}$ is a concrete category we call two elements of M **behaviourally equivalent**, and write $x \simeq_M y$, iff $\eta_M(x) = \eta_M(y)$. Intuitively, (1) expresses that every model M has a quotient BM which we call the **behaviour** of M; (2) is pictured as the left-hand diagram below

$$\begin{tikzcd} BM \arrow[r, "f^\sharp"] & BN \\ M \arrow[u, "\eta_M"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} BM \arrow[r, "(\eta_N \circ g)^\sharp"] & BN \\ M \arrow[u, "\eta_M"] \arrow[r, "g"'] & N \arrow[u, "\eta_N"'] \end{tikzcd}$$

and expresses that *morphisms in $\mathcal{C}$ preserve behavioural equivalence.* Indeed, for $g : M \to N$ in $\mathcal{C}$, we have, as indicated in the right-hand diagram above, $x \simeq_M y \;\Leftrightarrow\; \eta_M(x) = \eta_M(y) \;\Rightarrow\; (\eta_N \circ g)^\sharp \circ \eta_M(x) = (\eta_N \circ g)^\sharp \circ \eta_M(y) \;\Leftrightarrow\; \eta_N(g(x)) = \eta_N(g(y)) \;\Leftrightarrow\; g(x) \simeq_N g(y)$.

Here are some straightforward examples, more will follow later.

Example 1. 1. A well-known example is provided by the category of deterministic automata (without initial states, see [20] for details), BM being the minimal realisation of an automaton M and $\eta_M : M \to BM$ mapping a state in M to the state in BM that accepts the same language.

2. In the category of labelled transition systems, a notion of behaviour is given by the quotient BM of a transition system M wrt the largest bisimulation.
3. More generally, subsuming the above, for any set-endofunctor T, the category $\mathsf{Coalg}(T)$ of T-coalgebras [21] has a canonical notion of behaviour. For any T-coalgebra M, BM is given by the image factorisation $M \to BM \to Z$ of the unique arrow $M \to Z$ into the final T-coalgebra Z.[1]
4. Consider the signature for stacks over elements E given by $new : 1 \to X$, $push : E \times X \to X$, $pop : X \to X + 1$, $top : X \to E + 1$. Define a relation of observational (or behavioural) equivalence by $x \simeq y$ iff $top(pop^n(x)) = top(pop^n(y))$ for all $n \in \mathbb{N}$. Let $\mathcal{C}$ be the category consisting of the structures $(X, top, pop, push, new)$ for which $\simeq$ is a congruence.[2] Define BM to be the structure identifying all observationally equal states. Note that the array-pointer implementation and the standard model have the same behaviour.

Note that the last example is different from its predecessors in that $\mathcal{C}$ there has no final object due to the liberty in implementing *new* and *push*.

A notion of behaviour on $\mathcal{C}$ gives rise to two related categories, the category $\mathcal{C}^B$ consisting of all behaviours and the category $\mathcal{C}_B$ consisting of all models but seen from the point of view of their behaviours.

Definition B 2 *Consider a notion of behaviour B on $\mathcal{C}$.*

- *$\mathcal{C}^B$ is the full subcategory of $\mathcal{C}$ consisting of all objects M with $M \cong BM$.*
- *$\mathcal{C}_B$ has the same objects as $\mathcal{C}$ and morphisms $\mathcal{C}_B(M, N) = \mathcal{C}(BM, BN)$.*

Intuitively, $\mathcal{C}$ consists of all possible realizations of a specification whereas $\mathcal{C}^B$ only contains the black box views. $\mathcal{C}_B$ combines both aspects. The models are the same as in $\mathcal{C}$ but the morphisms incorporate the black box view, $\mathcal{C}_B(M, N) = \mathcal{C}(BM, BN)$.

Example 2 (Observational Logic). Our motivating example is observational logic [13]. In observational logic, each signature Σ determines a set of state sorts and a notion of observational equality for each state sort. The class $\mathcal{C}$ of models, called observational algebras, of a signature allows to take for any algebra $A \in \mathcal{C}$ the quotient BA of A wrt the observational equality induced by Sig. The important point to note about the category $\mathcal{C}_B$ is that its morphisms coincide with the observational morphisms of [13]. Since, on state sorts, a morphism maps a state to an equivalence class of states, these morphisms can be described as relations as in [13].

The reader not familiar with monads [19] can skip the next proposition and continue with its corollary and the following example. The proposition characterises notions of behaviour as *monads whose unit is componentwise epi* and shows that these monads are idempotent. In our context, idempotence means idempotence of taking behaviours, ie, $BBM \cong BM$.

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[1] The final coalgebra always exists [1] although its carrier may be a proper class. Avoiding the final coalgebra, BM can be defined as the quotient wrt the largest behavioural equivalence or as the co-intersection of all quotients of M.

[2] That $\simeq$ is a congruence means that using all operations does not allow us to distinguish more states than using only *top* and *pop*.

