Relation Liftings on Preorders and Posets

Marta Bílková[1], Alexander Kurz[2,⋆], Daniela Petrişan[2], and Jiří Velebil[3,⋆⋆]

[1] Faculty of Philosophy, Charles University, Prague, Czech Republic
marta.bilkova@ff.cuni.cz
[2] Department of Computer Science, University of Leicester, United Kingdom
{kurz,petrisan}@mcs.le.ac.uk
[3] Faculty of Electrical Engineering, Czech Technical University in Prague, Czech Republic
velebil@math.feld.cvut.cz

Abstract. The category $\mathsf{Rel(Set)}$ of sets and relations can be described as a category of spans and as the Kleisli category for the powerset monad. A set-functor can be lifted to a functor on $\mathsf{Rel(Set)}$ iff it preserves weak pullbacks. We show that these results extend to the enriched setting, if we replace sets by posets or preorders. Preservation of weak pullbacks becomes preservation of exact lax squares. As an application we present Moss's coalgebraic over posets.

1 Introduction

Relation lifting [Ba, CKW, HeJ] plays a crucial role in coalgebraic logic, see eg [Mo, Bal, V].

On the one hand, it is used to explain bisimulation: If $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ is a functor, then the largest bisimulation on a coalgebra $\xi : X \longrightarrow TX$ is the largest fixed point of the operator $(\xi \times \xi)^{-1} \circ \overline{T}$ on relations on X, where $\overline{T}$ is the lifting of T to $\mathsf{Rel(Set)} \longrightarrow \mathsf{Rel(Set)}$. (The precise meaning of 'lifting' will be given in the Extension Theorem 5.3.)

On the other hand, Moss's coalgebraic logic [Mo] is given by adding to propositional logic a modal operator ∇, the semantics of which is given by applying $\overline{T}$ to the forcing relation $\Vdash \subseteq X \times \mathcal{L}$, where $\mathcal{L}$ is the set of formulas: If $\alpha \in T(\mathcal{L})$, then $x \Vdash \nabla\alpha \;\Leftrightarrow\; \xi(x)\; \overline{T}(\Vdash)\; \alpha$.

In much the same way as Set-coalgebras capture bisimulation, Pre-coalgebras and Pos-coalgebras capture simulation [R, Wo, HuJ, Kl, L, BK]. This suggests that, in analogy with the Set-based case, a coalgebraic understanding of logics for simulations should derive from the study of Pos-functors together with on the one hand their predicate liftings and on the other hand their ∇-operator. The study of predicate liftings of Pos-functors was begun in [KaKuV], whereas here we lay the foundations for the ∇-operator of a Pos-functor. In order to do this, we start with the notion of monotone relation for the following reason. Let

⋆ Alexander Kurz acknowledges the support of EPSRC, EP/G041296/1.
⋆⋆ Marta Bílková and Jiří Velebil acknowledge the support of the grant No. P202/11/1632 of the Czech Science Foundation.

A. Corradini, B. Klin, and C. Cîrstea (Eds.): CALCO 2011, LNCS 6859, pp. 115–129, 2011.

$(X, \leq)$ and $(X', \leq')$ be the carriers of two coalgebras, with the preorders $\leq, \leq'$ encoding the simulation relations on X and X', respectively. Then a simulation between the two systems will be a relation $R \subseteq X \times X'$ such that $\geq ; R ; \geq' \subseteq R$, that is, R is a monotone relation. Similarly, $\Vdash$ will be a monotone relation. To summarise, the relations we are interested in are monotone, which enables us to use techniques of enriched category theory (of which no prior knowledge is assumed of the reader).

For the reasons outlined above, the purpose of the paper is to develop the basic theory of relation liftings over preorders and posets. That is, we replace the category Set of sets and functions by the category Pre of preorders or Pos of posets, both with monotone (i.e. order-preserving) functions. Section 2 introduces notation and shows that (monotone) relations can be presented by spans and by arrows in an appropriate Kleisli-category. Section 3 recalls the notion of exact squares. Section 4 characterises the inclusion of functions into relations $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel}(\mathsf{Pre})$ by a universal property and shows that the relation lifting $\overline{T}$ exists iff T satisfies the Beck-Chevalley-Condition (BCC), which says that T preserves exact squares. The BCC replaces the familiar condition known from $\mathsf{Rel}(\mathsf{Set})$, namely that T preserves weak pullbacks. Section 5 lists examples of functors (not) satisfying the BCC and Section 6 gives the application to Moss's coalgebraic logic over posets.

Related Work. The universal property of the embedding of a (regular) category to the category of relations is stated in Theorem 2.3 of [He]. Theorem 4.1 below generalizes this in passing from a category to a simple 2-category of (pre)orders.

Liftings of functors to categories of relations within the realm of regular categories have also been studied in [CKW].

2 Monotone Relations

In this section we summarize briefly the notion of monotone relations on preorders and we show that their resulting 2-category can be perceived in two ways:

1. Monotone relations are certain *spans*, called *two-sided discrete fibrations*.
2. Monotone relations form a *Kleisli category* for a certain *KZ doctrine* on the category of preorders.

Definition 2.1. *Given preorders $\mathscr{A}$ and $\mathscr{B}$, a* monotone relation R from $\mathscr{A}$ to *$\mathscr{B}$, denoted by*

$$\mathscr{A} \xrightarrow{R} \mathscr{B}$$

is a monotone map $R : \mathscr{B}^{op} \times \mathscr{A} \longrightarrow 2$ where by 2 we denote the two-element poset on $\{0, 1\}$ with $0 \leq 1$.

Remark 2.2. Unravelling the definition: for a binary relation R, $R(b, a) = 1$ means that a and b are related by R. Monotonicity of R then means that if $R(b, a) = 1$ and $b_1 \leq b$ in $\mathscr{B}$ and $a \leq a_1$ in $\mathscr{A}$, then $R(b_1, a_1) = 1$.

Relations compose in the obvious way. Two relations as on the left below

$$\mathscr{A} \xrightarrow{R} \mathscr{B} \qquad \mathscr{B} \xrightarrow{S} \mathscr{C} \qquad\qquad \mathscr{A} \xrightarrow{S \cdot R} \mathscr{C}$$

compose to the relation on the right above by the formula

$$S \cdot R(c,a) = \bigvee_b R(b,a) \wedge S(c,b) \tag{2.1}$$

hence the validity of $S \cdot R(c,a)$ is witnessed by at least one b such that both $R(b,a)$ and $S(c,b)$ hold.

