Equational Coalgebraic Logic

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Abstract

Coalgebra develops a general theory of transition systems, parametric in a functor T; the functor T specifies the possible one-step behaviors of the system. A fundamental question in this area is how to obtain, for an arbitrary functor T, a logic for T-coalgebras. We compare two existing proposals, Moss's coalgebraic logic and the logic of all predicate liftings, by providing one-step translations between them, extending the results in [21] by making systematic use of Stone duality. Our main contribution then is a novel coalgebraic logic, which can be seen as an equational axiomatization of Moss's logic. The three logics are equivalent for a natural but restricted class of functors. We give examples showing that the logics fall apart in general. Finally, we argue that the quest for a generic logic for T-coalgebras is still open in the general case.

Keywords: coalgebra, coalgebraic logic, Stone duality, predicate liftings, Moss-modality, nabla-modality

1 Introduction

When Aczel [1] introduced the idea of coalgebras for a functor T as a generalization of transition systems, he made three crucial observations: (1) coalgebras come with a canonical notion of observational or *behavioral equivalence* (induced by the functor T); (2) this notion of behavioral equivalence generalizes the notion of *bisimilarity* from computer science and modal logic; (3) any 'domain equation' $X \cong TX$ has a canonical solution, namely the *final* coalgebra, which is fully abstract wrt behavioral equivalence.

This idea of a type of dynamic systems being represented by a functor T and an individual system being a T-coalgebra, led Rutten [26] to the theory of universal coalgebra which, parametrized by T, applies in a uniform way to a large class of different types of systems.

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In particular, final semantics and the associated proof principle of coinduction (which are dual to initial algebra semantics and induction) find their natural place here.

These ideas have been proved very successful. Coalgebras encompass such diverse systems as, for example, labelled transition systems [1], deterministic automata [25], π-calculus processes [8], HD-automata [7], stochastic systems [6], neighborhood frames [9].

Very early on in this endeavour the following question arose. If universal coalgebra can cover a wide range of models of computation uniformly and parametric in the type-functor T, can the same be done for logics for coalgebras? The first positive answer was given by Moss [22]. His fascinating idea was, roughly, to take T itself as a modality. More precisely, if $\mathcal{M}$ is the set of formulas of his language and $\alpha \in T\mathcal{M}$ then $\nabla\alpha \in \mathcal{M}$.

In the case of the power-set functor $\mathcal{P}$, this modality, denoted as ∇, can be defined using the standard box and diamond: With $\alpha \in \mathcal{P}\mathcal{M}$ a set of formulas, the formula $\nabla\alpha$ can be seen as an abbreviation $\nabla\alpha = \Box \bigvee \alpha \wedge \bigwedge \Diamond\alpha$, where $\Diamond\alpha$ denotes the set $\{\Diamond a \mid a \in \alpha\}$.

Independently of Moss's work, Janin and Walukiewicz [11] already observed that the connectives ∇ and $\vee$ may replace the connectives $\Box, \Diamond, \wedge, \vee$. This observation, which is closely linked to fundamental automata-theoretic constructions, lies at the heart of the theory of the modal μ-calculus, and has many applications, see for instance [5,27]. Generalizing the link between fix-point logics and automata theory to the coalgebraic level of generality, Kupke & Venema [15] generalized some of these observations to show that many fundamental results in automata theory are really theorems of universal coalgebra.

A shortcoming of Moss's logic is that the connective ∇ is un-intuitive for writing out specifications. [16] was the first paper to propose a standard modal logic for coalgebras. Pattinson [23] discovered how to describe such modal logics for coalgebras in general via *predicate liftings*. The logic $\mathcal{L}$ of all predicate liftings was first investigated by Schröder [28] and Klin [12].

The second author's [21] started a systematic investigation of the relationship of Moss's logic $\mathcal{M}$ and the logic $\mathcal{L}$ of all predicate liftings. In particular, [21] introduced a special notion of predicate liftings, the *singleton liftings*, and observed that 1) they generate all other predicate liftings and 2) they can be translated into Moss's logic for all Kripke Polynomial functors.

We continue this line of research and summarize our contributions as follows:

- Coalgebraic logics can extend different underlying propositional logics. We investigate how this choice influences translations between Moss's logic and logic with predicate liftings.
- If the underlaying logic is classical, i.e. based on Boolean algebras, we
  - improve on the result of [21] by showing that all singleton liftings for any functor T can be translated into Moss's logic, establishing a one-step translation $\mathcal{L} \longrightarrow \mathcal{M}$,
  - give a simple description of a one-step translation of $\mathcal{M}$ to $\mathcal{L}$,
  - show that all expressive coalgebraic logics for a finitary functor that preserves finite sets are mutually translatable.
- We show that Moss's logic can be given a more standard equational (or modal) logic style by replacing the modal operator ∇ by a set of conventional modal operators. This is based on the well-known fact that any set-functor T has a presentation by operations and equations [2].

2 Notation and Preliminaries

We use $\mathcal{Q} : \mathsf{Set} \longrightarrow \mathsf{Set}^{op}$ for the contra-variant power set functor [5] . $\mathcal{P}$ denotes the covariant power set functor and $\mathcal{B}_{\mathbb{N}}$ the finite multiset functor: $\mathcal{B}_{\mathbb{N}}X$ consists of all maps (also knwon as 'bags') $B : X \longrightarrow \mathbb{N}$ with finite support; for $f : X \longrightarrow Y$, the function $\mathcal{B}_{\mathbb{N}}(f)$ maps a bag $B : X \longrightarrow \mathbb{N}$ to $\mathcal{B}_{\mathbb{N}}(f)(B) : Y \longrightarrow \mathbb{N}$ given by $y \mapsto \Sigma_{x \in f^{-1}(\{y\})} B(x)$. The finite distribution functor $\mathcal{D}$ follows the same idea: $\mathcal{D}X$ is the set of probability distributions $X \longrightarrow [0, 1]$ with finite support; similarly, $\mathcal{D}_{\leq}$ denotes the subdistribution functor, which maps X to $\{\mu : X \longrightarrow [0, 1] \mid \mu \text{ has finite support and } \Sigma_{x \in X} \mu(x) \leq 1\}$; on functions, both functors act in the same way as $\mathcal{B}_{\mathbb{N}}$.

BA denotes the category of Boolean algebras and Boolean homomorphisms, BA_{ω} the category of finite Boolean algebras and all Boolean homomorphisms between them, and Set_{ω} the category of finite sets and all functions between them.

Two properties of Boolean algebras will play an important role in our approach: First, (Stone duality) the contra variant power set functor can be seen as a functor into boolean algebras $P : \mathsf{Set} \longrightarrow \mathsf{BA}^{op}$ and it has a right adjoint $S : \mathsf{BA}^{op} \longrightarrow \mathsf{Set}$, which maps a Boolean algebra to the set of ultrafilters (an ultrafilter is a maximal consistent propositional theory). On maps, both functors map a function to its inverse image. Moreover, the restriction of P and S to BA_{ω} and Set_{ω} is a dual equivalence. Second, every Boolean algebra is the directed union of finite Boolean algebras, or, more formally, the finite Boolean algebras are precisely the finitely presentable ones.

Other relevant categories for this paper are: the category of distributive latices and lattice homomorphisms, denoted DL; the category of frames and frame homomorphisms, denoted Frm; the category of κ-complete boolean algebras, denoted BA^{κ}.

2.1 Coalgebras

Definition 2.1 The category $\mathsf{Coalg}(T)$ of *coalgebras* for a functor T on a category $\mathcal{X}$ has as objects arrows $\xi : X \longrightarrow TX$ in $\mathcal{X}$ and morphisms $f : (X, \xi) \longrightarrow (X', \xi')$ are arrows $f : X \longrightarrow X'$ such that $Tf \circ \xi = \xi' \circ f$.

Examples are provided by

Definition 2.2 Let Γ be a collection of set endofunctors. A Γ-*Kripke polynomial functor*, or Γ-KPF for short, is a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ built according to

$$T ::= Id \mid K_C \mid G \mid T + T \mid T \times T \mid \mathcal{P}T$$

where Id is the identity functor, K_C is the constant functor that maps all sets to the set C, $G \in \Gamma$, and $\mathcal{P}$ is covariant powerset functor. If Γ is empty, we just talk about Kripke polynomial functors [6] or KPFs.

Example 2.3 Coalgebras for the covariant power set functor are Kripke frames, also known as non-deterministic transitions systems [1]. Slight variations allow to add labels to transitions or states. Coalgebras for the finite multiset functor are directed graphs with $\mathbb{N}$-weighted edges, often referred as multigraphs [30]. Coalgebras for the finite distribution

[5] $\mathcal{Q}$ is intended to remind of 2, because of $\mathcal{Q}X = 2^X$.

[6] The term Kripke polynomial functor was coined in Rößiger [24].

functor are finitely branching discrete time Markov chains [3]. $\mathcal{QQ}$-coalgebras are known as neighborhood frames in modal logic [9].

As shown in the references above, the traditional notion of bisimilarity can be captured coalgebraically as follows.