Proposition B 3 *There is an bijection between notions of behaviour and monads whose unit is componentwise epi. Moreover these monads are idempotent.*

Proof. Given a notion of behaviour (B, η) with lifting $(\cdot)^\sharp$, we show that $(B, \eta, (\cdot)^\sharp)$ is a Kleisli triple.[3] By definition of a Kleisli triple one has to check for all $g : M \to BN, f : L \to BM$ the laws (i) $f = f^\sharp \circ \eta_M$, (ii) $(\eta_M)^\sharp = \mathrm{id}_{BM}$, (iii) $(g^\sharp \circ f)^\sharp = g^\sharp \circ f^\sharp$. This is straight forward. Conversely, it is immediate that every Kleisli-triple $(B, \eta, (\cdot)^\sharp)$ (and hence monad) with η_M epi gives rise to a notion of behaviour (B, η). Finally, to see that these monads are idempotent, that is, that the multiplication μ_M is iso, apply Condition (2) to $f = \mathrm{id}_{BM}$ and use Condition (1). (Details of the proof can be found in [16], Lemma 2.2).

The fact that B is an idempotent monad determines the structure described in the following corollary.

Corollary B 4 *Let (B, η) be a notion of behaviour for $\mathcal{C}$. There are functors*

$$\begin{tikzcd}
& \mathcal{C}^B \arrow[dl, hook, bend right, "I'"'] \\
\mathcal{C} \arrow[loop left, "B"] \arrow[ur, bend right, "B'"'] \arrow[ur, phantom, "top"] \arrow[dr, bend right, "H"'] \arrow[dr, phantom, "top"] & \\
& \mathcal{C}_B \arrow[uu, bend right, "B''"'] \arrow[ul, bend right, "G"']
\end{tikzcd}$$

where B', B'', and G map an object to its behaviour, I' is the inclusion of behaviours, and H is the identity on objects, all satisfying $I'B' = B = GH$, $B''H = B'$, $I'B'' = G$. Moreover, η is a natural transformation, behaviour is left adjoint to inclusion ($B' \dashv I'$), B'' is an equivalence of categories, and $\mathcal{C}^B$ is a full reflective subcategory of $\mathcal{C}$.

Proof. The category of algebras for the monad is (isomorphic to) $\mathcal{C}^B$ since the multiplication is iso. The Kleisli-category is (isomorphic to) $\mathcal{C}_B$. It now follows from B being a monad: functoriality, the equations, the adjunctions, naturality of η, and B'' full and faithful. Since the multiplication is iso, every object in $\mathcal{C}^B$ is isomorphic to an object in the image of B'', hence B'' is an equivalence. $\mathcal{C}^B$ is a full reflective subcategory due to the bijection between full reflective subcategories and idempotent monads. (See [7], Vol.2, Proposition 4.1.6 and Proposition 4.2.3 for proofs of the facts we used).

Example 3. 1. In the case of deterministic automata, $\mathcal{C}^B$ is (equivalent to) the category of languages,[4] B' can be understood as mapping an automaton to the language it accepts and I' as mapping a language to the minimal realising automaton. The insight that behaviour is left-adjoint to minimal realisation, $B' \dashv I'$, is due to Goguen [11].

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[3] There is a well-known bijection between Kleisli triples and monads. The monad (B, η, μ) associated to the Kleisli triple $(B, \eta, (\cdot)^\sharp)$ is given as follows: B is defined on arrows as $Bf = (\eta_N \circ f)^\sharp$ for $f : M \to N$ and $\mu_M = (\mathrm{id}_{BM})^\sharp$ for $M \in \mathcal{C}$. Conversely, given a monad (B, η, μ), the Kleisli triple is $(B, \eta, (\cdot)^\sharp)$ with $f^\sharp = \mu_M \circ Bf$ for $f : M \to BN$.

[4] More precisely, given an alphabet Σ, objects are subsets $L \subseteq \Sigma^*$ ordered by inclusion.

2. Continuing Example 2, the 'observational black-box functor' of observational logic appears here as $\mathcal{C}_B \xrightarrow{B''} \mathcal{C}^B \hookrightarrow \mathsf{Alg}(\Sigma)$. It is full and faithful since B'' is full and faithful. Moreover, the category $\mathcal{C}^B$ of fully abstract algebras is a full reflective subcategory of the category $\mathcal{C}$ of observational algebras.

1.2 Reachability

(R, ε) is a notion of reachable-part for $\mathcal{C}$ iff $(R^{\mathrm{op}}, \varepsilon^{\mathrm{op}})$ is a notion of behaviour for $\mathcal{C}^{\mathrm{op}}$.[5] In detail:

Definition R 1 *Let $\mathcal{C}$ be a category. Given an operation R on the objects of $\mathcal{C}$ and a family of arrows $\varepsilon = (\varepsilon_M : RM \to M)_{M \in \mathcal{C}}$, we call (R, ε) a* ***notion of reachable-part*** *for $\mathcal{C}$ iff*

$$\text{all } \varepsilon_M \text{ are mono}$$
$$\forall N, M \in \mathcal{C} \text{ there is } (\cdot)^\sharp : \mathcal{C}(RN, M) \to \mathcal{C}(RN, RM) \text{ such that } \varepsilon_M \circ f^\sharp = f$$

In case that $\mathcal{C}$ is a concrete category we call an element x of M **reachable** iff x is in the image of ε_M. Intuitively, the dual of (1) expresses that every model has a submodel RM which we call the **reachable-part** of M. The dual of (2), see the left-hand diagram below,

$$\begin{tikzcd} RM \arrow[d, "\varepsilon_M"'] & RN \arrow[l, "f^\sharp"'] \arrow[dl, "f"] \\ M & \end{tikzcd} \qquad \begin{tikzcd} RM \arrow[d, "\varepsilon_M"'] & RN \arrow[l, "(g \circ \varepsilon_N)^\sharp"'] \arrow[d, "\varepsilon_N"] \\ M & N \arrow[l, "g"'] \end{tikzcd}$$

expresses that morphisms in $\mathcal{C}$ preserve reachability. Indeed, with the notation of the right-hand diagram above, an element y in N is reachable iff there is y' in RN with $\varepsilon_N(y') = y$. Now, for $g : N \to M$ it holds that $g(y) = g \circ \varepsilon_N(y') = \varepsilon_M((g \circ \varepsilon_N)^\sharp(y'))$, that is, $g(y)$ is reachable.

