Logical Relations

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Part I

Midlands Graduate School 2012

Introduction

A versatile proof technique for higher-order frameworks

- Definability problems
- Relationships between various kinds of semantics
- Strong normalisability
- Observational equivalence
- $\cdots$

The aim of the course is provide an introduction to logical relations along with some fundamental case studies.

> “This can be proved using the logical relations technique.”

Finitary Relations

Definition 1. *Let* $n \in \mathbb{N}$*.*

R *is called an* n*-ary relation over (sets)* $X_1, \cdots, X_n$ *if*

$$R \subseteq X_1 \times \cdots \times X_n.$$

- We shall write $R(x_1, \cdots, x_n)$ for $(x_1, \cdots, x_n) \in R$.
- Unary relations ($n = 1$) are also known as *predicates* ($R \subseteq X_1$).
- Binary ($n = 2$), ternary ($n = 3$), quaternary ($n = 4$) relations.

Higher-Order Types

Let $Types$ be the collection of types generated by the following grammar.

$$\theta \quad ::= \quad o \quad | \quad \theta \to \theta$$

Let X be a set. Consider the following assignment of sets to types:

- $[\![o]\!] = X$,
- $[\![\theta_1 \to \theta_2]\!] = [\![\theta_1]\!] \Rightarrow [\![\theta_2]\!]$.

Here $\Rightarrow$ stands for the set-theoretic function space.

This yields a family $\{[\![\theta]\!]\}_{\theta \in Types}$ of sets parameterised by $Types$.

Logical Relations

Definition 2. *An n-ary* **logical relation** *is a family $\mathcal{R} = \{R_\theta\}_{\theta \in Types}$ of n-ary relations such that $R_\theta \subseteq \underbrace{[\![\theta]\!] \times \cdots \times [\![\theta]\!]}_{n}$ for any θ and*

$$R_{\theta_1 \to \theta_2}(f_1, \cdots, f_n)$$

$$\Longleftrightarrow$$

for all $(d_1, \cdots, d_n) \in [\![\theta_1]\!]^n$,
if $R_{\theta_1}(d_1, \cdots, d_n)$ then $R_{\theta_2}(f_1(d_1), \cdots, f_n(d_n))$.

A family like this is uniquely determined by $R_o \subseteq [\![o]\!] \times \cdots \times [\![o]\!]$.

Warm-up Exercises

- Let $\{R_\theta\}$ be an n-ary logical relation. What can we say about R_θ in the following cases?
  - $R_o = [\![o]\!]^n$
  - $R_o = \emptyset$
- Let $\{R_\theta\}$ and $\{S_\theta\}$ be logical of the same arity. Are $\{R_\theta \cup S_\theta\}$ and $\{R_\theta \cap S_\theta\}$ logical? Arbitrary sums and intersections?
- Suppose $\{R_\theta\}$ and $\{S_\theta\}$ are binary logical relations. Is $\{R_\theta; S_\theta\}$ logical?

$$(R_1; R_2)(x, y) \iff \exists z.\ (R_1(x, z) \text{ and } R_2(z, y))$$

Many Other Variants

Twists possible:

- more parameters
- changing arities
- other function spaces
- other relational domains (syntactic or semantic)
- $\cdots$

Binding slogan:

Entities related at function types take **related inputs** to **related results**.

Some history

Good ideas occur independently in many places!

- **Early years**: Tait, Howard, Friedman, Milne, Reynolds
- **Formative years**: Plotkin (”logical relations”), Statman (”fundamental theorem”)
- **Subsequent years**:

Jung, Tiuryn, Sieber, O’Hearn, Riecke, Pitts, Reddy,
Benton, Birkedal, Dreyer, Ahmed, Hofmann, Kennedy, ...

Lambda Calculus

Syntax

$$\frac{(x:\theta)\in\Gamma}{\Gamma\vdash x:\theta}\qquad\frac{\Gamma,x:\theta\vdash M:\theta'}{\Gamma\vdash\lambda x^{\theta}.M:\theta\to\theta'}\qquad\frac{\Gamma\vdash M:\theta\to\theta'\quad\Gamma\vdash N:\theta}{\Gamma\vdash MN:\theta'}$$

Semantics

Let $\Gamma=\{x_1:\theta_1,\cdots,x_m:\theta_m\}$. Terms $\Gamma\vdash M:\theta$ can be interpreted by functions $[\![\Gamma\vdash M:\theta]\!]:[\![\theta_1]\!]\times\cdots\times[\![\theta_n]\!]\to[\![\theta]\!]$ according to the following recipe. If $n=0$, the domain degenerates to $\{\star\}$.

Let $\rho=(d_1,\cdots,d_n)\in[\![\theta_1]\!]\times\cdots\times[\![\theta_n]\!]$.

- $[\![\Gamma\vdash x_i]\!](\rho)=d_i$
- $[\![\Gamma\vdash MN]\!](\rho)=([\![\Gamma\vdash M]\!](\rho))([\![\Gamma\vdash N]\!](\rho))$
- $[\![\Gamma\vdash\lambda x^{\theta}.M]\!](\rho)(d)=[\![\Gamma,x:\theta\vdash M]\!](\rho,d)$

Remarks

Closed terms $\vdash M : \theta$ are represented by functions from $\{\star\}$ to $[\![\theta]\!]$, i.e. they are essentially elements of $[\![\theta]\!]$. We shall often regard them as such for succinctness ($[\![\vdash M]\!] \in [\![\theta]\!]$).