Definition 2.4 Two states x_i in two coalgebras X_i are *T-bisimilar*, or *T-behaviorally equivalent*, if there is a coalgebra (Z, ζ) and there are coalgebra morphisms $f_i : (X_i, \xi_i) \longrightarrow (Z, \zeta)$ such that $f_1(x_1) = f_2(x_2)$.

2.2 Coalgebraic Logic

The Stone duality approach to coalgebraic logic associates, in a systematic way, to a given category of coalgebras a suitable category of modal algebras. This category of modal algebras then embodies a suitable modal logic for coalgebras. The basic example is to consider coalgebras over Set and logics which extend Boolean propositional logic, that is, we are in the following situation:

$$L \circlearrowleft \mathsf{BA} \underset{S}{\overset{P}{\rightleftarrows}} \mathsf{Set} \circlearrowright T \tag{1}$$

A coalgebraic logic is a functor L together with a natural transformation $\delta : LP \longrightarrow PT$. Using δ we can associate to a T-coalgebra $\xi : X \longrightarrow TX$ its dual L-algebra

$$\widehat{P}(\xi) = LPX \xrightarrow{\delta_X} PTX \xrightarrow{P(\xi)} PX. \tag{2}$$

The logic is given by the initial L-algebra $LI \longrightarrow I$, the semantics by the unique arrow $[\![\cdot]\!]_{(X,\xi)} : I \longrightarrow \widehat{P}(\xi)$, mapping a formula $\varphi \in I$ to the set of states $[\![\varphi]\!] = \{x \in X \mid x \Vdash \varphi\}$.

Remark 2.5 It is important to understand that L only describes how to add one layer of modalities: If A consists of Boolean formulas, then LA consists of modal formulas in which each formula $a \in A$ is under the scope of precisely one modal operator. The initial L-algebra is obtained by iterating this construction and contains modal formulas of arbitrary depth. Moreover, L can take into account not only the syntax, but also the axiomatisation of the logic, as revealed in (3) below. To capture these by a functor, it is essential to consider L on BA and not simply on Set.

One advantage of this functorial approach to modal logic is that (L, δ) gives us a syntax-free description of the logic. Properties of the logic can be expressed by abstract properties of (L, δ), for example, (L, δ) is complete iff δ is injective [13,18]. It is also possible, without any consideration of syntax, to generalize from Kripke frames to all KPFs (and beyond) the Jónsson-Tarski theorem [20] and the Goldblatt-Thomason theorem [19].

Concrete descriptions of logics (L, δ) are usually obtained by presenting the functor L. Presentations of functors are analogous to presentations of algebras and studied in detail in [4,20]. For our purposes, the following example should suffice.

Example 2.6 Consider $T = \mathcal{P}$, ie, coalgebras are Kripke frames (unlabelled transition systems). So we expect to have standard modal logic, given by one finite-meet preserving modal operator $\Box$. Accordingly, we define $L_{\mathcal{P}} : \mathsf{BA} \longrightarrow \mathsf{BA}$ to map an algebra A to the

algebra $L_{\mathcal{P}}(A)$ generated by $\Box a$, $a \in A$, and quotiented by the relation stipulating that $\Box$ preserves finite meets, that is,

$$\Box\top = \top \qquad \Box(a \wedge b) = \Box a \wedge \Box b \tag{3}$$

$(\delta_{\mathcal{P}})_X : L_{\mathcal{P}}PX \longrightarrow P\mathcal{P}X$ is defined by

$$\Box a \mapsto \{b \in \mathcal{P}X \mid b \subseteq a\}, \tag{4}$$

so that we obtain the usual semantics of $\Box$ stating that a set b of successors satisfies $\Box a$ iff $b \subseteq a$.

The previous example shows how modal operators correspond to generators, and how modal axioms correspond to quotienting freely generated algebras. It is not difficult to check that with $\delta_{\mathcal{P}}$ and the semantics of (2) we obtain the usual semantics of modal logic. The details of how the example can be inductively extended to obtain a logic (L_T, δ_T) for each KPF T can be found in [19]. Although these logics are defined by syntax and axioms, all the good properties they enjoy flow from the following syntax-free characterisation.

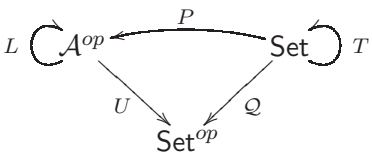
Proposition 2.7 *Let T be a KPF. Then $(\delta_T)_X : L_TPX \longrightarrow PTX$ is an isomorphism for all finite sets X.*

As a consequence one obtains for example:

Corollary 2.8 *For any KPF T the logic (L_T, δ_T) is sound and complete.*

3 Two Coalgebraic Languages

The Stone duality approach, presented in the previous section, can be generalized to any concrete category $\mathcal{A}$ equipped with a functor $P : \mathsf{Set} \longrightarrow \mathcal{A}^{op}$ such that $UP = \mathcal{Q}$. Intuitively this means that power-sets are algebras in $\mathcal{A}$ and the inverse image of a function is a morphism of $\mathcal{A}$-algebras.



(5)

Definition 3.1 A category $\mathcal{A}$ is said to be a category *with power-set algebras* if (1) it is a concrete category over Set. (2) The forgetful functor $U_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathsf{Set}$ has a left adjoint $F_{\mathcal{A}} : \mathsf{Set} \longrightarrow \mathcal{A}$. (3) There exists a functor $P : \mathsf{Set} \longrightarrow \mathcal{A}^{op}$ such that $U_{\mathcal{A}}P = \mathcal{Q}$. (We will often drop the subscripts.)

Examples of categories with power-set algebras are: sets, semi-lattices, distributive lattices, frames, κ-complete Boolean algebras, completely distributive lattices, complete atomic Boolean algebras. This general perspective on coalgebraic languages will help us to illustrate the importance of the underlying logic to define translations.

3.1 *Moss's Logic*

Moss's logic can be given for an arbitrary functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ preserving weak pullbacks. Examples of such functors are all $(\mathcal{B}_{\mathbb{N}}, \mathcal{D})$-KPFs, but not the functor $\mathcal{QQ}$.

In the original version [22], Moss showed that his coalgebraic logic characterizes bisimilarity of T-coalgebras. Because T may permit unbounded branching, this purpose needs infinitary conjunctions in the logic. Here our interests are slightly different: To specify properties of coalgebras we want all Boolean connectives, but only finitary ones. Accordingly, we will work with the finitary version T_ω of T.

For convenience, and without loss of generality [2], we assume that T is **standard**, that is T preserves inclusions and the equalizer $0 \longrightarrow 1 \rightrightarrows 2$. Under these assumptions we can define the finitary version of T by $T_\omega X = \bigcup\{TY \mid Y \subseteq X, Y \text{ finite }\}$. A functor is said to be **finitary** iff $T = T_\omega$. For example, $\mathcal{P}_\omega X$ is the set of finite subsets of X.

Definition 3.2 Moss's language $\mathcal{M}_T$ is the smallest set closed under Boolean operations and under the formation rule 'if $\alpha \in T_\omega(\mathcal{M}_T)$ then $\nabla\alpha \in \mathcal{M}_T$' (we will often drop the subscript T).

Following Diagram (1), we now cast this definition in terms of a functor on BA. Moreover, we generalise from BA to a category $\mathcal{A}$ with power-set algebras.

Definition 3.3 Let $\mathcal{A}$ be a category with power set algebras, and let $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ be a weak pullback preserving functor. *Moss's logic for T in $\mathcal{A}$* is given by the functor

$$FT_\omega U = M_T : \mathcal{A} \longrightarrow \mathcal{A}.$$

Following Diagram (2), to define the semantics $M_T P \longrightarrow PT$, it is enough to give a natural transformation $T_\omega UPX \longrightarrow UPTX$, or $T\mathcal{Q} \longrightarrow \mathcal{Q}T$. To this end, we let [7] $\boldsymbol{\nabla} : T_\omega\mathcal{Q} \longrightarrow \mathcal{Q}T$ be the natural transformation with the following components: an element $\Phi \in T_\omega\mathcal{Q}X$ is mapped to

$$\boldsymbol{\nabla}(\Phi) = \{\alpha \in TX \mid \alpha \; \overline{T}(\in_X) \; \Phi\}, \tag{6}$$

where $\overline{T}(\in_X)$ is the relation $TX \leftarrow T(\in_X) \rightarrow T\mathcal{Q}X$ obtained from applying T to the membership relation $X \leftarrow\in_X\rightarrow \mathcal{Q}X$.

Remark 3.4 The above procedure can be applied to any binary relation $R \subseteq X \times Y$, yielding a new relation $\overline{T}(R) \subseteq TX \times TY$, which is called the **relation lifting** of R.

Definition 3.3 and (6) give us syntax and semantics of Moss's logic over various propositional base logics. We would like to make the following

Remark 3.5 (i) In the case $\mathcal{A}$ is the category of Boolean algebras, the carrier of the initial M_T algebra is the quotient of $\mathcal{M}_T$ under Boolean equivalence, so both M_T and $\mathcal{M}_T$ give us essentially the same information.