Here is a typical example of reachability:

Example 4. 1. In the category of deterministic automata (with initial state), there is an operation R mapping an automaton M to the automaton RM which contains only states reachable from the initial state.
2. More generally, given a signature for algebras, there is the operation R mapping an algebra M to the algebra RM which contains only elements which are denoted by a term.

Dualising Definition B 2, we obtain categories $\mathcal{C}^R$ and $\mathcal{C}_R$. The objects of $\mathcal{C}^R$ are called **reachable**, an interpretation of the morphisms of $\mathcal{C}_R$ is given in the remark following the definition.

Definition R 2 *Consider a notion of reachable-part R on $\mathcal{C}$.*

- *$\mathcal{C}^R$ is the full subcategory of $\mathcal{C}$ consisting of all objects M with $M \cong RM$.*
- *$\mathcal{C}_R$ has the same objects as $\mathcal{C}$ and morphisms $\mathcal{C}_R(M, N) = \mathcal{C}(RM, RN)$.*

[5] The $(\cdot)^{\mathrm{op}}$-notation is explained in appendix A.

Remark 1. In categorical language a partial map $X \to Y$ with domain D can be described, see the left-hand diagram,

$$
\begin{tikzcd}
& D \arrow[dl] \arrow[d] \\
Y & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& RN \arrow[dl] \arrow[d, "\varepsilon_N"] \\
M & N
\end{tikzcd}
$$

as a mono $D \to X$ and a map $D \to Y$. Because a morphism $RN \to RM$ of $\mathcal{C}_R$ can be considered as a morphism $RN \to M$,[6] the right-hand diagram shows that morphisms in $\mathcal{C}_R$ are partial morphisms defined on reachable-parts.

Example 5 (Constructor-Based Logic). Our motivating example is constructor-based logic [6]. In constructor-based logic, each signature Σ determines a set of *constrained sorts* along with a set of *constructor* operations (defining how to construct elements of constrained sorts). The class $\mathcal{C}$ of models, called constructor-based algebras, of a signature allows to take for any $A \in \mathcal{C}$ the reachable-part (called generated part in [5]) RA of A.[7] The constructor-based morphisms of [6] coincide with the morphisms in $\mathcal{C}_R$. Their distinguishing feature is that they are, on constrained sorts, partial functions defined only on the reachable-part.

It follows from the duality principle and Proposition B 3 that notions of reachable-part coincide with comonads having a counit that is componentwise mono and that these comonads are, moreover, idempotent. Idempotence means here idempotence of taking reachable-parts, ie, $RRM \cong RM$.

Proposition R 3 *There is a bijection between notions of reachable-part and comonads having a counit that is componentwise mono. Moreover, this comonads are idempotent.*

Corollary R 4 *Let (R, ε) be a notion of behaviour for $\mathcal{C}$. There are functors*

$$
\begin{tikzcd}
& \mathcal{C}^R \arrow[dl, bend right, hook', "I'"'] \\
\mathcal{C} \arrow[loop left, "R"] \arrow[ur, bend right, "R'"'] \arrow[ur, phantom, "\perp"] \arrow[dr, bend right, "H"'] \arrow[dr, phantom, "\perp"] & \\
& \mathcal{C}_R \arrow[ul, bend right, "G"'] \arrow[uu, "R''"']
\end{tikzcd}
$$

where R', R'', and G map an object to its reachable-part, I' is the inclusion of reachable-parts, and H is the identity on objects, all satisfying $I'R' = R = GH$, $R''H = R'$, $I'R'' = G$. Moreover, ε is a natural transformation, restriction to reachable-parts is right adjoint to inclusion ($I' \dashv R'$), R'' is an equivalence of categories, and $\mathcal{C}^R$ is a full coreflective subcategory of $\mathcal{C}$.

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[6] It follows from Corollary R 4 below (or directly from the dual of Conditions (1) and (2)) that $\mathcal{C}(RN, RM) \to \mathcal{C}(RN, M)$, $f \mapsto \varepsilon_M \circ f$, is a bijection, the inverse given by $(\cdot)^\sharp$.

[7] Compared to earlier treatments of reachability, the distinguishing feature of constructor-based logic is that models are not required to be reachable, operations only have to preserve reachability.

2 Notions of Behaviour and Reachable-Part for Indexed Categories

In a second step we extend our notions of behaviour/reachable-part from categories to indexed categories.

2.1 Observability

We require a notion of behaviour for each index and that reindexing preserves behaviours.