Remark 2.3. The supremum in formula (2.1) is, in fact, exactly a coend in the sense of enriched category theory, see [Ke].

The above composition of relations is associative and it has monotone relations $\mathscr{A} \xrightarrow{\mathscr{A}} \mathscr{A}$ as units, where $\mathscr{A}(a,a')$ holds, if $a \leq a'$. Moreover, the relations can be ordered pointwise: $R \longrightarrow S$ means that $R(b,a)$ entails $S(b,a)$, for every a and b. Hence we have a 2-category of monotone relations $\mathsf{Rel}(\mathsf{Pre})$.

Remark 2.4. Observe that one can form analogously the 2-category $\mathsf{Rel}(\mathsf{Pos})$ of monotone relations on *posets*. In all what follows one can work either with preorders or posets. We will focus on preorders in the rest of the paper, the modifications for posets always being straightforward. Observe that both $\mathsf{Rel}(\mathsf{Pre})$ and $\mathsf{Rel}(\mathsf{Pos})$ have the crucial property: The only isomorphism 2-cells are identities.

2.A The Functor $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel}(\mathsf{Pre})$

We describe now the functor $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel}(\mathsf{Pre})$ and show its main properties. The case of posets is completely analogous. For a monotone map $f : \mathscr{A} \longrightarrow \mathscr{B}$ define two relations

$$\mathscr{A} \xrightarrow{f_\diamond} \mathscr{B} \qquad\qquad \mathscr{B} \xrightarrow{f^\diamond} \mathscr{A}$$

by the formulas $f_\diamond(b,a) = \mathscr{B}(b,fa)$ and $f^\diamond(a,b) = \mathscr{B}(fa,b)$.

Lemma 2.5. *For every $f : \mathscr{A} \longrightarrow \mathscr{B}$ in Pre there is an adjunction in $\mathsf{Rel}(\mathsf{Pre})$*

$$f_\diamond \dashv f^\diamond : \mathscr{B} \longrightarrow \mathscr{A} .$$

Remark 2.6. Left adjoint morphisms in $\mathsf{Rel}(\mathsf{Pre})$ can be characterized as *exactly* those of the form $f_\diamond$ for some monotone map f having a *poset* as its codomain. Therefore, if $L \dashv R : \mathscr{B} \longrightarrow \mathscr{A}$ in $\mathsf{Rel}(\mathsf{Pre})$ and $\mathscr{B}$ is a poset, then there exists a monotone map $f : \mathscr{A} \longrightarrow \mathscr{B}$ such that $f_\diamond = L$ and $f^\diamond = R$.

Observe that if $f \longrightarrow g$, then $f_\diamond \longrightarrow g_\diamond$ holds. For if $\mathscr{B}(b,fa) = 1$ then $\mathscr{B}(b,ga) = 1$ holds by transitivity, since $fa \leq ga$ holds. Moreover, taking the lower diamond clearly maps an identity monotone map $id_\mathscr{A} : \mathscr{A} \longrightarrow \mathscr{A}$ to the identity monotone relation $\mathscr{A} \xrightarrow[\quad]{\mathscr{A}=(id_{\mathscr{A}})_\diamond} \mathscr{A}$ Further, taking the lower diamond preserves composition:

$$(g \cdot f)_\diamond(c,a) = \mathscr{C}(c, gfa) = \bigvee_b \mathscr{C}(c, gb) \wedge \mathscr{B}(b, fa) = g_\diamond \cdot f_\diamond(c,a)$$

Hence we have a functor $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel}(\mathsf{Pre})$ enriched in preorders. Moreover, $(-)_\diamond$ is *locally fully faithful*, i.e., $f_\diamond \longrightarrow g_\diamond$ holds iff $f \longrightarrow g$ holds.

2.B Rel(Pre) as a Kleisli Category

The 2-functor $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel}(\mathsf{Pre})$ is a *proarrow equipment with power objects* in the sense of Section 2.5 [MRW]. This means that $(-)_\diamond$ has a right adjoint $(-)^\dagger$ such that the resulting 2-monad on Pre is a KZ doctrine and $\mathsf{Rel}(\mathsf{Pre})$ is (up to equivalence) the corresponding Kleisli 2-category. All of the following results are proved in the paper [MRW], we summarize it here for further reference.

The 2-functor $(-)^\dagger$ works as follows:

1. On objects, $\mathscr{A}^\dagger = [\mathscr{A}^{op}, \mathbb{2}]$, the lowersets on $\mathscr{A}$, ordered by inclusion.
2. For a relation R from $\mathscr{A}$ to $\mathscr{B}$, the functor $R^\dagger : [\mathscr{A}^{op}, \mathbb{2}] \longrightarrow [\mathscr{B}^{op}, \mathbb{2}]$ is defined as the left Kan extension of $a \mapsto R(-,a)$ along the Yoneda embedding $\mathsf{y}_{\mathscr{A}} : \mathscr{A} \longrightarrow [\mathscr{A}^{op}, \mathbb{2}]$. This can be expressed by the formula:

$$R^\dagger(W) = b \mapsto \bigvee_a Wa \wedge R(b,a)$$

i.e., b is in the lowerset $R^\dagger(W)$ iff there exists a in W such that $R(b,a)$ holds.

It is easy to prove that $(-)^\dagger$ is a 2-functor and that $(-)^\dagger \dashv (-)_\diamond$ is a 2-adjunction of a KZ type. The latter means that if we denote by

$$(\mathbb{L}, \mathsf{y}, \mathfrak{m}) \tag{2.2}$$

the resulting 2-monad on Pre, then we obtain the string of adjunctions $\mathbb{L}(\mathsf{y}_{\mathscr{A}}) \dashv \mathfrak{m}_{\mathscr{A}} \dashv \mathsf{y}_{\mathbb{L}\mathscr{A}}$, see [M$_1$], [M$_2$], for more details.