(ii) As indicated in the introduction (see also the example below), in case of $T = \mathcal{P}$ and $\mathcal{A} = \mathsf{BA}$, we obtain a logic which is equivalent to the standard modal logic of $\Box$ and $\Diamond$. It is well-known that this finitary logic does not characterise bisimilarity for

[7] $\boldsymbol{\nabla}$ is the **semantics** of ∇.

infinitely branching transition systems. But the following version of Moss's result still holds: If the functor T is finitary, then $\mathcal{M}_T$ characterizes T-bisimilarity (behavioral equivalence). This result remains true if we instantiate $\mathcal{A}$ with DL or meet-semi-lattices.

Example 3.6 (i) In the case of the identity functor Id, the operator $\boldsymbol{\nabla} : Id\mathcal{Q} \longrightarrow \mathcal{Q}Id$ is the identity and Moss's logic is just that of deterministic transition systems ($\nabla\varphi \equiv \Box\varphi \equiv \Diamond\varphi$). Explicitely, a state x in a coalgebra ξ satisfies $\nabla\varphi$ iff $\xi(x) \in [\![\varphi]\!]$.

(ii) In the case of a constant functor K_C, the operator $\boldsymbol{\nabla} : K_C\mathcal{Q} \longrightarrow \mathcal{Q}K_C$ maps an element $d \in C$ to the set $\{d\}$. A state x in a coalgebra ξ satisfies ∇d iff $\xi(x) = d$.

(iii) In the case of the covariant power set functor $\boldsymbol{\nabla}$ is given by

$$\alpha \in \boldsymbol{\nabla}(\Phi) \text{ iff } (\forall \varphi \in \Phi \,.\, \exists x \in \alpha \,.\, x \in \varphi) \text{ and } (\forall x \in \alpha \,.\, \exists \varphi \in \Phi \,.\, x \in \varphi).$$

It is well-known (and not difficult to check) that in this case Moss's logic (over BA) is equivalent to classical modal logic, that is, there are *translations* in both directions: $\nabla\alpha = \Box \bigvee \alpha \wedge \bigwedge \Diamond\alpha$ and $\Box\varphi = \nabla\{\varphi\} \vee \nabla\emptyset$, $\Diamond\varphi = \nabla\{\varphi, \top\}$.

(iv) In the case of the finite distribution functor, we can describe the operator $\boldsymbol{\nabla}$ noticing that for $b \in \mathcal{D}(X)$ and $B \in \mathcal{D}(\mathcal{Q}X)$ the relation $b\,\overline{\mathcal{D}}(\in_X)\,B$ can be described as follows. First note that $b = (x_i, p_i)_{1\leq i\leq n}$ for some $x_i \in X, p_i \in [0,1], p_i > 0, n \in \mathbb{N}$; similarly $B = (\varphi_j, q_j)_{1\leq j\leq m}$ for $\varphi_j \in \mathcal{Q}X, q_j \in [0,1], q_j > 0, m \in \mathbb{N}$. Then $b\,\overline{\mathcal{D}}(\in_X)\,B$ iff there are $(r_{ij})_{1\leq i\leq n, 1\leq j\leq m}$, $r_{ij} \in [0,1]$ such that $x_i \notin \varphi_j \;\Rightarrow\; r_{ij} = 0$ and $\sum_i r_{ij} = q_j$ and $\sum_j r_{ij} = p_i$.

For example, a state x in a coalgebra ξ satisfies $\nabla\{(\varphi, q), (\top, 1-q)\}$ iff the probability of going to a successor satisfying φ is larger or equal to q. That is, ∇ (together with Boolean operators) can express the usual modal operators of probability logic [10].

In the case of the finite multiset functor we have the same description, just replacing $[0,1]$ by $\mathbb{N}$. For example, a state x in a coalgebra ξ satisfies

- $\nabla\{(\top, n)\}$ iff x has exactly n successors;
- $\nabla\{(\varphi, m), (\top, n)\}$ iff x has at least m successors satisfying φ and exactly $m + n$ successors in total;

In fact, each ∇-formula specifies the total number of successors. The usual graded modalities can therefore not be expressed.

As mentioned in the introduction, there are at least two motivations for a ∇-based approach towards modal logic: (1) In applications to automata theory, ∇-based modal logic works better because one may almost eliminate conjunctions from the language. This observation, which is closely linked to fundamental automata-theoretic constructions, lies at the heart of the theory of the modal μ-calculus, and has many applications, see for instance [5,11,27]. (2) Moreover, as we saw, ∇ allows coalgebraic generalizations. This has been used, see [15,29], to show that many fundamental results in automata theory are really theorems of universal coalgebra.

3.2 The Logic of All Predicate Liftings

For any endofunctor T on Set, we define an endofunctor L_T on the category BA of Boolean algebras. The idea is the following. Going back to Example 2.6, we defined δ and then

proved that, on finite X, $\delta : LP \longrightarrow PL$ is an iso (Proposition 2.7). We now turn this around and essentially use it as a definition.

Definition 3.7 $L_T : \mathsf{BA} \longrightarrow \mathsf{BA}$ is defined, on finite A, as $L_T A = PTSA$. Since every Boolean algebra A is the directed union of finite subalgebras A_i we let $L_T A$ be the directed union of the $L_T A_i$. δ is given by $L_T P = PTSP \cong PT$ on finite X and extended uniquely to arbitrary X.

Example 3.8 For $\mathcal{P}$ the functor L has been described explicitly in Example 2.6

As shown in [20], L can be represented by operations, where we take $ULFn$ as the set of operations of arity n (identifying the number n with a set of n-elements). Calculating $ULFn = UPTSFn = \mathcal{Q}T(\mathcal{Q}n)$, which, by the Yoneda Lemma, is the set of natural transformations $\mathcal{Q}^n X \longrightarrow \mathcal{Q}TX$, or, in more familiar notation, $(2^n)^X \longrightarrow 2^{TX}$. Explicitly this is:

Proposition 3.9 *There is a natural isomorphism (natural in n and $\mathcal{Q}T$)*

$$Y_{(n,T)} : \mathcal{Q}T\mathcal{Q}(n) \longrightarrow \mathsf{Nat}(\mathcal{Q}^n, \mathcal{Q}T).$$

Proof. Recall that $\mathcal{Q}^n X = Hom(X, \mathcal{Q}n)$. We define a bijection between $\mathcal{Q}T(\mathcal{Q}n)$ and natural transformations $\mathcal{Q}^n X \longrightarrow \mathcal{Q}TX$ as follows: any $p \in \mathcal{Q}T(\mathcal{Q}n)$ gives a predicate lifting that maps $v : X \longrightarrow \mathcal{Q}n$ to $\mathcal{Q}Tv(p)$. Conversely, for each $\lambda_X : \mathcal{Q}^n X \longrightarrow \mathcal{Q}TX$ we have $\lambda_{\mathcal{Q}n}(\mathrm{id}) \in \mathcal{Q}T(\mathcal{Q}n)$. □

The proposition holds for any contravariant functor F and not just for for the functor $\mathcal{Q}T$; this fact is known as Yoneda Lemma. It shows that our operations of arity n are precisely the *predicate liftings* of arity n in Schröder [28]:

Definition 3.10 Given a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, an n-ary predicate lifting is a natural transformation $\mathcal{Q}^n X \longrightarrow \mathcal{Q}TX$.

Proposition 3.9 tells us that predicate liftings of arity n can be identified with subsets of $T(2^n)$; this is particularly useful to present examples of predicate liftings.

Example 3.11 (i) Let T be the covariant power set functor and let $2 = \{\bot, \top\}$. The existential modality $\Diamond$ can be presented using an homonymous predicate lifting $\Diamond : \mathcal{Q} \longrightarrow \mathcal{Q}\mathcal{P}$, with the followings components $\Diamond_X(A) = \{U \subseteq X \mid A \cap U \neq \emptyset\}$. Using Proposition 3.9, we can see that this corresponds to the set $\{\{\top\}, \{\top, \bot\}\}$. Similarly, the universal modality $\Box$ can be presented as a predicate lifting $\Box_X(A) = \{B \subseteq X \mid B \subseteq A\}$ (compare this with (4)). By Proposition 3.9, this predicate lifting is associated to the set $\{\emptyset, \{\top\}\}$.

(ii) Let T be the finite distribution functor. The modality $\Diamond_p \varphi$ specifies a probability of at least p for the event of going to a successor satsifying φ. It can be described by the predicate lifting $\mathcal{Q}X \longrightarrow \mathcal{Q}TX$, $a \mapsto \{d \in \mathcal{D}X \mid \mu_d(a) \geq p\}$, where $\mu_d(a) = \sum_{x \in a} d(x)$ is the measure associated with d. By Proposition 3.9, this predicate lifting corresponds to a subset of $\mathcal{D}(2)$. If we describe a probability distribution $d : 2 \longrightarrow [0,1]$ by its value on $\top$ ($d(\bot) = 1 - d(\top)$), we find that $\Diamond_p$ corresponds to the set $\{q \in [0,1] \mid q \geq p\}$. Similarly, the predicate lifting associated to an interval $(q, q') \subseteq [0,1]$ maps a set $a \subseteq X$ to the set of probability distributions over X that assign a probability between q and q' to the set a.