Definition B 5 (Notion of Behaviour for Indexed Categories) *Given a functor $Mod : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$ we call a family $\mathsf{B} = (B_\Sigma, \eta_\Sigma)_{\Sigma \in \mathsf{Sig}}$ a* notion of behaviour *for Mod iff each (B_Σ, η_Σ) is a notion of behaviour for $Mod(\Sigma)$ and for all arrows $\sigma : \Sigma \to \Sigma'$ and all $M' \in Mod(\Sigma')$ an isomorphism making the diagram*

$$\begin{tikzcd}
\mathsf{B}(Mod(\sigma)(M')) \arrow[rr, "\cong"] & & Mod(\sigma)(\mathsf{B}(M')) \\
& Mod(\sigma)(M') \arrow[ul, "\eta_{Mod(\sigma)(M')}"] \arrow[ur, "Mod(\sigma)(\eta_{M'})"'] &
\end{tikzcd} \tag{3}$$

commute. We write B for $B_{\Sigma'}, B_\Sigma$, respectively, the subscripts being clear from the context.

Intuitively, (3) states that $Mod(\sigma)$ preserves behaviours. The habit of avoiding the subscripts of $B_{\Sigma'}, B_\Sigma$ comes from an equivalent formulation of the above definition in terms of fibred categories (aka fibrations). The reader not familiar with fibred categories may continue with Corollary B 8.

The following definition generalises the characterisation of notions of behaviour given by Proposition B 3 from categories to fibred categories.

Definition B 6 (Notion of Behaviour for Fibrations) *Let $p : \mathcal{C} \to \mathcal{X}$ be a fibration. A fibred monad (B, η, μ) on p is called a notion of behaviour for p iff all η_M, $M \in \mathcal{C}$, are epi.*

Remark 2. The concept of a monad with componentwise epi unit makes sense for any 2-category. Definition B 6 formulates the instance in the 2-category of fibrations over $\mathcal{X}$ whereas the previous section dealt with the instance in the 2-category of categories.

One advantage of Definition B 6 is that monads on fibrations are easier to define than monads on indexed categories and hence the analogue of Corollary B 4 comes more natural in terms of fibrations. But before going into this let us first sketch the equivalence of the two definitions.

Proposition B 7 *Definition B 5 and Definition B 6 are equivalent.*

Proof. The proof is a corollary to the equivalence of (non-split) indexed categories and cloven fibrations, see eg [12]. We sketch the argument for our special case. Let $Mod : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$ be a functor and $p : \mathcal{C} \to \mathsf{Sig}$ the corresponding fibration obtained via the Grothendieck-construction. We give a bijection between notions of behaviour for Mod and p. First, given (B, η, μ) on p we define B_Σ and η_Σ as the restriction of B and η to the fibre over Σ. Condition (3) is satisfied since B preserves cartesian arrows. Second, given (B_Σ, η_Σ), we can extend B to all of $\mathcal{C}$ as follows. Every morphism $f : M \to N$ in $\mathcal{C}$ over $\sigma : \Sigma \to \Sigma'$ can be factored as $\bar{\sigma} \circ f^{|}$ where $\bar{\sigma}$ is cartesian over σ and $f^{|}$ is vertical. Let $Bf = \bar{\bar{\sigma}} \circ \hat{\sigma}_N \circ Bf^{|}$ where $\bar{\bar{\sigma}}$ is cartesian over σ in BN and $\hat{\sigma}_N$ is the (unique) iso given by Condition (3). Thanks to Condition (1), the isomorphism $\hat{\sigma}_N$ given by Condition (3) is natural in N and satisfies $\hat{\sigma}^{\mathrm{id}}_N = \mathrm{id}_{BN}$ and $\widehat{\sigma \circ \varrho}_N = Mod(\varrho)(\hat{\sigma}_N) \circ \hat{\varrho}_{Mod(\sigma)(N)}$. Chasing the obvious big diagram now shows that B is indeed a functor. B is fibrewise a monad due to Proposition B 3 and preserves cartesian liftings by construction. $\square$

A pleasant consequence of the proposition is that we now obtain a version of Corollary B 4 for indexed categories. Indeed, let $\mathsf{B} = (B_\Sigma, \eta_\Sigma)_{\Sigma \in \mathsf{Sig}}$ be a notion of behaviour for $Mod : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$. By Proposition B 7, we can extend B to a monad B on the total category $\mathcal{C}$ of the fibration p corresponding to Mod. Now we can apply Corollary B 4 to $\mathcal{C}$ and B.

$$
\begin{tikzcd}
 & \mathcal{C}^B \arrow[dl, hook', bend right, "I'"'] \\
\mathcal{C} \arrow[loop left, "B"] \arrow[ur, bend right, "B'"'] \arrow[ur, phantom, "\top"] \arrow[dr, "H"'] \arrow[dr, phantom, "\top"] & \\
 & \mathcal{C}_B \arrow[ul, bend right, "G"'] \arrow[uu, "B''"']
\end{tikzcd}
$$

Moreover, we know (see [16], Proposition 4.18) that $\mathcal{C}^B$ and $\mathcal{C}_B$ are fibred over Sig, that is, there are fibrations p^B and p_B making the diagram above into a diagram in the (2-)category of fibrations over Sig. Translating the diagram back to indexed categories, it follows that the constructions of Corollary B 4 extend from categories to indexed categories.

For future reference we record (recall Definition B 2 for the operations $(\cdot)^{B_\Sigma}$ and $(\cdot)_{B_\Sigma}$)

Corollary B 8 *Let* $\mathsf{B} = (B_\Sigma, \eta_\Sigma)_{\Sigma \in \mathsf{Sig}}$ *be a notion of behaviour for* $Mod : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$. *Then there are functors* $Mod^{\mathsf{B}} : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$, $\Sigma \mapsto Mod(\Sigma)^{B_\Sigma}$ *and* $Mod_{\mathsf{B}} : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$, $\Sigma \mapsto Mod(\Sigma)_{B_\Sigma}$.