The unit of the above KZ doctrine is the Yoneda embedding $\mathsf{y}_{\mathscr{A}} : \mathscr{A} \longrightarrow [\mathscr{A}^{op}, \mathbb{2}]$ and the multiplication $\mathfrak{m}_A : [[\mathscr{A}^{op}, \mathbb{2}]^{op}, \mathbb{2}] \longrightarrow [\mathscr{A}^{op}, \mathbb{2}]$ is the left Kan extension of identity on $[\mathscr{A}^{op}, \mathbb{2}]$ along $\mathsf{y}_{[\mathscr{A}^{op}, \mathbb{2}]}$. In more detail:

$$\mathfrak{m}_{\mathscr{A}}(\mathscr{W}) = a \mapsto \bigvee_W \mathscr{W}(W) \wedge W(a)$$

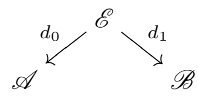
where $\mathscr{W}$ is in $[[\mathscr{A}^{op}, \mathbb{2}]^{op}, \mathbb{2}]$ and W is in $[\mathscr{A}^{op}, \mathbb{2}]$. Hence a is in the lowerset $\mathfrak{m}_{\mathscr{A}}(\mathscr{W})$ iff there exists a lowerset W in $\mathscr{W}$ such that a is in W. The following result is proved in Section 2.5 of [MRW]:

Proposition 2.7. *The 2-functor* $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel}(\mathsf{Pre})$ *exhibits* $\mathsf{Rel}(\mathsf{Pre})$ *as a Kleisli category for the KZ doctrine* $(\mathbb{L}, \mathsf{y}, \mathfrak{m})$.

2.C Relations as Spans

Monotone relations are going to be exactly certain spans, called *two-sided discrete fibrations*. For more information see [S_4].

Definition 2.8. *A* span $(d_0, \mathscr{E}, d_1) : \mathscr{B} \longrightarrow \mathscr{A}$ *from* $\mathscr{B}$ *to* $\mathscr{A}$ *is a diagram*



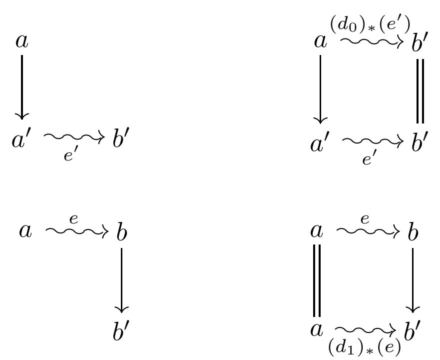
of monotone maps. The preorder $\mathscr{E}$ *is called the* vertex *of the span* $(d_0, \mathscr{E}, d_1)$.

Remark 2.9. Given a span $(d_0, \mathscr{E}, d_1) : \mathscr{B} \longrightarrow \mathscr{A}$, the following intuitive notation might prove useful: a typical element of $\mathscr{E}$ will be denoted by a wiggly arrow

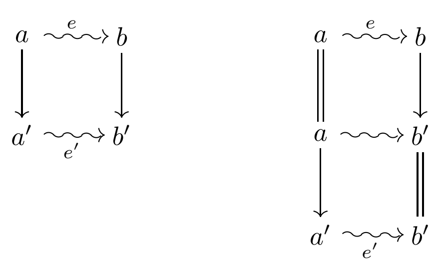
$$d_0(e) \overset{e}{\rightsquigarrow} d_1(e)$$

and $d_0(e)$ will be the *domain* of e and $d_1(e)$ the *codomain* of e.

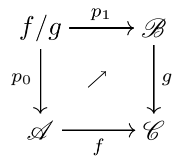
Definition 2.10. *A span* $(d_0, \mathscr{E}, d_1) : \mathscr{B} \longrightarrow \mathscr{A}$ *in* Pre *is a* two-sided discrete fibration *(we will say just* fibration *in what follows), if the following three conditions are satisfied. For every situation below on the left, there is a unique fill in on the right, denoted by* $(d_0)_*(e')$*, respectively* $(d_1)_*(e)$*:*



Every situation on the left can be written as depicted on the right:



Definition 2.11. *A* comma object *of monotone maps* $f : \mathscr{A} \longrightarrow \mathscr{C}$*,* $g : \mathscr{B} \longrightarrow \mathscr{C}$ *is a diagram*



where elements of the preorder f/g *are pairs* (a, b) *with* $f(a) \leq g(b)$ *in* $\mathscr{C}$*, the preorder on* f/g *is defined pointwise and* p_0 *and* p_1 *are the projections. The whole "lax commutative square" as above will be called a* comma square*.*

Example 2.12. Every span $(p_0, f/g, p_1) : \mathscr{A} \longrightarrow \mathscr{B}$ arising from a comma object of $f : \mathscr{A} \longrightarrow \mathscr{C}$, $g : \mathscr{B} \longrightarrow \mathscr{C}$ is a fibration.

A monotone relation $\mathscr{B} \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} \mathscr{A}$ induces a fibration $(d_0, \mathscr{E}, d_1) : \mathscr{B} \longrightarrow \mathscr{A}$ with $\mathscr{E} = \{(a, b) \mid R(a, b) = 1\}$ ordered by $(a, b) \leq (a', b')$, if $a \leq a'$ and $b \leq b'$; and $(d_0, \mathscr{E}, d_1)$ induces the relation $R(a, b) = 1 \;\Leftrightarrow\; \exists e \in \mathscr{E} \,.\, d_0(e) = a, d_1(e) = b$.

Proposition 2.13. *Fibrations in* Pre *correspond exactly to monotone relations. Moreover, if* $(d_0, \mathscr{E}, d_1) : \mathscr{B} \longrightarrow \mathscr{A}$ *is the fibration corresponding to a relation* $R : \mathscr{B} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathscr{A}$*, then* $R = (d_0)_\diamond \cdot (d_1)^\diamond$*.*

Remark 2.14. The proposition can be extended to any category enriched in Pre.

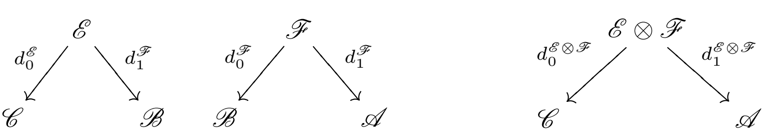
Example 2.15. Suppose that $f : \mathscr{A} \longrightarrow \mathscr{B}$ is monotone. Recall the relations $f_\diamond : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathscr{B}$ and $f^\diamond : \mathscr{B} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathscr{A}$. Their corresponding fibrations are the spans

arising from the respective comma squares.

Example 2.16. The relation $(\mathsf{y}_{\mathscr{A}})^\diamond$ from $\mathbb{L}\mathscr{A}$ to $\mathscr{A}$ will be called the *elementhood* relation and denoted by $\in_{\mathscr{A}}$, since $(\mathsf{y}_{\mathscr{A}})^\diamond(a, A) = \mathbb{L}\mathscr{A}(\mathsf{y}_{\mathscr{A}} a, A) = A(a)$ holds by the Yoneda Lemma.