We can also define a logic for any set of predicate liftings Λ as follows:

Definition 3.12 Let Λ be a set of predicate liftings. The functor $\bar{L}_\Lambda : \mathcal{A} \longrightarrow \mathcal{A}$ maps A to the free $\mathcal{A}$-algebra generated by all $\lambda(a_1, \ldots, a_n)$ where $\lambda \in \Lambda$, n is the arity of λ, and $a_i \in UA$. We write $\bar{L}_T$, or just $\bar{L}$, if Λ is the set of all predicate liftings.

More explicitly, we can describe the functor $\bar{L}_\lambda$, for a single predicate lifting λ, to be $L_\lambda = FU^{n_\lambda}$, where n_λ is the arity of λ. In general we have $\bar{L}_\Lambda = \coprod_{\lambda\in\Lambda} \bar{L}_\lambda$. We can also define the language of all predicate liftings $\mathcal{L}_T$ based on Boolean logic as follows (in future, we will usually drop the subscript T).

Definition 3.13 $\mathcal{L}_T$ is the smallest set closed under Boolean operations and under the rule if $n < \omega, 1 \leq i \leq n, \varphi_i \in \mathcal{L}_T, \lambda \in \mathcal{Q}T\mathcal{Q}n \;\; \Rightarrow \;\; \lambda(\varphi_1, \ldots \varphi_n) \in \mathcal{L}_T$.

This perspective of languages with predicate liftings will prove to be useful to generalize the results in [21].

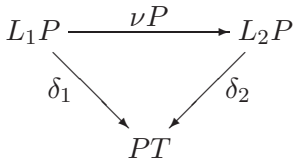
4 Translators and $\mathcal{A}$-Translators

In this section we will investigate under what circumstances we can find translation from the ∇-logic $\mathcal{M}$ into the logic of all predicate liftings $\mathcal{L}$ and vice versa. Let us note first that we are not interested in showing only that every formula in $\mathcal{L}$ has an equivalent formula in $\mathcal{M}$ (and v.v.). Rather we want an inductive definition of the translation, which respects the one-step nature (see Remark 2.5). This stronger property of one-step translations is captured by natural transformations $\bar{L} \longrightarrow M$ and $M \longrightarrow L$.

4.1 One-step translations

We start by defining translations between coalgebraic logics. Our notion of coalgebraic logic assumes a category $\mathcal{A}$ of power-set algebras as in Definition 3.1, a functor $L : \mathcal{A} \longrightarrow \mathcal{A}$ and a natural transformation $\delta : LP \longrightarrow PL$, as explained in Section 2.2.

Definition 4.1 Given two coalgebraic logics (L_1, δ_2) and (L_2, δ_2), a natural transformation $\nu : L_1 \longrightarrow L_2$ is a *one-step translation* if it commutes with the semantics:

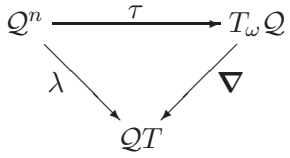


A one-step translation can be understood as an inductive definition of a translation between the associated logics. Indeed, given any L_2-algebra $L_2A \longrightarrow A$ we obtain an L_1-algebra $L_1A \xrightarrow{\nu_A} L_2A \longrightarrow A$; moreover, since ν is a natural transformation any morphism $f : A \longrightarrow A'$ of L_2 algebras is also a morphism between the corresponding L_1-algebras. Denote by $L_iI_i \longrightarrow I_i$ the initial L_i-algebras. Using this observation, we find, by initiality of I_1, an inductively defined morphism of L_1-algebras $I_1 \longrightarrow I_2$ which translates formulas in I_1 to formulas in I_2. Notice that it is important that ν is a natural because this allows to map a morphism of L_2-algebras $I_2 \longrightarrow A$ to a morphism of L_1-algebras.

4.2 Translating Predicate Liftings

We are looking for a natural transformation $\bar{L} \longrightarrow M$ (see Definitions 3.12 and 3.3). To do this, we need to translate predicate liftings λ into $\boldsymbol{\nabla}$. Note that every predicate lifting $\lambda : \mathcal{Q}^n \longrightarrow \mathcal{Q}T$ and the natural transformation $\boldsymbol{\nabla} : T_\omega \mathcal{Q} \longrightarrow \mathcal{Q}T$ share the codomain but have different domains (the subscript ω below corresponds to the one in Definition 3.3). This motivates the following definition [21].

Definition 4.2 A translator for a predicate lifting λ is a natural transformation $\tau : (\mathcal{Q})^n \longrightarrow T\mathcal{Q}$ such that



Example 4.3 Consider the predicate lifting associated with the existential modality $\Diamond$ of the covariant power set functor (Example 3.11). It has the translator $\tau_X : \mathcal{Q}X \longrightarrow \mathcal{P}_\omega \mathcal{Q}X$ mapping an element $a \subseteq X$ to $\tau_X(a) = \{a, X\}$. Compare with the equivalence $\Diamond\varphi = \nabla\{\varphi, \top\}$ discussed in Example 3.6.

For more see Example 4.6 below. The idea of a translator is to define a one-step translation tr via

$$tr(\lambda\varphi) = \nabla\tau(tr(\varphi)). \tag{7}$$

Unfortunately not all predicate liftings have translators.

Example 4.4 Let K_C be a constant functor where C has at least two distinct elements c_1, c_2. Using Proposition 3.9 (see also Example 3.11), predicate liftings correspond to subsets of C. The predicate lifting λ_E corresponding to $E = \{c_1, c_2\}$ does not have a translator. This is because the components of a natural transformation $\tau : \mathcal{Q} \longrightarrow K_C$ are constant functions, hence the cardinality of $\boldsymbol{\nabla}\tau(X)$ is always 1, but $\lambda_E X = E$. Nevertheless, notice that the formula $\nabla c_1 \vee \nabla c_2$ translates the predicate lifting λ_E.

If we look back at Equation (7), we can see that translators produce "simple" translations not involving operators such as $\vee$. Accordingly, translations will involve translators as well as propositional operators. First, we need to know a big enough class of predicate liftings that do have translators.

Definition 4.5 ([21]) An n-ary predicate lifting λ is called a *singleton predicate lifting*, or a *singleton lifting* for short, if it is associated (via Proposition 3.9) with a single element $p \in T(2^n)$, ie, if the following holds: Given $\varphi : n \longrightarrow 2^X$

$$\lambda_X(\varphi) = \{t \in TX \mid T(\chi_\varphi)(t) = p\}, \tag{8}$$

where $\chi_\varphi : X \longrightarrow 2^n$ is the transpose of φ. If λ is a singleton lifting, we write it λ_p or just p, where p is the associated element of $T(2^n)$.

Example 4.6 (i) If T is a constant functor with value C, then the singleton liftings for T are associated with elements $c \in C$. The X-component of a singleton lifting λ_c is the function $\lambda_c : \mathcal{Q}X \longrightarrow \mathcal{Q}K_C$ with constant value $\{c\}$.

(ii) If T is the identity functor and we assume $2 = \{\top, \bot\}$, then there are two singleton liftings of arity 1 for Id. The X-component of $\lambda_\top$ is the identity. Similarly, the X-component of $\lambda_\bot$ is the function $(\lambda_\bot)_X : \mathcal{Q}X \longrightarrow \mathcal{Q}X$ mapping a set $\varphi \subseteq X$ to $\lambda_\bot(\varphi) = \neg_X\varphi$ to its complement.

(iii) The covariant power set functor has four singleton liftings of arity 1, explicitely these are associated with $\mathcal{P}(2) = \{\emptyset, \{\top\}, \{\bot\}, \{\top, \bot\}\}$. Given a set $\varphi \subseteq X$, the action of these predicate liftings is (we drop the subscripts X):

$$\lambda_{\{\top\}}(\varphi) = \{U \in \mathcal{P}X \mid \emptyset \neq U \subseteq \varphi\}; \quad \lambda_{\{\bot\}}(\varphi) = \{U \in \mathcal{P}X \mid \emptyset \neq U \subseteq \neg_X\varphi\};$$
$$\lambda_{\emptyset}(\varphi) = \{\emptyset\}; \quad \lambda_{\{\top,\bot\}}(\varphi) = \{U \in \mathcal{P}X \mid U \cap \neg_X\varphi \neq \emptyset \neq U \cap \varphi\};$$

Note that they all have translators, corresponding to $\nabla\{\varphi\}$, $\nabla\{\neg_X\varphi\}$, $\nabla\emptyset$, $\nabla\{\varphi, \neg_X\varphi\}$, respectively.

(iv) If T is the finite multiset functor, a singleton lifting is given by a pair of natural numbers (n, m). Its X component, $(n, m) : \mathcal{Q}X \longrightarrow \mathcal{Q}\mathcal{B}_{\mathbb{N}}X$, maps a set $\varphi \subseteq X$ to the set of bags over X with $n + m$ elements, n of which are in φ and m are in the complement of φ. Such a predicate lifting has a translator as it corresponds to $\nabla\{(\varphi, n), (\neg_X\varphi, m)\}$, in the notation of Example 3.6.