2.2 Reachability

$(R_\Sigma, \varepsilon_\Sigma)_{\Sigma \in \mathsf{Sig}}$ is a notion of reachable-part for the indexed category $Mod : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$ iff $({R_\Sigma}^{\mathrm{op}}, {\varepsilon_\Sigma}^{\mathrm{op}})$ is a notion of behaviour for the dual indexed category Mod^{co}.[8] In detail:

[8] The definition of $(\cdot)^{\mathrm{co}}$ is recalled in appendix B.

Definition R 5 *Given a functor* $Mod : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$ *we call a family* $\mathsf{R} = (R_\Sigma, \varepsilon_\Sigma)_{\Sigma \in \mathsf{Sig}}$ *a* ***notion of reachable-part*** *for Mod iff each* $(R_\Sigma, \varepsilon_\Sigma)$ *is a notion of reachable-part for* $Mod(\Sigma)$ *and for all arrows* $\sigma : \Sigma \to \Sigma'$ *and all* $M' \in Mod(\Sigma')$ *an isomorphism making the diagram*

$$\begin{tikzcd} Mod(\sigma)(\mathsf{R}(M')) \arrow[rr, "\cong"] \arrow[dr, "Mod(\sigma)(\varepsilon_{M'})"'] & & \mathsf{R}(Mod(\sigma)(M')) \arrow[dl, "\varepsilon_{Mod(\sigma)(M')}"] \\ & Mod(\sigma)(M') & \end{tikzcd}$$

commute. We write R *for* $R_{\Sigma'}, R_\Sigma$*, respectively, the subscripts being clear from the context.*

Intuitively, the dual of (3) states that $Mod(\sigma)$ preserves reachability.

A notion of reachable-part on a cofibration q is dual to a notion of behaviour on the fibration q^{op}, in detail:

Definition R 6 *Let* $q : \mathcal{C} \to \mathcal{X}$ *be a cofibration. A cofibred comonad* (R, ε, δ) *on* q *is called a notion of reachable-part for* q *iff all* ε_M*,* $M \in \mathcal{C}$*, are mono.*

Proposition R 7 *Definition R 5 and Definition R 6 are equivalent.*

Proof. Let $Mod : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$ be a functor and $q : \mathcal{C} \to \mathsf{Sig}$ the corresponding co-fibration obtained via the co-Grothendieck-construction. Apply Proposition B 7 to Mod^{co} and q^{op}.

Corollary R 8 *Let* $\mathsf{R} = (R_\Sigma, \varepsilon_\Sigma)_{\Sigma \in \mathsf{Sig}}$ *be a notion of reachable-part for* $Mod : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$*. Then there are functors* $Mod^{\mathsf{R}} : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$*,* $\Sigma \mapsto Mod(\Sigma)^{R_\Sigma}$ *and* $Mod_{\mathsf{R}} : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$*,* $\Sigma \mapsto Mod(\Sigma)_{R_\Sigma}$*.*

3 Institutions for Observability and Reachability

In this section, we show how to obtain institutions whose satisfaction relation respects a given notion of behaviour or reachable-part. In order to treat both cases simultaneously, we call the behaviour/reachable-part of a model the black-box view of the model.

We assume that we are given a 'standard' logic as eg first-order logic which we use to reason about black-box views and that we want to extend this logic to all models in a way such that formulas of the logic can not distinguish between a model and its black box view.

Definition 9 *Let* $\mathcal{C}$ *be a class (of models),* V *an operation on* $\mathcal{C}$ *(with* $\mathsf{V}M$ *called the black-box view of* $M \in \mathcal{C}$*),* $\mathcal{L}$ *a class (of formulas) and* $\models, \Vdash\!\models \subseteq \mathcal{C} \times \mathcal{L}$ *two relations. Then* $\Vdash\!\models$ *is called the* ***black-box view*** *relation induced by* $(\mathsf{V}, \models)$ *iff*

$$M \Vdash\!\models \varphi \iff \mathsf{V}M \models \varphi \tag{4}$$

We can now define the notion of a black-box institution. Recall that a notion of behaviour (reachable-part) V for an indexed category $Mod : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$ is given by functors V_Σ on $Mod(\Sigma)$ satisfying (the dual of) Conditions (1)–(3). According to Definition B(R) 2 and Corollary B(R) 8, V and Mod give rise to the indexed categories Mod^{V} and Mod_{V}, Mod^{V} being the category of the black-box views.

Definition 10 *Let* V *be a notion of behaviour (reachable-part) for* Mod *and* $(Mod^{\mathsf{V}}, Sen, \models)$ *be an institution and* $\Vvdash$ *be the black-box view relation induced by* $(\mathsf{V}, \models)$*. Then* $(Mod, Sen, \Vvdash)$ *and* $(Mod_{\mathsf{V}}, Sen, \Vvdash)$ *are the* ***black-box institutions*** *induced by* $(\mathsf{V}, \models)$*.*

The definition is justified by the following theorem (we assume that institutions have satisfaction relations that do not distinguish isomorphic models).