2.D Composition of Fibrations

Suppose that we have two fibrations as on the left below. We want to form their composite $\mathscr{E} \otimes \mathscr{F}$ *as a fibration.*



The idea is similar to the ordinary relations: the composite is going to be a quotient of a pullback of spans, this time the quotient will be taken by a map that is surjective on objects, hence *absolutely dense*.

Remark 2.17. A monotone map $e : \mathscr{A} \longrightarrow \mathscr{B}$ is called *absolutely dense* (see [ABSV] and [BV]) iff there is an isomorphism

$$\mathscr{B}(b, b') \cong \bigvee_a \mathscr{B}(b, ea) \wedge \mathscr{B}(ea, b')$$

natural in b and b'. Clearly, every monotone map surjective on objects has this property. The converse is true if $\mathscr{B}$ is a poset. If $\mathscr{B}$ is a preorder, then e is absolutely dense when each strongly connected component of $\mathscr{B}$ contains at least an element in the image of e.

In defining the composition of fibrations we proceed as follows: construct the pullback

$$\begin{array}{ccc} \mathscr{E} \circ \mathscr{F} & \xrightarrow{q_1} & \mathscr{F} \\ {\scriptstyle q_0}\downarrow & & \downarrow{\scriptstyle d_0^{\mathscr{F}}} \\ \mathscr{E} & \xrightarrow[d_1^{\mathscr{E}}]{} & \mathscr{B} \end{array}$$

and define $\mathscr{E} \otimes \mathscr{F}$ to be the following preorder:

1. Objects are wiggly arrows of the form $c \rightsquigarrow a$ such that there exists $b \in \mathscr{B}$ with $(c \rightsquigarrow b, b \rightsquigarrow a) \in \mathscr{E} \circ \mathscr{F}$.
2. Put $c \rightsquigarrow a$ to be less or equal to $c' \rightsquigarrow a'$ iff $c \leq c'$ and $a \leq a'$.

Define a monotone map $w : \mathscr{E} \circ \mathscr{F} \longrightarrow \mathscr{E} \otimes \mathscr{F}$ in the obvious way and observe that it is surjective on objects.

We equip now $\mathscr{E} \otimes \mathscr{F}$ with the obvious projections $d_0^{\mathscr{E} \otimes \mathscr{F}} : \mathscr{E} \otimes \mathscr{F} \longrightarrow \mathscr{C}$ and $d_1^{\mathscr{E} \otimes \mathscr{F}} : \mathscr{E} \otimes \mathscr{F} \longrightarrow \mathscr{A}$. Then the following result is obvious.

Lemma 2.18. *The span* $(d_0^{\mathscr{E} \otimes \mathscr{F}}, \mathscr{E} \otimes \mathscr{F}, d_1^{\mathscr{E} \otimes \mathscr{F}}) : \mathscr{A} \longrightarrow \mathscr{C}$ *is a fibration.*

3 Exact Squares

The notion of *exact squares* replaces the notion of weak pullbacks in the preorder setting and exact squares will play a central rôle in our extension theorem. Exact squares were introduced and studied by René Guitart in [Gu].

Definition 3.1. *A lax square in* Pre

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{p_1} & \mathscr{B} \\ {\scriptstyle p_0}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ \mathscr{A} & \xrightarrow[f]{} & \mathscr{C} \end{array} \tag{3.3}$$

is exact *iff the canonical comparison in* Rel(Pre) *below is an iso (identity).*

$$\begin{array}{ccc} \mathscr{P} & \xleftarrow{(p_1)^{\diamond}} & \mathscr{B} \\ {\scriptstyle (p_0)_{\diamond}}\downarrow & \searrow & \downarrow{\scriptstyle g_{\diamond}} \\ \mathscr{A} & \xleftarrow[f^{\diamond}]{} & \mathscr{C} \end{array} \tag{3.4}$$

Remark 3.2. In defining the canonical comparison, we use the adjunctions $(p_1)_{\diamond} \dashv (p_1)^{\diamond}$ and $f_{\diamond} \dashv f^{\diamond}$ guaranteed by Lemma 2.5.

Using the formula (2.1) we obtain an equivalent criterion for exactness: there is an isomorphism, natural in a and b,

$$\mathscr{C}(fa, gb) \cong \bigvee_{w} \mathscr{A}(a, p_0 w) \wedge \mathscr{B}(p_1 w, b) \tag{3.5}$$

Remark 3.3 ([Gu], Example 1.14). Exact squares can be used to characterise order embeddings, absolutely dense morphisms, (relative) adjoints, and absolute Kan extensions. Further, (op-)comma squares are exact.

Example 3.4. Every square (3.3) where f and p_1 are *left* adjoints, is exact iff $p_0 \cdot p_1^r \cong f^r \cdot g$, where we denote by f^r and p_1^r the respective right adjoints.

Example 3.5. If the square on the left is exact, then so is the square on the right:

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{p_1} & \mathscr{B} \\ {\scriptstyle p_0}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ \mathscr{A} & \xrightarrow[f]{} & \mathscr{C} \end{array} \qquad\qquad \begin{array}{ccc} \mathscr{P}^{op} & \xrightarrow{p_0^{op}} & \mathscr{A}^{op} \\ {\scriptstyle p_1^{op}}\downarrow & \nearrow & \downarrow{\scriptstyle f^{op}} \\ \mathscr{B}^{op} & \xrightarrow[g^{op}]{} & \mathscr{C}^{op} \end{array}$$

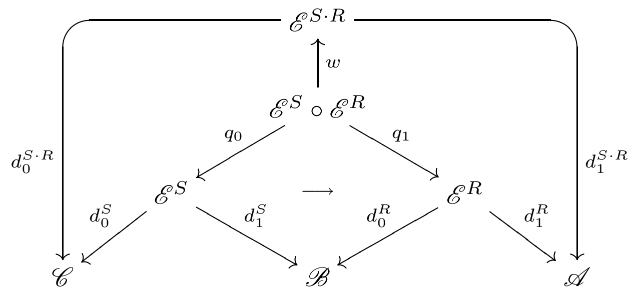
Lemma 3.6. *Suppose that* $(d_0^S, \mathscr{E}^S, d_1^S)$ *and* $(d_0^R, \mathscr{E}^R, d_1^R)$ *are two-sided discrete fibrations. Then the pullback*

$$\begin{array}{ccc} \mathscr{E}^S \circ \mathscr{E}^R & \xrightarrow{q_1} & \mathscr{E}^R \\ {\scriptstyle q_0}\downarrow & & \downarrow{\scriptstyle d_0^R} \\ \mathscr{E}^S & \xrightarrow[d_1^S]{} & \mathscr{B} \end{array}$$

considered as a lax commutative square where the comparison is identity, is exact.