(v) If T is the finite distribution functor, a singleton lifting is given by a real number $q \in [0, 1]$. The X-component of q maps a set $\varphi \subseteq X$ to the set of probability distributions over X that assign probability q to the set φ. Such predicate liftings have translators as they correspond to $\nabla\{(\varphi, q), (\neg_X\varphi, 1 - q)\}$, in the notation of Example 3.6.

The second author's [21] started the study of singleton liftings because: (1) In the case of KPFs they can be presented inductively over the complexity of the functor, and (2) by Proposition 3.9 they generate all the other predicate liftings:

Proposition 4.7 ([21]) *If λ is an n-ary predicate lifting associated with a set $P \subseteq T(2^n)$, then for every set X and every n-sequence $\varphi : n \longrightarrow \mathcal{Q}X$ we have: $\lambda_X(\varphi) = \bigcup_{p\in P}(\lambda_p)_X(\varphi)$. In other words, every n-ary predicate lifting can be obtained as a (possibly infinite) join of singleton predicate liftings.*

Example 4.8 Going back to Example 3.11, the predicate lifting for $\Box$ is $\lambda_{\{\emptyset,\{\top\}\}}$. It does not have a translator but is the union $\lambda_\emptyset \cup \lambda_{\{\top\}}$ of two singleton liftings, which have a translator by Example 4.6. Similarly, the predicate lifting for $\Diamond$ is $\lambda_{\{\{\top,\bot\},\{\bot\}\}} = \lambda_{\{\top,\bot\}} \cup \lambda_{\{\bot\}}$. Incidentally, $\Diamond$ does have a translator, see Example 4.3.

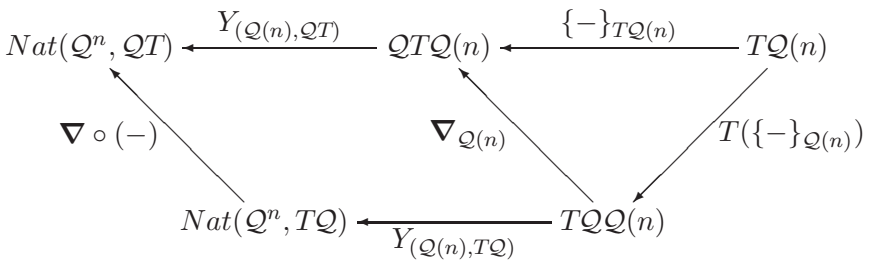
The starting point of the present paper was the discovery that singleton liftings always have translators. The proof is based on the following lemma, which also plays a crucial role in [14]. The proof of the lemma is immediate from the fact that the composition $X \overset{\{-\}}{\longrightarrow} \mathcal{Q}X \overset{\ni}{\longrightarrow} X$ of relations is the identity.

Lemma 4.9 *Consider $\{-\}_A : A \longrightarrow \mathcal{Q}A$, to be the singleton function i.e. $a \mapsto \{a\}$. Then $\boldsymbol{\nabla}_{\mathcal{Q}} \circ T(\{-\}_{\mathcal{Q}}) = \{-\}_{T\mathcal{Q}}$.*

Using this we can prove the following result.

Proposition 4.10 *Let T be a weak pullback preserving functor. Then each singleton lifting λ_p has a translator. Moreover, the translator is associated with $T(\{-\}_Q)(p)$.*

Proof. Consider the following diagram



The parallelogram on the left expresses the naturality of Yoneda Lemma 3.9, hence commutes, the triangle on the right commutes by Lemma 4.9. The upper edge maps an element in $p \in T\mathcal{Q}(n)$ to the associated singleton lifting λ_p. The commutativity of the diagram implies that the natural transformation associated with $T(\{-\}_{\mathcal{Q}(n)})(p)$, which is a natural transformation $\mathcal{Q}^n \longrightarrow T\mathcal{Q}$, is a natural translator for λ_p. □

Notice that translators almost define one-step translators. However, we have to make sure that τ in Equation (7) is definable in the logic. This is not always possible as the following example shows.

Example 4.11 Suppose we replace, in Definition 3.3, $\mathcal{A}$ by the category DL of distributive lattices, that is, we work with a positive Moss logic without negation. Consider T to be the identity functor and the predicate lifting $\lambda_\perp : \mathcal{Q} \longrightarrow \mathcal{Q}$ given by complementation. In this example, ∇ is the identity and complementation $\neg : \mathcal{Q} \longrightarrow \mathcal{Q}$ is a translator for λ. However, all the operators in $\mathcal{M}_{Id}$ are monotone, therefore all the definable terms are monotone, which implies that negation is not definable. In other words, we cannot translate $\lambda_\perp$ into $\mathcal{M}_{Id}$.

The example shows that the underlying category plays a role to whether translations are possible: $\lambda_\perp\varphi$ can be translated into a ∇-formula over BA but not over DL. This leads us to refine the notion of translator to that of an $\mathcal{A}$-translator. Intuitively, an $\mathcal{A}$-translator is a translator that is natural wrt to $\mathcal{A}$-morphisms. We will show later that all translators can be extended to BA-translators.

Definition 4.12 Let λ be an n-ary predicate lifting, $\mathcal{A}$ a category with power-set algebras, and $U : \mathcal{A} \longrightarrow \mathsf{Set}$ the forgetful functor. An *$\mathcal{A}$-translator* τ for λ is a natural transformation $\tau : U^n \longrightarrow T_\omega U$ such that τ_P is a translator for λ (recall that $UP = \mathcal{Q}$).

If the category $\mathcal{A}$ is clear from the context, we often call an $\mathcal{A}$-translator a *logical translator*. We say that the logical translator τ extends the translator τ_P. If there exists an $\mathcal{A}$-translator for λ, we say that the predicate lifting λ is $\mathcal{A}$-translatable.

Example 4.13 (i) In Example 4.11, $\tau = \neg$ extends to a BA-translator, but not to a DL-translator.

(ii) Consider the predicate lifting associated with the existential modality $\Diamond$ as in Example 4.3. We define a BA-translator τ: Given a Boolean algebra $\mathfrak{A}$, with carrier A, the function $\tau_{\mathfrak{A}} : A \longrightarrow \mathcal{P}A$ maps an element $x \in A$ to $\tau_{\mathfrak{A}}(x) = \{x, \top\}$. Notice that this BA-translator is also an $\mathcal{A}$-for any category $\mathcal{A}$ of power-set algebras and induces the following translation $tr(\Diamond\varphi) = \nabla\{tr(\varphi), \top\}$.

(iii) We can ask which predicate liftings have $\mathcal{A}$-translators for all categories $\mathcal{A}$ of power-set algebras. These are precisely what we call the Moss liftings, see Remark 6.7.

The main property of logical translators, as suggested by the previous examples, is that they produce translations:

Proposition 4.14 *Let Λ be a set of predicate liftings, each of which has a logical translator. Then can we find a one-step translation $\bar{L}_\Lambda \longrightarrow M$.*

Proof. For each $\lambda \in \Lambda$ let $\tau_\lambda : U^{n_\lambda} \longrightarrow T_\omega U$ be a logical translator. Combining those we obtain a map $\coprod_{\lambda\in\Lambda} U^{n_\lambda} \longrightarrow T_\omega U$, the image of this map under the left adjoint of U is the required translation $\bar{L}_\Lambda \longrightarrow M$. □

5 Translating with Classical Logic

In this section, we will produce one-step translations (Definition 4.1) between Moss's logic $\mathcal{M}_T$ (Definition 3.2) and the logic $\mathcal{L}_T$ (Definition 3.13) of all predicate liftings. The main technical result is that that translators (Definition 4.2) can always be extended to BA-translators (Definition 4.12).

The translations rely on some conditions on the type functor T and on the propositional logic being Boolean. Accordingly, in this section we always assume our logic is based on BA. If we would like to extend the results to DL, we should modify the notion of predicate lifting by working with endofunctors T over the category of ordered sets and replace the functor $\mathcal{Q}$ by the down-set functor. We do not pursue this issue here.