Theorem 1. *If* V *is a notion of behaviour or reachable-part for Mod then* Mod^{V} *and* Mod_{V} *are functors. If, moreover,* $(Mod^{\mathsf{V}}, Sen, \models)$ *is an institution then* $(Mod, Sen, \Vvdash)$ *and* $(Mod_{\mathsf{V}}, Sen, \Vvdash)$ *are institutions.*

Proof. The first part of the theorem is Corollary B 8. For the second part we have to show that for $(Mod, Sen, \Vvdash)$ and $(Mod_{\mathsf{V}}, Sen, \Vvdash)$ the satisfaction condition of institutions holds. The proof being the same in both cases, we pick the first. Let $\sigma : \Sigma \to \Sigma'$ be a signature morphism, $\varphi \in Sen(\Sigma)$, $M' \in Mod(\Sigma')$. We have

$$Mod(\sigma)(M') \Vvdash \varphi \;\Leftrightarrow\; \mathsf{V}(Mod(\sigma)(M')) \models \varphi \;\Leftrightarrow\; Mod(\sigma)(\mathsf{V}M') \models \varphi \;\Leftrightarrow\; \mathsf{V}M' \models Sen(\sigma)(\varphi) \;\Leftrightarrow\; M' \Vvdash Sen(\sigma)(\varphi).$$

The equivalences are due to, respectively, Condition (4), Condition (3), the satisfaction condition for $\models$, and Condition (4). □

In the same way, we obtain theorems for combining notions of behaviour and reachable-part. For example, consider a situation where we start with some indexed category of models Mod and assume that we have a notion of reachable-part R for Mod and then a notion of behaviour on reachable algebras. The black-box view of a model M is then given by $\mathsf{BR}M$ and all black-box views are assembled in the indexed category $(Mod^{\mathsf{R}})^{\mathsf{B}}$. As before, we suppose that we have a standard logic $\models$ on black-box views that we want to transfer to all models, organised in the indexed categories Mod or $(Mod_{\mathsf{R}})_{\mathsf{B}}$.

Theorem 2. *If* R *is a notion of reachable-part for Mod and* B *a notion of behaviour for* Mod^{R} *then* $(Mod^{\mathsf{R}})^{\mathsf{B}}$ *and* $(Mod_{\mathsf{R}})_{\mathsf{B}}$ *are functors. If, moreover,* $((Mod^{\mathsf{R}})^{\mathsf{B}}, Sen, \models)$ *is an institution and* $\Vvdash$ *is the black-box view relation induced by* $(\mathsf{BR}, \models)$*, then* $(Mod, Sen, \Vvdash)$ *and* $((Mod_{\mathsf{R}})_{\mathsf{B}}, Sen, \Vvdash)$ *are institutions.*

Proof. For the first part, note that a notion of behaviour B for Mod^{R} induces a notion of behaviour for the equivalent category Mod_{R}, justifying to consider $(Mod_{\mathsf{R}})_{\mathsf{B}}$. The second part is shown as for the theorem above.

We have shown how to reduce the proof of institutions to the verification of Conditions (1)–(4). In all applications we will consider in the next section the verification of (1) and (2) is straightforward. Similarly, $\Vvdash$ is usually designed to satisfy (4). So the point is to carefully choose Sig in order to guarantee (3).

4 Applications

We sketch how our framework applies to the examples that motivated our presentation, namely observational logic, constructor-based logic, and COL.

Observational Logic Observational logic was introduced in [13]. Our discussion refers to the presentation in [5]. Conditions (1) and (2) are satisfied, because the class of models, called observational algebras, is defined in such a way that the quotient wrt observational equality is again a model (cf. Definition 2.6 in [5]). Technically this is expressed by the requirement that observational equality is a congruence ([5], Definition 2.6) which in turn formalises the intuition that non-observer operations are not allowed to contribute to observations. Condition (3) is satisfied because the category Sig has been defined in a way such that reindexing preserves behaviours ([5], Corollary 2.19). Condition (3) can be understood as requiring that a signature morphism $\sigma : \Sigma \to \Sigma'$ from an 'old' signature Σ to a 'new' signature Σ' may not introduce new observations on old sorts (formalised as [5], Definition 2.14). Condition (4) appears as Theorem 2.12 in [5].

Constructor-Based Logic Constructor-based logic was introduced in [6]. Our discussion refers to the presentation in [5]. Conditions (1) and (2) are satisfied, because the class of models, called constructor-based algebras, is defined in such a way that the sub*sets* of elements which are generated by constructors form a sub*algebra* ([5], Definition 3.6 and 3.7). This formalises the intuition that the use of non-constructors cannot lead to elements which are not reachable by constructors. Condition (3) is [5], Corollary 3.19, and a consequence of the definition of Sig which can be phrased as 'new constructors may not generate new elements on old sorts'. Condition (4) is Theorem 3.12 in [5].

Constructor-Based Observational Logic (COL) COL is introduced in [4]. COL consists of an integration of the two concepts above. Let us denote by $Mod : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$ the indexed category consisting of all COL-algebras with standard algebra morphisms. Constructors and observers determine a notion of reachable-part R. For any COL-algebra A, $\mathsf{R}A$ consists of all elements which are observably equal to an element generated by constructors. This yields an functor Mod_{R}. (Taking into account only constructible experiments,) observers then determine a notion of behaviour B on Mod_{R}, yielding the indexed category $(Mod_{\mathsf{R}})_{\mathsf{B}}$ of COL-algebras with COL-morphisms. That COL now forms an institution follows from Theorem 2 by establishing Conditions (3) and (4).

5 Conclusion

The aim of this paper was to give a semantic analysis of how to obtain institutions respecting notions of observability and reachability. We formalised observability and reachability as *notions of behaviour and reachable-part* and showed how they give rise to *black-box institutions*. We believe that the simplicity of Conditions (1)–(4) defining black-box institutions, together with the duality inherent in our approach, provide a nice and unifying picture for several known examples of institutions.