Given monotone relations $\mathscr{A} \overset{R}{\looparrowright} \mathscr{B}$ and $\mathscr{B} \overset{S}{\looparrowright} \mathscr{C}$, the two-sided fibration corresponding to the composition $S \cdot R$ is the composition of the fibrations corresponding to S and R as described in Section 2.D. The properties described in the next Corollary are essential for the proof of Theorem 4.1.

Corollary 3.7. *Form, for a pair R, S, of monotone relations the following commutative diagram*



where the lax commutative square in the middle is a pullback square (hence the comparison is the identity), and w is a map, surjective on objects, coming from composing $\mathscr{E}^S$ and $\mathscr{E}^R$ as fibrations. Then the square is exact and w is an absolutely dense monotone map.

4 The Universal Property of $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel(Pre)}$

We prove now that the 2-functor $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel(Pre)}$ has an analogous universal property to the case of sets. From that, the result on a unique lifting of T to $\overline{T}$ will immediately follow, see Theorem 5.3 below.

Theorem 4.1. *The 2-functor $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel(Pre)}$ has the following three properties:*

1. *Every $f_\diamond$ is a left adjoint.*
2. *For every exact square (3.3) the equality $f^\diamond \cdot g_\diamond = (p_0)_\diamond \cdot (p_1)^\diamond$ holds.*
3. *For every absolutely dense monotone map e, the relation $e_\diamond$ is a split epimorphism with the splitting given by $e^\diamond$.*

Moreover, the functor $(-)_\diamond$ is universal w.r.t. these three properties in the following sense: if K is any 2-category where the isomorphism 2-cells are identities, to give a 2-functor $H : \mathsf{Rel(Pre)} \longrightarrow \mathsf{K}$ is the same thing as to give a 2-functor $F : \mathsf{Pre} \longrightarrow \mathsf{K}$ with the following three properties:

1. *Every Ff has a right adjoint, denoted by $(Ff)^r$.*
2. *For every exact square (3.3) the equality $Ff^r \cdot Fg = Fp_0 \cdot (Fp_1)^r$ holds.*
3. *For every absolutely dense monotone map e, Fe is a split epimorphism, with the splitting given by $(Fe)^r$.*

Proof (Sketch.). It is trivial to see that $(-)_\diamond$ has the above three properties.

Given a 2-functor $H : \mathsf{Rel(Pre)} \longrightarrow \mathsf{K}$, define F to be the composite $H \cdot (-)_\diamond$. Such F clearly has the above three properties, since 2-functors preserve adjunctions.

Conversely, given $F : \mathsf{Pre} \longrightarrow \mathsf{K}$, define $H\mathscr{A} = F\mathscr{A}$ on objects, and on a relation $R = (d_0^R)_\diamond \cdot (d_1^R)^\diamond$ define $H(R) = Fd_0^R \cdot (Fd_1^R)^r$, where $(Fd_1^R)^r$ is the right adjoint of Fd_1^R in K.

That H is a well-defined functor follows using Corollary 3.7 and our assumption on F. □

5 The Extension Theorem

Definition 5.1. *We say that a locally monotone functor T : $\mathsf{Pre} \longrightarrow \mathsf{Pre}$ satisfies the* Beck-Chevalley Condition *(BCC) if it preserves exact squares.*

Remark 5.2. A functor satisfying the BCC has to preserve order-embeddings, absolutely dense monotone maps and absolute left Kan extensions. This follows from Example 1.14 of [Gu], see also Remark 3.3. Examples of functors (not) satisfying the BCC can be found in Section 6.

Theorem 5.3. *For a 2-functor $T : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ the following are equivalent:*

1. *There is a 2-functor $\overline{T} : \mathsf{Rel}(\mathsf{Pre}) \longrightarrow \mathsf{Rel}(\mathsf{Pre})$ such that*

$$\begin{array}{ccc} \mathsf{Rel}(\mathsf{Pre}) & \xrightarrow{\overline{T}} & \mathsf{Rel}(\mathsf{Pre}) \\ {\scriptstyle (-)_\diamond}\uparrow & & \uparrow{\scriptstyle (-)_\diamond} \\ \mathsf{Pre} & \xrightarrow[T]{} & \mathsf{Pre} \end{array} \tag{5.6}$$

2. *The functor T satisfies the BCC.*
3. *There is a distributive law $T \cdot \mathbb{L} \longrightarrow \mathbb{L} \cdot T$ of T over the KZ doctrine $(\mathbb{L}, \mathsf{y}, \mathsf{m})$ described in (2.2) above.*

Proof. The equivalence of 1. and 3. follows from general facts about distributive laws, using Proposition 2.7 above. See, e.g., [S_1]. For the equivalence of 1. and 2., observe that T satisfies the BCC iff

$$\mathsf{Pre} \xrightarrow{T} \mathsf{Pre} \xrightarrow{(-)_\diamond} \mathsf{Rel}(\mathsf{Pre})$$

satisfies the three properties of Theorem 4.1 above. □

Corollary 5.4. *If T is a locally monotone functor satisfying the BCC, the lifting $\overline{T}$ is computed as follows: $\overline{T}(R) = (Td_0)_\diamond \cdot (Td_1)^\diamond$ where $(d_0, \mathscr{E}, d_1)$ is the two-sided discrete fibration corresponding to R.*

6 Examples

Example 6.1. All the "Kripke-polynomial" functors satisfy the Beck-Chevalley Condition. This means the functors defined by the following grammar:

$$T ::= \mathit{const}_{\mathscr{X}} \mid \mathit{Id} \mid T^\partial \mid T + T \mid T \times T \mid \mathbb{L}T$$

where $\mathit{const}_{\mathscr{X}}$ is the constant-at-$\mathscr{X}$, T^∂ is the *dual* of T, defined by putting

$$T^{\partial}\mathscr{A} = (T\mathscr{A}^{op})^{op}$$

and $\mathbb{L}\mathscr{X} = [\mathscr{X}^{op}, 2]$ (the lowersets on $\mathscr{X}$, ordered by inclusion). Observe that $\mathbb{L}^{\partial}\mathscr{X} = [\mathscr{X}, 2]^{op}$, hence $\mathbb{L}^{\partial}\mathscr{X} = \mathbb{U}\mathscr{X}$ (the uppersets on $\mathscr{X}$, ordered by reversed inclusion).