5.1 From $\mathcal{L}$ to $\mathcal{M}$

Lemma 5.1 *Every translator $\tau : \mathcal{Q}^n \longrightarrow T_\omega\mathcal{Q}$ can be extended to a BA-translator, i.e. a natural transformation $U^n \longrightarrow T_\omega U$, where U : BA $\longrightarrow$ Set is the forgetful functor.*

Proof. (Sketch.) Recall that (1) every Boolean algebra is the directed colimit of finite Boolean algebras and (2) every Boolean algebra morphism between finite BAs arises from the inverse image of a function between sets. Because of (2) we have that $\tau : U^n_\omega \longrightarrow T_\omega U_\omega$ is natural where U_ω : $\mathsf{BA}_\omega \longrightarrow$ Set is the restriction of U to finite Boolean algebras. Because of (1), we can extend τ from U_ω to U. This makes the translator τ into a BA-translator. □

The lemma does not hold for other categories of power-set algebras. But for BA, we obtain

Theorem 5.2 *If T preserves finite sets and weak pullbacks, there is a one-step translation $\bar{L}_T \longrightarrow M_T$.*

Proof. (Sketch.) Let $\bar{L}_s$ be the functor given as in Definition 3.12, but using only singleton liftings. Because T preserves finite sets, every predicate lifting can be expressed as a finite join of singleton liftings (Proposition 4.7), hence we have an isomorphism $\bar{L} \cong \bar{L}_s$. Now let λ be a singleton lifting and let τ be the corresponding translator (Proposition 4.10). Extend τ to a logical translator $U^n \longrightarrow T_\omega U$ as in the previous lemma. This gives a natural transformation $U^n \longrightarrow M_T$. Doing this for each singleton lifting and combining all of these logical translators we obtain a translation $\bar{L}_s \longrightarrow M_T$. □

Notice that Examples 4.11, and 4.4 show that to translate all predicate liftings, we need at least classical logic. The following example shows that the condition of T preserving finite sets can not be avoided.

Example 5.3 If the functor T does not preserve finite sets, not every predicate lifting can be translated into Moss's language. Let T be the constant functor with value $\mathbb{N}$, let $E \subseteq \mathbb{N}$ be the set of even numbers. If we are working over BA, the predicate lifting λ_E can not be translated into Moss' language. Consider the coalgebra $N = (\mathbb{N}, 1_{\mathbb{N}})$ and the formula $\lambda_E \top$. On the one hand, this formula defines the set of even numbers, i.e. $[\![\lambda\top]\!] = E$. On the other hand, we can check that using Moss' language we can only define finite and cofinite sets; therefore we conclude that the predicate lifting λ_E can not be expressed into Moss' language over BA.

The following translations are illustrations of the previous theorem.

Example 5.4 • Example 4.11 is a BA-translator obtained using the previous theorem.

- Let (n, m) be singleton lifting for the finite multiset functor (Example 4.6). We define a BA-translator for (n, m) as follows: Given a Boolean algebra $\mathfrak{A}$, with carrier A, the function $\tau_{\mathfrak{A}} : A \longrightarrow \mathcal{B}_{\mathbb{N}}A$ maps an element $x \in A$ to the following bag: $B_{(x,n,m)} : A \longrightarrow \mathbb{N}$

$$B_{(x,n,m)}(x) = n, B_{(x,n,m)}(\neg_{\mathfrak{A}} x) = m \text{ and } B_{(x,n,m)}(a) = 0 \text{ for any other element.}$$

This logical translator induces the following translation $t((n, m)a) = \nabla B_{(t(a),n,m)}$.

5.2 *From $\mathcal{M}$ to $\mathcal{L}$*

Our next step is to find a translation $M_T \longrightarrow L_T$. Note that we do not expect a natural transformation $M_T \longrightarrow \bar{L}_T$ because each ∇-formula corresponds to many different formulas of $\mathcal{L}_T$ (see also the next section). But L (Definition 3.7) already quotients out by one-step logical equivalence, thus identifying all equivalent formulas.

Theorem 5.5 *For all weak pullback preserving functors T there exists a one-step translation $M_T \longrightarrow L_T$.*

Proof. If we restrict to finite sets and finite boolean algebras, we have an isomorphism $\iota : Id \longrightarrow SP$ (see Diagram (1)) and then the following natural transformation

$$T_\omega UP \xrightarrow{\nabla} UPT \xrightarrow{\iota} UPTSP$$

which can be freely extended to a natural transformation $\theta_\omega : M_T P \longrightarrow L_T P$. Since every BA is a directed colimit of finite Boolean algebras of the form PX for finite X and since M_T preserves directed colimits, we can extend θ_ω to a natural transformation $\theta : M_T \longrightarrow L_T$

$$\begin{array}{ccc} M_T A & \xrightarrow{\theta_A} & L_T A \\ \uparrow & & \uparrow \\ M_T PX & \xrightarrow{(\theta_\omega)_X} & L_T PX \end{array}$$

which is a one-step translation (Definition 4.1). □

Again, the theorem is specific to BA. On the other hand, it is a particular instance of a more general result. Namely, if we are working over BA, all expressive coalgebraic logics for a finitary functor that preserves finite sets are equivalent, i.e. mutually translatable. This Lindström like theorem is formulated in the next theorem.

Theorem 5.6 *Assume that T that preserves finite sets and that L is a complete and expressive coalgebraic logic. Then for all coalgebraic logics L' there is a one-step translation $\tau : L' \longrightarrow L$. Moreover, if L' is complete and expressive as well, then τ is an isomorphism.*

The proof is exactly as the one used in the case of Moss's logic. The natural transformation τ can be obtained as the extension of $L'PX \xrightarrow{\delta'} PTX \xrightarrow{\delta^{-1}} L_2PX$, where δ' and δ are the natural transformations inducing the respective semantics. This relies on the fact that L is complete and expressive iff δ_X is iso on finite X.

6 Equational Coalgebraic Logic

The aim of this section is to apply our translation. We do this by presenting Moss's logic using only conventional operators, i.e. predicate liftings and showing how the axiomatization of Moss's logic from [14], gives rise to a standard modal axiomatization. One advantage of such an equational version of Moss's logic is that one can reuse known logical methods. For example, in a logic given by predicate liftings, the subformulas of a formula $\lambda(\varphi_1, \dots \varphi_n)$ are the φ_i. But what should be the subformulas of $\nabla\alpha$, if all we know about α is that $\alpha \in T_\omega(\mathcal{M}_T)$? Or how to state that ∇ is monotone? Or what does congruence mean? All these questions can be answered [14], but this requires some technical work, which can be avoided in the equational presentation.

To present $\mathcal{M}_T$ we use the fact that every finitary functor T_ω is a quotient of a polynomial functor Σ

$$\Sigma X = \coprod_{n<\omega} \Sigma_n \times X^n \xrightarrow{E_X} T_\omega X. \tag{9}$$

Such quotient is called a **presentation** $\langle \Sigma, E \rangle$ of T_ω by operations and equations: Σ_n is called the set of operations of arity n and the equations defining T_ω are the kernel of E_X (for some countably infinite set of 'variables' X) (for more on set-functors and their presentations see Adámek and Trnková [2]).

Example 6.1 $\mathcal{P}$ is a quotient of the list-functor $\mathsf{List}(X) = \coprod_{n<\omega} X^n$. E_X maps lists $(x_0, \dots x_n)$ to sets $\{x_0, \dots x_n\}$. The equations given by E are the usual equations defining sets from lists (expressing that order and repetitions don't matter).

Remark 6.2 Every finitary functor T_ω has a **canonical presentation** given by $\Sigma_n = T_\omega(n)$ and $E_X(p, v) = T(v)(p)$ for $p \in T(n)$ and $v : n \longrightarrow X$.

Using the presentation, we can compute relation liftings (Remark 3.4). The following lemma is the key stone for our development of equational coalgebraic logic

Lemma 6.3 *Let R be a relation between X and Y and T a finitary endofunctor on* Set. *For every $t_x \in TX$ and $t_y \in TY$ the following conditions are equivalent:*

- $t_x \, \overline{T}(R) \, t_y$.
- *There exists $k < \omega$, $r \in T(k)$, $a : k \longrightarrow X$, and $b : k \longrightarrow Y$ such that $T(a)(r) = t_x$, $T(b)(r) = t_y$, and $(\forall i \in k)(a_i \, R \, b_i)$.*

More informally, we read the lemma as

$$t_x \,\overline{T}(R)\, t_y \quad \text{iff} \quad t_x = r(a_1, \ldots a_k) \;\text{ and }\; t_y = r(b_1, \ldots b_k) \;\text{ and }\; a_i R b_i$$

where '$=$' refers to the equational theory of the presentation of T.

Proof. The proof is straightforward from contemplating the following commuting diagram

$$\begin{array}{ccccc} \Sigma X & \xleftarrow{\Sigma(\pi_X)} & \Sigma R & \xrightarrow{\Sigma(\pi_Y)} & \Sigma Y \\ {\scriptstyle E_X}\downarrow & & {\scriptstyle E_R}\downarrow & & {\scriptstyle E_Y}\downarrow \\ TX & \xleftarrow[T(\pi_X)]{} & TR & \xrightarrow[T(\pi_Y)]{} & TY \end{array} \tag{10}$$

and taking into account that E_R is surjective (due to T finitary). □

6.1 The logic $\mathcal{K}_T$

Given a presentation $\langle \Sigma, E \rangle$ of T_ω, every $p \in \Sigma_n$ gives rise to an n-ary predicate lifting

$$\begin{array}{ccc} \mathcal{Q}^n & \xrightarrow{E_{\mathcal{Q}}(p,-)} & T_\omega \mathcal{Q} \\ & {\scriptstyle \lambda^p} \searrow \quad \swarrow {\scriptstyle \nabla} & \\ & \mathcal{Q}T & \end{array} \tag{11}$$

If $\langle \Sigma, E \rangle$ is the canonical presentation, we call a predicate lifting arising in this way a **Moss lifting**. The set of all Moss liftings can be identified with $\coprod_{n<\omega} T_\omega(n)$ (notice that if $\langle \Sigma, E \rangle$ is any presentation of T there is a canonical function $\Sigma_n \longrightarrow T_\omega(n)$).