Nevertheless, it would certainly be interesting to find more instances of our framework. One possibility might be to look at generalisations. For example, our approach relied on the notion of an idempotent monad which is slightly more general than a notion of behaviour since the unit is not required to be componentwise epi. Moreover, most results would also hold for monads in general.

Institutions and Indexed Categories On the one hand, we followed the tradition to say that, in an institution, $Mod : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$ is a functor (= split indexed category) and not a pseudo-functor (= indexed category). On the other hand, in Definition B 5 it seemed more natural to consider isomorphisms in (3) rather than identity. But this means that B is a functor in the category of (non-split) indexed categories and not in the category of split indexed categories. It therefore seems natural to allow pseudo-functors in the definition of an institution.[9]

Induction and Coinduction It is well-known that notions of behaviour and reachable-part give rise to coinduction and induction proof principles. On our level of abstraction, however, we cannot deal with these more concrete[10] issues. On a more concrete level this has been done eg in [5].

Related Work The duality of observability and reachability goes back to Kalman [15] in the context of linear systems in control theory. His results were generalised by Arbib and Manes [2,3] to, essentially, algebras for an endofunctor on an arbitrary category. In [6] we took up their idea of duality of observability and reachability and applied it to specification formalisms. This paper continues [6] by showing that also the institution proofs for the observability and reachability case can be obtained from each other by using a formal duality principle.

The idea to use Conditions (1)–(4) to prove a general theorem about the existence of institutions appears in [16] where it was extracted from the proof of the observational logic institution in [13]. Compared to [16], which contains most of the technicalities used here, we have simplified matters by insisting on Condition (1) instead of working with idempotent monads in general and by moving the fibrations more into the background. More importantly, we show here that this approach fits nicely with the duality of observability and reachability and can also be applied to institutions which integrate both observability and reachability.

Let us also note that our approach has been designed for cases where we start from a standard logic which is an institution but does not respect the black-box view. A different approach is to start with a logic which respects the black-box view and directly build an institution. This approach is natural, for instance, if modal logic is used as a specification language and the black-box view is given by bisimulation, for examples see [9,8,17].

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[9] In the three examples of Section 4, the isomorphism of Condition (3) is an identity.

[10] We can take *concrete* here in the technical sense as referring to categories having a forgetful functor to sets: induction is about sub*sets* being closed under operations and coinduction is about *elements* being bisimilar.

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A Categorical Duality

We briefly review categorical duality, for background see [19]. A category $\mathcal{C}$ consists of a class of objects, also denoted by $\mathcal{C}$, and for all $A, B \in \mathcal{C}$ of a set of arrows (or morphisms) $\mathcal{C}(A,B)$. The *dual* (or opposite) category $\mathcal{C}^{\mathrm{op}}$ has the same objects and arrows $\mathcal{C}^{\mathrm{op}}(A,B) = \mathcal{C}(B,A)$. We write A^{op} and f^{op} for $A \in \mathcal{C}$ and $f \in \mathcal{C}(B,A)$ to indicate when we think of A as an object in $\mathcal{C}^{\mathrm{op}}$ and of f as an arrow in $\mathcal{C}^{\mathrm{op}}(A,B)$. Duality can now be formalised as follows. Let P be a property of objects or arrows in $\mathcal{C}$. We then say that:

An object A (arrow f, respectively) in $\mathcal{C}$ has property co-P
iff A^{op} (f^{op}, respectively) has property P.

For example, a morphism $f \in \mathcal{C}(A,B)$ is co-epi (usually called mono) iff f^{op} is epi.

The duality principle extends to functors and natural transformations. The dual of a functor $F : \mathcal{C} \to \mathcal{D}$ is the functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ which acts on objects and morphisms as F does. The dual of a natural transformation $\eta = (\eta_A)$ is $\eta^{\mathrm{op}} = (\eta_{A^{\mathrm{op}}})$. For instance, for an endofunctor F, the category of F-coalgebras is dual to the category of F^{op}-algebras; a functor F is left adjoint to G iff F^{op} is right adjoint to G^{op}; and (T, η, ε) is a comonad on $\mathcal{C}$ iff $(T^{\mathrm{op}}, \eta^{\mathrm{op}}, \varepsilon^{\mathrm{op}})$ is a monad on $\mathcal{C}^{\mathrm{op}}$.

B Indexed Categories and Institutions

For our purposes, unless stated otherwise, an *indexed category* over a (large) category $\mathcal{I}$ is a functor $H : \mathcal{I}^{\mathrm{op}} \to \mathsf{CAT}$ to the (superlarge) category CAT of all large categories.

An institution $(\mathit{Mod}, \mathit{Sen}, \models)$ consists of an indexed category Mod over a category of *signatures* Sig, a functor $\mathit{Sen} : \mathsf{Sig} \to \mathsf{Set}$, and for each signature $\Sigma \in \mathsf{Sig}$ a *satisfaction relation* $\models_\Sigma \subseteq \mathit{Mod}(\Sigma) \times \mathit{Sen}(\Sigma)$ such that for all $\sigma : \Sigma \to \Sigma'$, all $\varphi \in \mathit{Sen}(\Sigma)$, and all $M' \in \mathit{Mod}(\Sigma')$

$$\mathit{Mod}(\sigma)(M') \models_\Sigma \varphi \quad \Leftrightarrow \quad M' \models_{\Sigma'} \mathit{Sen}(\sigma)(\varphi).$$

This condition is called the *satisfaction condition* of institutions. Moreover, we require that $M_1 \models_\Sigma \varphi \;\Leftrightarrow\; M_2 \models_\Sigma \varphi$ whenever M_1, M_2 are isomorphic. We usually write $M \models \varphi$ instead of $M \models_\Sigma \varphi$ because the signature Σ can always be inferred from the signature of the model M. For more on institutions see [23].