Example 6.2. Recall the adjunction $Q \dashv I : \mathsf{Pos} \longrightarrow \mathsf{Pre}$, where I is the inclusion functor and $Q(\mathscr{A})$ is the quotient of $\mathscr{A}$ obtained by identifying a and b whenever $a \leq b$ and $b \leq a$. The functors Q and I are locally monotone and map exact squares to exact squares. Hence, if $T : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ satisfies the BCC, so does $QTI : \mathsf{Pos} \longrightarrow \mathsf{Pos}$.

Example 6.3. The *powerset functor* $\mathbb{P} : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ is defined as follows. The order on $\mathbb{P}\mathscr{A}$ is the Egli-Milner preorder, that is, $\mathbb{P}(A, B) = 1$ if and only if

$$\forall a \in A\ \exists b \in B\ a \leq b \text{ and } \forall b \in B\ \exists a \in A\ a \leq b \tag{6.7}$$

$\mathbb{P}f(A)$ is the direct image of A. The functor $\mathbb{P}$ is locally monotone and satisfies the BCC.

The *finitary powerset functor* $\mathbb{P}_\omega$ is defined similarly: $\mathbb{P}_\omega\mathscr{A}$ consists of the finite subsets of $\mathscr{A}$ equipped with the Egli-Milner preorder. $\mathbb{P}_\omega$ is locally monotone and satisfies the BCC.

Example 6.4. Given a preorder $\mathscr{A}$, a subset $A \subseteq \mathscr{A}$ is called *convex* if $x \leq y \leq z$ and $x, z \in A$ imply $y \in A$.

The *convex powerset functor* $\mathbb{P}^c : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ is defined as follows. $\mathbb{P}^c\mathscr{A}$ is the set of convex subsets of $\mathscr{A}$ endowed with the Egli-Milner order. $\mathbb{P}^c f(A)$ is the direct image of A. This is a well defined locally monotone functor. Notice that $\mathbb{P}^c \simeq Q\mathbb{P}I$, so by Example 6.2, $\mathbb{P}^c$ satisfies the BCC.

The *finitely-generated convex powerset* $\mathbb{P}^c_\omega$ is defined similarly to $\mathbb{P}^c$. The only difference is that the convex sets appearing in $\mathbb{P}^c_\omega\mathscr{A}$ are convex hulls of finitely many elements of $\mathscr{A}$. Then $\mathbb{P}^c_\omega$ is locally monotone and is isomorphic to $Q\mathbb{P}_\omega I$, thus it also satisfies the BCC.

Observe that both functors are self-dual: $(\mathbb{P}^c)^{\partial} = \mathbb{P}^c$ and $(\mathbb{P}^c_\omega)^{\partial} = \mathbb{P}^c_\omega$.

Example 6.5. Since the lowerset functor $\mathbb{L} : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ satisfies the Beck-Chevalley Condition by Example 6.1, we can compute its lifting $\overline{\mathbb{L}} : \mathsf{Rel}(\mathsf{Pre}) \longrightarrow \mathsf{Rel}(\mathsf{Pre})$. We show how $\overline{\mathbb{L}}$ works on the relation $\mathscr{A} \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;} \mathscr{B}$. The value $\overline{\mathbb{L}}(R)$ is, by Theorems 4.1 and 5.3, given by $(\mathbb{L}d_0)_\diamond \cdot (\mathbb{L}d_1)^\diamond$ where $(d_0, \mathscr{E}^R, d_1) : \mathscr{A} \longrightarrow \mathscr{B}$ is the two-sided discrete fibration corresponding to R. Using the formula (2.1) for relation composition, we can write

$$\overline{\mathbb{L}}(R)(B, A) = \bigvee_W \mathbb{L}\mathscr{B}(B, \mathbb{L}d_0(W)) \wedge \mathbb{L}\mathscr{A}(\mathbb{L}d_1(W), A) \tag{6.8}$$

where $B : \mathscr{B}^{op} \longrightarrow 2$ and $A : \mathscr{A}^{op} \longrightarrow 2$ are arbitrary lowersets. Since $\mathbb{L}d_1$ is a left adjoint to restriction along $d_1^{op} : (\mathscr{E}^R)^{op} \longrightarrow \mathscr{A}^{op}$, we can rewrite (6.8) to

$$\overline{\mathbb{L}}(R)(B, A) = \bigvee_W \mathbb{L}\mathscr{B}(B, \mathbb{L}d_0(W)) \wedge \mathbb{L}\mathscr{E}^R(W, A \cdot d_1^{op})$$

and, by the Yoneda Lemma, to

$$\overline{\mathbb{L}}(R)(B,A) = \mathbb{L}\mathscr{B}(B, \mathbb{L}d_0(A \cdot d_1^{op}))$$

Hence the lowersets B and A are related by $\overline{\mathbb{L}}(R)$ if and only if the inclusion

$$B \subseteq \mathbb{L}d_0(A \cdot d_1^{op})$$

holds in $[\mathscr{B}^{op}, 2]$. Recall that

$$\mathbb{L}d_0(A \cdot d_1^{op}) = b \mapsto \bigvee_w \mathscr{B}(b, d_0 w) \wedge (A \cdot d_1^{op})(w)$$

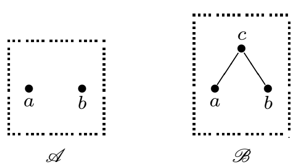
Therefore the inclusion $B \subseteq \mathbb{L}d_0(A \cdot d_1^{op})$ is equivalent to the statement: For all b in B there is (b_1, a_1) such that $R(b_1, a_1)$ and $b \leq b_1$ and a_1 in A.

Observe that the above condition is reminiscent of one half of the Egli-Milner-style of the relation lifting of a powerset functor. This is because $\mathbb{L}$ is the "lower half" of two possible "powerpreorder functors". The "upper half" is given by $\mathbb{U} : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ where $\mathbb{U} = \mathbb{L}^{\partial}$.