Example 6.4 (i) Let $T = 1 + Id$ (deterministic transition systems with termination). For each arity n there is a Moss lifting λ^*_n, which indicates termination; this lifting corresponds to the unique element of 1. All other Moss liftings of arity n correspond to the elements of n. For $p \in n$, the Moss lifting λ^p maps a sequence $\varphi : n \longrightarrow \mathcal{Q}X$ to the set φ_p. Using Moss liftings we can see that if the system is deterministic there is no need go beyond arity 1.

(ii) Let $T = \mathcal{P}$ (non-deterministic transition systems). Moss liftings of arity n are associated with subsets of n. Let p be one of those subsets. The Moss lifting λ^p maps a sequence $\varphi : n \longrightarrow \mathcal{Q}X$ to the set

$$\lambda^p(\varphi) = \{\alpha \in \mathcal{P}X \mid (\forall x \in \alpha)(\exists i \in p)(x \in \varphi_i) \wedge (\forall i \in p)(\exists x \in \alpha)(x \in \varphi_i)\}.$$

(iii) Let T be the finite multiset functor. Moss liftings of arity n corresponds to bags $p : n \longrightarrow \mathbb{N}$. The predicate lifting associated with such a bag p maps a sequence $\varphi : n \longrightarrow \mathcal{Q}X$ as follows. The pair (p, φ) can be considered as multiset over $\mathcal{Q}X$ (p_i being the multiplicity of φ_i). It is then mapped by ∇ according to Example 3.6.

Definition 6.5 Given a presentation $\langle \Sigma, E \rangle$ of T_ω, the logic $\mathcal{K}_T^{\langle \Sigma, E \rangle}$ is the logic (Definition 3.12) given by the set of predicate liftings λ^p, $p \in \Sigma_n$. We simply write $\mathcal{K}_T$ if the

presentation is clear from the context or the canonical presentation. The corresponding functor is denoted by $K_T : \mathsf{BA} \longrightarrow \mathsf{BA}$.

In [21] the natural transformation E was used to translate Moss's logic into the language of predicate liftings. The translation is based on the fact for each $\alpha \in T_\omega X$ there exists a pair $(p, v) \in T_\omega(n) \times X^n$ such that $E_X(p, v) = \alpha$. Thus formulas $\nabla\alpha$ can be replaced by formulas $\lambda^p(\varphi_1, \ldots \varphi_n)$. More formally we have.

Proposition 6.6 ([21]) *For every formula in $\mathcal{M}_T$ there exists an equivalent formula $\mathcal{K}_T$.*

Remark 6.7 By definition, see (11), every Moss' lifting can be translated into Moss's language, or more technically, $E_{\mathcal{Q}X}(p, -)$ is a translator for λ^p in the sense of Definition 4.2. Moreover, it is also an $\mathcal{A}$-translator (Definition 4.12) for all category $\mathcal{A}$ with power-set algebras (Definition 5). Conversely, instantiating $\mathcal{A}$ with Set, we find that Moss liftings are the only predicate liftings for which we can find $\mathcal{A}$-translators for any $\mathcal{A}$. Thus we may say: The Moss liftings are precisely the totally translatable predicate liftings.

Another important property of Moss liftings is that they are monotone:

Proposition 6.8 *Let $\lambda^p : \mathcal{Q}^n \longrightarrow \mathcal{Q}T$ be a Moss lifting; let $\varphi, \psi : n \longrightarrow \mathcal{Q}X$ be sequences of sets. If $(\forall i)(\varphi_i \subseteq \psi_i)$ then $\lambda^p(\varphi) \subseteq \lambda^p(\psi)$.*

Proof. Let $E(p, -)$ be the translator of λ^p. Using Lemma 6.3 we see that $(\forall i)(\varphi_i \subseteq \psi_i)$ implies $E_{\mathcal{Q}}(p, \varphi)\, \overline{T}(\subseteq)\, E_{\mathcal{Q}}(p, \psi)$. Applying $\boldsymbol{\nabla}$ on both sides of the previous inequality will transform $\overline{T}(\subseteq)$ into $\subseteq$; we conclude $\lambda^p(\varphi) \subseteq \lambda^p(\psi)$. □

This has the following important corollary.

Corollary 6.9 *For every weak pullbacks preserving functor T there exists a set Λ of monotone predicate liftings such that the logic L_Λ is expressive. The set Λ is that of Moss liftings.*

Remark 6.10 Finding a monotone set of predicate liftings is important in coalgebraic modal logic, as it opens the possibility of adding fix points operators. The previous proposition solves this problem in the case of weak-pullback preserving functors. As far as we know, the general problem for non-weak pullback preserving functors is still open.

6.2 A complete equational proof system for $\mathcal{K}$

Now we will present a proof system to describe logical equivalence between formulas built from Moss liftings. [14] presents the following complete and sound system for the ∇-logic.

$$(\nabla 1)\quad \textstyle\bigwedge\{\nabla\alpha \mid \alpha \in A\} \leq \bigvee\{\nabla T(\bigwedge)\Phi \mid \Phi \in SRD(A)\}.$$

$$(\nabla 2)\quad \nabla T(\textstyle\bigvee)\Phi \leq \bigvee\{\nabla\alpha \mid \alpha \overline{T}(\in)\Phi\}. \qquad (\nabla 3)\quad \text{From } \alpha \overline{\leq} \beta \text{ infer } \vdash_\nabla \nabla\alpha \leq \nabla\beta$$

$$\text{where } \alpha \in T_\omega\mathcal{M},\ A \in \mathcal{P}_\omega T_\omega\mathcal{M},\ \Phi \in T_\omega\mathcal{P}_\omega\mathcal{M}$$

Space forces us to refer to [14] for details. Intuitively, $(\nabla 1)$ eliminates conjunctions, $(\nabla 2)$ distributes disjunctions over the ∇ and $(\nabla 3)$ is congruence. But note that these intuitions are not expressed in standard logical concepts, e.g. $(\nabla 1)$ involves applying T to the map $\bigwedge : \mathcal{P}_\omega\mathcal{M} \longrightarrow \mathcal{M}$ and the congruence rule uses relation lifting instead of simply substituting terms into operation symbols. This can be avoided by moving from $\mathcal{M}_T$ to $\mathcal{K}_T$, as we show in the following.

To emphasize the equational axiomatisation of T we introduce

Notation 6.11 *Given* $(p,a),(q,b) \in \Sigma_n \times X^n$*, we write* $p(a)$ *for* (p,a) *and* $q(b)$ *for* (q,b) [8] *and*

$$p(a) \approx_T q(b) \quad \textit{iff} \quad T(a)(p) = T(b)(q) \quad (\textit{ ie iff } E_X(p,a) = E_X(q,b)\ ).$$

Example 6.12 In the case of the canonical presentation we have.

(i) If $T = 1 + \mathrm{id}$, $p(a) \approx_T q(b)$ iff $p = q = *$ or $a_p = b_q$.

(ii) If $T = \mathcal{P}$, $p(a) \approx_T q(b)$ iff $\{a_i \,|\, i \in p\} = \{q_j \,|\, j \in q\}$.

(iii) If $T = \mathcal{B}_{\mathbb{N}}$, $p(a) \approx_T q(b)$ for $p : n \longrightarrow \mathbb{N}$, $q : m \longrightarrow \mathbb{N}$ iff there is a matrix $(r_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$ such that $r_{ij} = a_i = b_j$ and $\sum_i r_{ij} = q_j$ and $\sum_j r_{ij} = p_i$, see Example 3.6.

The key concept behind $(\nabla 1)$ is that of a redistribution.

Definition 6.13 A *redistribution* of a set $A \subseteq \Sigma X$ is an element $(q,\psi) \in T_\omega(n) \times (\mathcal{Q}X)^n$ such that: for each $(p,a) \in A$ there exists $k \leq n$, $r \in T(k)$, $b : k \longrightarrow X$ and $\varphi : k \longrightarrow \mathcal{Q}X$ such that

$$r(b) \approx_T p(a) \ \wedge\ r(\varphi) \approx_T q(\psi) \ \wedge\ (\forall i)(b_i \in \varphi_i). \tag{12}$$

Let $|A| = \{a_i \mid (p,a) \in A\}$. A redistribution (q,ψ) is *slim* if $n \leq 2^{|A|}$ and $\bigcup_{i \in n} \psi_i \subseteq |A|$. The set of slim redistributions of A is denoted $\Sigma RD(A)$.

'Slim' makes sure that $\Sigma RD(A)$ is finite if A finite. [9] $(\nabla 1)$ now becomes

$$(\Sigma 1) \qquad \bigwedge \{\lambda^p(a) \,|\, (p,a) \in A\} \leq \bigvee \{\lambda^q(\bigwedge \psi) \,|\, (q,\psi) \in \Sigma RD(A)\}.$$

where $\bigwedge \psi$ is short for $(\bigwedge \psi_1 \ldots \bigwedge \psi_n)$.