Dual Notions The *dual of an indexed category* $H : \mathcal{I}^{\mathrm{op}} \to \mathsf{CAT}$ is the indexed category $H^{\mathrm{co}} : \mathcal{I}^{\mathrm{op}} \to \mathsf{CAT}$ given by $H^{\mathrm{co}}(u) = (Hu)^{\mathrm{op}} : (HJ)^{\mathrm{op}} \to (HI)^{\mathrm{op}}$ for $u : I \to J$. Note that the dual H^{co} of H as an indexed category is different from the dual H^{op} of H as a functor.

C Fibrations

We briefly recall the basic notions of fibred category. This material is only needed for Definition B 6 and Proposition B 7. For more on fibred categories see [14,7,22].

A functor $H : \mathcal{I}^{\mathrm{op}} \to \mathsf{CAT}$ is called an indexed category, since for each 'index' $I \in \mathcal{I}$ there is a category $H(I)$ and for each arrow $u : I \to J$ in $\mathcal{I}$ there is a functor $H(u) : H(J) \to H(I)$ 'reindexing' objects in $H(J)$ along u.

Alternatively, one can use a functor $p : \mathcal{A} \to \mathcal{I}$ to describe how objects of $\mathcal{A}$ are indexed over objects of $\mathcal{I}$. Consider an arrow $u : I \to J$ in $\mathcal{I}$. We say that A is *over* I [f is over u] if $p(A) = I$ [$p(f) = u$] and that f is *vertical* if $p(f)$ is the identity. $\mathcal{A}$ is called the *total category* and the *fibre* over I is the category consisting of the objects over I and the vertical arrows between them.

Using this terminology, we define a *fibration* over $\mathcal{I}$ to be a functor $p : \mathcal{A} \to \mathcal{I}$ such that for all $u : I \to J$ in $\mathcal{I}$ and all A over J there is an object $u^*(A)$ and a so-called *cartesian arrow* $\bar{u} : u^*(A) \to A$ over u, that is, for all f over u there is a unique vertical g such that $\bar{u} \circ g = f$. Moreover, cartesian arrows are required to be closed under composition.

A fibration is called a *split fibration* if it comes equipped with a choice $u^*(A) \to A$ of cartesian arrow for each u and A and, moreover, $(id_I)^*$ is the identity functor and $(u \circ v)^* = v^* \circ u^*$.

There is an equivalence between functors $\mathcal{I}^{\mathrm{op}} \to \mathsf{CAT}$ and split fibrations over $\mathcal{I}$. Given a split fibration $p : \mathcal{A} \to \mathcal{I}$, define $H(I)$ to be the fibre over I and, for $u : I \to J$, let $H(u) = u^*$. Given a functor $H : \mathcal{I}^{\mathrm{op}} \to \mathsf{CAT}$ the corresponding fibration $p : \mathcal{A} \to \mathcal{I}$ is obtained via the *Grothendieck construction* as follows. The objects of $\mathcal{A}$ are pairs $(I, A), I \in \mathcal{I}, A \in H(I)$. Morphisms of $\mathcal{A}$ are pairs $(u, f) : (I, A) \to (J, B)$ with $u : I \to J \in \mathcal{I}$ and $f : A \to (Hu)(B)$. $p : \mathcal{A} \to \mathcal{I}$ is the first projection.

A *fibred functor* $F : p \to p'$ between fibrations $p : \mathcal{A} \to \mathcal{I}$, $p' : \mathcal{A}' \to \mathcal{I}$ is a functor $F : \mathcal{A} \to \mathcal{A}'$ such that $p'F = p$ and F preserves cartesian liftings. A *fibred natural transformation* between fibred functors is a natural transformation which has vertical components.

Dual Notions The functor $q : \mathcal{A} \to \mathcal{I}$ is a *(split) cofibration* (over I) iff $q^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{I}^{\mathrm{op}}$ is a (split) fibration. There is an equivalence between indexed categories over $\mathcal{I}$ and split cofibrations over $\mathcal{I}^{\mathrm{op}}$. Given a functor $H : \mathcal{I}^{\mathrm{op}} \to \mathsf{CAT}$ the corresponding cofibration $q : \mathcal{A} \to \mathcal{I}^{\mathrm{op}}$ is obtained via the *co-Grothendieck construction* as follows. The objects of $\mathcal{A}$ are pairs $(I, A), I \in \mathcal{I}, A \in H(I)$. The morphisms of $\mathcal{A}$ are pairs $(u, f) : (J, B) \to (I, A)$ with $u : J \to I \in \mathcal{I}^{\mathrm{op}}$ and $f : (Hu)(B) \to A$.

Note that, given an indexed category H, the corresponding fibration p and cofibration q have identical fibres but the non-vertical arrows are generally different. The cofibration obtained from H^{co} is dual to the fibration obtained from H.

For examples of institutions which are more naturally associated with cofibrations than fibrations, so called co-institutions, see [18].