Example 6.6. The relation liftings $\overline{\mathbb{P}}$, $\overline{\mathbb{P}^c}$, $\overline{\mathbb{P}_\omega}$, $\overline{\mathbb{P}^c_\omega}$ of the (convex) powerset functor and their finitary versions yield the "Egli-Milner" style of the relation lifting. More precisely, for a relation $\mathscr{B} \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;} \mathscr{A}$ we have $\overline{\mathbb{P}}(R)(B,A)$ (respectively $\overline{\mathbb{P}_\omega}(R)(B,A)$, $\overline{\mathbb{P}^c}(R)(B,A)$, $\overline{\mathbb{P}^c_\omega}(R)(B,A)$) if and only if

$$\forall a \in A\ \exists b \in B\ R(b,a) \text{ and } \forall b \in B\ \exists a \in A\ R(b,a).$$

Example 6.7. To find a functor that does not satisfy the BCC, it suffices, by Remark 5.2, to find a locally monotone functor $T : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ that does not preserve order-embeddings. For this, let T be the *connected components functor*, i.e., T takes a preorder $\mathscr{A}$ to the discretely ordered poset of connected components of $\mathscr{A}$. T does not preserve embedding $f : \mathscr{A} \longrightarrow \mathscr{B}$ indicated below.



7 An Application: Moss's Coalgebraic Logic over Posets

We show how to develop the basics of Moss's coalgebraic logic over posets. For reasons of space, this development will be terse and assume some familiarity with, e.g., Sections 2.2 and 3.1 of [KuL].

Since the logics will have propositional connectives but no negation (to capture the semantic order on the logical side) we will use the category DL of bounded distributive lattices. We write $F \dashv U : \mathsf{DL} \longrightarrow \mathsf{Pos}$ for the obvious adjunction; and $P : \mathsf{Pos}^{op} \longrightarrow \mathsf{DL}$ where $UP\mathscr{X} = [\mathscr{X}, 2]$ and $S : \mathsf{DL} \longrightarrow \mathsf{Pos}^{op}$ where

$SA = \mathsf{DL}(A, \mathbb{2})$. Note that $UP = [-, \mathbb{2}]$ and recall $\mathbb{L} = [(-)^{op}, \mathbb{2}]$. Further, let $T : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ be a locally monotone finitary functor that satisfies the BCC.

We define coalgebraic logic abstractly by a functor $L : \mathsf{DL} \longrightarrow \mathsf{DL}$ given as

$$L = FT^{\partial}U$$

where the functor $T^{\partial} : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ is given by $T^{\partial}\mathscr{X} = (T(\mathscr{X}^{op}))^{op}$. By Example 6.1, T^{∂} satisfies the BCC. The formulas of the logic are the elements of the initial L-algebra $FT^{\partial}U(\mathcal{L}) \longrightarrow \mathcal{L}$. The formula given by some $\alpha \in T^{\partial}U(\mathcal{L})$ is written as $\nabla\alpha$. The semantics is given by a natural transformation

$$\delta : LP \longrightarrow PT^{op}$$

Before we define δ, we need for every preorder $\mathscr{A}$, the relation[1]

$$[\mathscr{A}, \mathbb{2}] \xrightarrow{\ni_{\mathscr{A}}} \mathscr{A}^{op}$$

given by the evaluation map $\mathsf{ev}_{\mathscr{A}} : \mathscr{A} \times [\mathscr{A}, \mathbb{2}] \longrightarrow \mathbb{2}$. Observe that

$$\ni_{\mathscr{A}} = (\mathbb{y}_{\mathscr{A}^{op}})^{\diamond} \tag{7.9}$$

since $(\mathbb{y}_{\mathscr{A}^{op}})^{\diamond}(a, V) = [\mathscr{A}, \mathbb{2}](\mathbb{y}_{\mathscr{A}^{op}}a, V) = Va$ holds by the Yoneda Lemma.

Lemma 7.1. *For every monotone map $f : \mathscr{A} \longrightarrow \mathscr{B}$ we have*

$$\begin{array}{ccc} [\mathscr{A}, \mathbb{2}] & \xrightarrow{\ \ni_{\mathscr{A}}\ } & \mathscr{A}^{op} \\ {\scriptstyle [f,\mathbb{2}]^{\diamond}}\uparrow & & \uparrow{\scriptstyle (f^{op})^{\diamond}} \\ [\mathscr{B}, \mathbb{2}] & \xrightarrow[\ \ni_{\mathscr{B}}\ ]{} & \mathscr{B}^{op} \end{array}$$

Corollary 7.2. *For every locally monotone functor T that satisfies the Beck-Chevalley Condition and for every monotone map $f : \mathscr{A} \longrightarrow \mathscr{B}$, we have*

$$\begin{array}{ccc} \overline{T}[\mathscr{A}, \mathbb{2}] & \xrightarrow{\ \overline{T}\ni_{\mathscr{A}}\ } & \overline{T}\mathscr{A}^{op} \\ {\scriptstyle \overline{T}[f,\mathbb{2}]^{\diamond}}\uparrow & & \uparrow{\scriptstyle \overline{T}(f^{op})^{\diamond}} \\ \overline{T}[\mathscr{B}, \mathbb{2}] & \xrightarrow[\ \overline{T}\ni_{\mathscr{B}}\ ]{} & \overline{T}\mathscr{B}^{op} \end{array}$$

Coming back to $\delta : LP \longrightarrow PT^{op}$. It suffices, due to $F \dashv U$, to give

$$\tau : T^{\partial}UP \longrightarrow UPT^{op}$$

Observe that, for every preorder $\mathscr{X}$, we have

$$UPT^{op}(\mathscr{X}) = [T^{op}\mathscr{X}, \mathbb{2}] = \mathbb{L}((T^{op}\mathscr{X})^{op})$$

By Proposition 2.7, to define $\tau_{\mathscr{X}}$ it suffices to give a relation from $T^{\partial}UP\mathscr{X}$ to $(T^{op}\mathscr{X})^{op}$, and we obtain it from Theorem 5.3 by applying $\overline{T^{\partial}}$ to the relation $\ni_{\mathscr{X}}$. That $\tau_{\mathscr{X}}$ so defined is natural, follows from Corollary 7.2. This follows [KKuV] with the exception that here now we need to use T^{∂}.

[1] The type of $\ni_{\mathscr{X}}$ conforms with the logical reading of $\ni$ as $\Vdash$. Indeed, $\ni(x, \varphi)$ & $\varphi \subseteq \psi \Rightarrow \ni(x, \psi)$ and $\ni(x, \varphi)$ & $x \leq y \Rightarrow \ni(y, \varphi)$, where φ, ψ are uppersets of $\mathscr{X}$.