Remark 6.14 $(\Sigma 1)$ simplifies some, but not all aspects of $(\nabla 1)$. In particular, it does not replace the notion of a redistribution in the sense of [14] by something fundamentally simpler: A ΣRD lives in the upper row of Diagram (10) and has been defined so that it matches the notion from [14] living in the lower row. One way to understand our axiomatisation in general, and $(\Sigma 1)$ and (12) in particular, is as an *implementation* of the axiomatisation in [14]. Indeed, given A as in $(\nabla 1)$ or $(\Sigma 1)$, to apply the axiom we need a join over a sufficiently large set of redistributions of A. (12) tells us how to compute this set using the equational theory $\approx_T$. For such computational purposes, one would not work with the canonical representation but rather a smaller one as eg given for the powerset in Example 6.1.

To translate $(\nabla 2)$ we make

Definition 6.15 A *coredistribution* of an element $(q,\psi) \in \Sigma\mathcal{Q}X$ is an element $(p,a) \in \Sigma X$ satisfying (12) and a injective. The set of coredistributions of (p,ψ) is denoted $CRD(p,\psi)$.

Now $(\nabla 2)$ can be written as follows:

$$(\Sigma 2) \qquad \lambda^p(\bigvee \psi) \leq \bigvee \{\lambda^q(a) \,|\, (q,a) \in CRD(p,\psi)\}.$$

[8] To emphasise that p and q denote operators acting on formulas.

[9] Our notion is derived from the corresponding notion of [14]. $|A|$ is the 'base' of $E_X[A]$ and the cardinality restriction on n derives from the one in [14] plus conditions (1-3) in the proof of the theorem below. If one wants to work with a non-canonical presentation of T_ω, one has to make sure that (1-3) still hold or modify the bound for n. It may also be possible to find better bounds for particular T.

One advantage of our equational axiomatisation is that the rule $(\nabla 3)$ reduces to the standard congruence rule of equational logic. In summary we have:

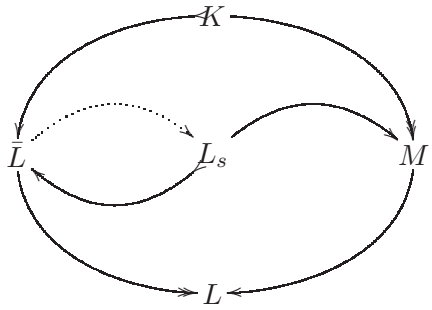
Theorem 6.16 *Let $\langle \Sigma, E \rangle$ be the canonical presentation of T_ω. The derivation system given by the equational logic for Σ and the axioms $\Sigma 1$ and $\Sigma 2$ on top of a complete equational presentation for classical propositional logic is sound and complete for the logic $\mathcal{K}_T$.*

Proof. (Sketch) Let tr be the translation of Moss liftings into the ∇-logic obtained from the translators $E(p, -)$. Since tr is onto, the axioms $\Sigma 1$ and $\Sigma 2$ are translated into instances of the axioms $\nabla 1$ and $\nabla 2$; and vice-versa.

To see this for $(\Sigma 1)$, we use Lemma 6.3 as well as the following observations on standard functors T_ω. (1) For all (p, v) there is $(q, w) \approx_T (p, v)$ with w injective. (2) $\mathrm{Elem}_x = \{(p, v) \mid E(p, v) = x, v \text{ injective } \}$ has an initial element (p, v) in the sense that $\forall (q, w) \in \mathrm{Elem}_x \,.\, \exists f : \mathrm{dom}(v) \longrightarrow \mathrm{dom}(w) \,.\, Tf(p) = q$. (3) If $(p, v) \in Tn \times X^n$ is such an initial element, then n, or more precisely, the image of v, is the *base* (see [14]) of $E(p, v)$. □

7 Conclusion

In this paper we have depicted a general relation between Moss's coalgebraic logic and the logic of all predicate liftings. Working over a Boolean base logic, the one-step translations we discussed are summarised below:



A solid arrow means that the translation works for all T, subject to the proviso that M is only defined if T preserves weak pullbacks. A dotted arrow means that T has to preserve finite sets. K is the logic given by Moss liftings, $\bar{L}$ is the logic given by all predicate liftings and L is obtained by quotienting $\bar{L}$ with a complete axiomatisation, M is Moss's coalgebraic logic. The translations $K \rightarrowtail \bar{L}$, $K \twoheadrightarrow M$, $\bar{L} \twoheadrightarrow L$ are immediate from the respective definitions, the translations $\bar{L} \longrightarrow M$ and $M \longrightarrow L$ are Theorems 5.2 and 5.5, respectively. Double arrowheads indicate that the translation is onto and can be reversed, albeit not necessarily by a natural transformation as choices of representatives are involved. Arrows with tails indicate that the translation is one-to-one.

The diagram above suggests that L is the canonical logic for T-coalgebras. L can be defined for any T, is always complete [18] and it is expressive if T is finitary [28]. If T preserves finite sets, then L and M are equivalent, L has only countably many formulas, and the formulas of L correspond precisely to subsets of the final sequence of T(see [17]). But if T does not preserve finite sets (as eg for the distribution or multi-set functor), it is

		Base Categories		
		BA	DL	BA^κ
Modalities	Moss' Modality	Translatable into the logic of all predicate liftings	Translatable into the logic of Moss liftings using choice	Translatable into the logic of Moss liftings using choice
	Moss liftings	BA-Translatable into Moss' logic	DL-Translatable into Moss' logic	BA^κ-Translatable into Moss' logic
	Singleton liftings	BA-Translatable into Moss' logic	Not translatable into Moss' logic	BA^κ-Translatable into Moss' logic for all $(\mathcal{D}, \mathcal{B}_{\mathbb{N}})$-KPF's. Unknown for other functors
	Predicate liftings	BA-Translatable into Moss' logic if T preserves finite sets	Not translatable into Moss' logic	BA^κ-Translatable into Moss' logic for all $(\mathcal{D}, \mathcal{B}_{\mathbb{N}})$-KPF's, if $\kappa > 2^{\aleph_0}$. Unknown for other functors

Table 1
Comparison Table: Modalities, base categories, and transaltions.

not so clear whether L is the best choice of logic for T in general: On the one hand, L is too expressive as it may have uncountably many formulas, on the other hand it is not maximally expressive in the sense that there may be modal predicates definable by subsets of the final sequence of T that do not correspond to formulas in L.

We also emphasised that these theorems depend on working over BA. This suggests that it would be worth studying *coalgebraic non-classical logic*. In particular, we do not know of a general relation between Moss's modality and predicate liftings if the underlying logic is not classical. A summary of the relations between Moss's modality and the three classes of predicate liftings that we studied with respect to different base logics is presented in Table 1.

The work of Venema on fix points logics suggests that many results on Moss's logic [15,29] will generalise to this new framework, at least for distributive lattices. There is not much work on non-classical logics of predicate liftings. Notice that there is already an issue at the basics, namely, what is the appropriate notion of predicate lifting if we don't work with Boolean algebras? A more technical issue is that the expressivity result in Schröder [28] does not seem to work if we leave classical logic. It is not clear to us how the existence of a separating set of predicate liftings implies the Hennessy-Milner property if the underlying logic does not have negations.

At the purely mathematical level, in this paper, we have developed the concepts of

translator, *logical translator* and *singleton lifting* introduced in [21]. We have shown that every singleton lifting has a translator (Proposition 4.10). Using these and properties of the category of Boolean algebras, we have shown that if the underlying logic is classical then all these translators can be made into BA-translators (Lemma 5.1) and then all singleton liftings are translatable into Moss's logic based on Boolean algebras. We have also shown that classical logic is a necessary requirement to be able to translate, see Examples 4.11 and 5.3. In the other direction, we have provided a compositional translation of Moss's language (Theorem 5.5). As an additional gain we have used our techniques to prove a Lindström Theorem for coalgebraic logics (Theorem 5.6). In Example 5.3, we showed that not all predicate liftings are translatable if T does not preserve finite sets, even if the underlying logic is classical logic. However, it would be interesting to give a general characterization of the predicate liftings that can be translated into Moss's logic and of those that can not be translated. Our conjecture is: A predicate lifting is translatable into Moss's logic iff it can be presented as a finite disjunction of singleton liftings.

Using our translation techniques, previously mentioned, we have developed a complete and sound equational logic for coalgebras (Section 6). We have shown that for every weak pullback preserving functor there exists a set of monotone predicate liftings, namely the set of Moss liftings (page 11), which is as expressive as Moss's coalgebraic logic. This opens the possibility to add fix points to logics of predicate liftings. Notice that we developed our equational logic using the canonical representation of a functor T (Remark 6.2). It seems that our work can be carried out using other more economical representations of T. It might be worth to study equational logics obtained from different representations of T.

Another issue that we have not studied is related to the computable properties of translators and logical translators. We don't know what is the actual computational cost of a translation using logical translators. This might be interesting in the case of an actual implementation of translators.

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