Example 7.3. Recall the functor $\mathbb{P}^c_\omega$ of Example 6.4 and consider a coalgebra $c : \mathscr{X} \longrightarrow \mathbb{P}^c_\omega \mathscr{X}$. On the logical side we allow ourselves to write $\nabla\alpha$ for any finite subset α of $U(\mathcal{L})$. Of course, we then have to be careful that the semantics of α agrees with the semantics of the convex closure of α. Interestingly, this is done automatically by the machinery set up in the previous section, since $\mathbb{P}^c_\omega = Q\mathbb{P}_\omega I$ and all these functors are self-dual. By Example 6.6, the semantics of $\nabla\alpha$ is given by

$$x \Vdash \nabla\alpha \quad \Leftrightarrow \quad \forall y \in c(x) \exists \varphi \in \alpha . y \Vdash \varphi \;\text{ and } \forall \varphi \in \alpha \exists y \in c(x) . y \Vdash \varphi .$$

8 Conclusions

We hope to have illustrated in the previous two sections that, after getting used to handle the $(-)_\diamond$, $(-)^\diamond$ and $(-)^{op}$, the techniques developed here work surprisingly smoothly and will be useful in many future developments. For example, an observation crucial for both [KKuV, KuL] is that composing the singleton map $X \longrightarrow \mathscr{P}X$, $x \mapsto \{x\}$, with the relation $\ni_X : \mathscr{P}X \longrightarrow\!\!\!\!\!\!\!+\;\; X$ is id_X. Referring back to (7.9), we find here the same relationship

$$\ni_{\mathscr{A}} \circ (\mathsf{y}_{\mathscr{A}^{op}})_\diamond = (\mathsf{y}_{\mathscr{A}^{op}})^\diamond \circ (\mathsf{y}_{\mathscr{A}^{op}})_\diamond = id_{\mathscr{A}^{op}}$$

The question whether the completeness proof of [KKuV] and the relationship between ∇ and predicate liftings of [KuL] can be carried over to our setting are a direction of future research.

Another direction is the generalisation to categories which are enriched over more general structures than $\mathbb{2}$, such as commutative quantales. Simulation, relation lifting and final coalgebras in this setting have been studied in [Wo].

References

[ABSV] Adámek, J., El Bashir, R., Sobral, M., Velebil, J.: On functors that are lax epimorphisms. Theory Appl. Categ. 8.20, 509–521 (2001)

[BK] Balan, A., Kurz, A.: Finitary Functors: from Set to Preord and Poset. In: Corradini, A., Klin, B., Crstea, C. (eds.) CALCO 2011. LNCS, vol. 6859, pp. 85–99. Springer, Heidelberg (2011)

[Bal] Baltag, A.: A logic for coalgebraic simulation. Electron. Notes Theor. Comput. Sci. 33, 41–60 (2000)

[Ba] Barr, M.: Relational algebras. In: Reports of the Midwest Category Seminar IV. Lecture Notes in Mathematics, vol. 137, pp. 39–55. Springer, Heidelberg (1970)

[BV] El Bashir, R., Velebil, J.: Reflective and coreflective subcategories of presheaves. Theory Appl. Categ. 10.16, 410–423 (2002)

[CKW] Carboni, A., Kelly, G.M., Wood, R.J.: A 2-categorical approach to change of base and geometric morphisms I. Cahiers de Top. et Géom. Diff. XXXII.1, 47–95 (1991)

[Gu] Guitart, R.: Relations et carrés exacts. Ann. Sci. Math. Québec IV.2, 103–125 (1980)

[He] Hermida, C.: A categorical outlook on relational modalities and simulations, preprint, http://maggie.cs.queensu.ca/chermida/papers/sat-sim-IandC.pdf

[HeJ] Hermida, C., Jacobs, B.: Structural induction and coinduction in the fibrational setting. Inform. and Comput. 145, 107–152 (1998)

[HuJ] Hughes, J., Jacobs, B.: Simulations in coalgebra. Theor. Comput. Sci. 327, 71–108 (2004)

[KaKuV] Kapulkin, K., Kurz, A., Velebil, J.: Expressivity of Coalgebraic Logic over Posets. In: CMCS 2010 Short contributions, CWI Technical report SEN-1004, pp. 16–17 (2010)

[Ke] Kelly, G.M.: Basic concepts of enriched category theory. London Math. Soc. Lecture Notes Series, vol. 64. Cambridge Univ. Press, New York (1982)

[Kl] Klin, B.: An Abstract Coalgebraic Approach to Process Equivalence for Well-Behaved Operational Semantics. University of Aarhus (2004)

[KKuV] Kupke, C., Kurz, A., Venema, Y.: Completeness of the finitary Moss logic. In: Advances in Modal Logic, pp. 193–217. College Publications (2008)

[KuL] Kurz, A., Leal, R.: Equational coalgebraic logic. Electron. Notes Theor. Comput. Sci. 249, 333–356 (2009)

[L] Levy, P.B.: Similarity quotients as final coalgebras. In: Hofmann, M. (ed.) FOSSACS 2011. LNCS, vol. 6604, pp. 27–41. Springer, Heidelberg (2011)

[M_1] Marmolejo, F.: Doctrines whose structure forms a fully faithful adjoint string. Theor. Appl. Categ. 3(2), 24–44 (1997)

[M_2] Marmolejo, F.: Distributive laws for pseudomonads. Theor. Appl. Categ. 5(5), 91–147 (1999)

[MRW] Marmolejo, F., Rosebrugh, R., Wood, R.J.: Duality for CCD lattices. Theor. Appl. Categ. 22(1), 1–23 (2009)

[Mo] Moss, L.: Coalgebraic logic. Ann. Pure Appl. Logic 96, 277–317 (1999)

[R] Rutten, J.: Relators and Metric Bisimulations (Extended Abstract). Electr. Notes Theor. Comput. Sci. 11, 252–258 (1998)

[S_1] Street, R.: The formal theory of monads. J. Pure Appl. Algebra 2, 149–168 (1972)

[S_2] Street, R.: Fibrations and Yoneda's lemma in a 2-category. In: Category Seminar, Sydney 1974. Lecture Notes in Mathematics, vol. 420, pp. 104–133 (1974)

[S_3] Street, R.: Elementary cosmoi I. In: Category Seminar, Sydney 1974. Lecture Notes in Mathematics, vol. 420, pp. 134–180. Springer, Heidelberg (1974)

[S_4] Street, R.: Fibrations in bicategories. Cahiers de Top. et Géom. Diff. XXI.2, 111–159 (1980)

[V] Venema, Y.: Automata and fixed point logic: a coalgebraic perspective. Inform. and Comput. 204.4, 637–678 (2006)

[Wo] Worrell, J.: Coinduction for recursive data types: partial orders, metric spaces and Ω-categories. Electron. Notes Theor. Comput. Sci. 33, 337–356 (2000)