Definition 3. *An element of $d \in [\![\theta]\!]$ is* λ-definable *if there exists $\vdash M : \theta$ such that $[\![\vdash M]\!] = d$.*

Definability Problem: can one characterize definable elements mathematically?

This question is not specific to the lambda calculus. By investigating definability in a particular framework for a specific programming language, we discover the mathematical meaning of programs.

Permutation Invariance

The lambda calculus has no means to distinguish between elements of $[\![o]\!]$ (data independence). We might want to capture this through permutation invariance.

Definition 4. *Let* $\pi : [\![o]\!] \to [\![o]\!]$ *be a bijection (permutation). Let* $\{\pi_\theta\}$ *be a family of bijections as follows.*

$$\begin{array}{rcl} \pi_o(d) & = & \pi(d) \\ \pi_{\theta_1 \to \theta_2}(d) & = & \pi_{\theta_2} \circ d \circ \pi_{\theta_1}^{-1} \end{array}$$

Theorem 5 (Laüchli)**.** *For any closed* λ*-term* $\vdash M : \theta$*,*

$$\pi_\theta([\![\vdash M : \theta]\!]) = [\![\vdash M : \theta]\!].$$

How precise is this?

Let $\theta \equiv o \to o$ and $[\![o]\!] = \{T, F\}$. Then $[\![o \to o]\!] = \{T, F\} \Rightarrow \{T, F\}$ contains four elements.

- identity
- negation
- constant T
- constant F

Theorem **??** yields: $[\![M]\!](\pi_o x) = \pi_o([\![M]\!](x))$.

If we take π_o to be the swap (negation) permutation, then the equation above rules out the constant functions as definable. The remaining two functions pass the invariance test. However, only the identity function is definable, so there is scope for improvement!

Permutation Invariance revisited

Definition 6. *Given* $d \in [\![\theta]\!]$ *and a logical relation* $\mathcal{R} = \{R_\theta\}$ *we shall say that* d *satisfies* $\mathcal{R}$ *(or is invariant under* $\mathcal{R}$*) if* $R_\theta(d, \cdots, d)$.

Let us define $R_\theta(x, y)$ by $\pi_\theta(x) = y$.

Exercise: show that $\{R_\theta\}$ is a logical relation.

Then $\pi_\theta([\![\vdash\ M : \theta]\!]) = [\![\vdash\ M : \theta]\!]$ amounts to saying that

$$R_\theta([\![\vdash\ M]\!], [\![\vdash\ M]\!])$$

i.e. that $[\![\vdash\ M]\!]$ satisfies $\{R_\theta\}$.

Mike Gordon suggested investigating invariance with respect to arbitrary relations (as opposed to mere permutation invariance).

Fundamental Property of Logical Relations

Theorem 7 (Plotkin)**.** *Let* $\{R_\theta\}$ *be a logical relation. For any closed* λ*-term* $\vdash M : \theta$*,* $R_\theta([\![\vdash M : \theta]\!], \cdots, [\![\vdash M : \theta]\!])$*.*

Lemma 8. *Let* $\Gamma \vdash M : \theta$*, where* $\Gamma = \{x_1 : \theta_1, \cdots, x_m : \theta_m\}$ *and* $f = [\![\Gamma \vdash M : \theta]\!]$*. Suppose* $\{R_\theta\}$ *is an* n*-ary logical relation and*

$$\rho_i = (d_{i1}, \cdots, d_{im}) \in [\![\theta_1]\!] \times \cdots \times [\![\theta_m]\!] \qquad i = 1, \cdots n$$

are such that $R_{\theta_j}(d_{1j}, \cdots, d_{nj})$ *(*$1 \leq j \leq m$*). Then*

$$R_\theta(f\rho_1, \cdots, f\rho_n).$$

Exercise: Prove Theorem 7 and Lemma 8.

Definability

This makes one wonder whether perhaps the definable elements are exactly those that satisfy all logical relations.

Recall that the negation map $\neg : \{T, F\} \to \{T, F\}$ was permutation-invariant.

$\neg$ is *not* invariant under all logical relations. For example, take R_o to be $\{(T, F)\}$.

- Then $R_o(T, F)$.
- However $R_o(F, T)$ does not hold.
- Consequently $R_{o \to o}(\neg, \neg)$ does not hold.

Type Order

$$\mathsf{ord}(o) = 0 \qquad \mathsf{ord}(\theta_1 \to \theta_2) = \max(\mathsf{ord}(\theta_1) + 1, \mathsf{ord}(\theta_2))$$

Examples

Order	Type
order 0	o
order 1	$o \to \cdots \to o$
order 2	$(o \to \cdots \to o) \to \cdots \to (o \to \cdots \to o) \to o$
order 3	$((o \to o) \to o) \to o$

Definability results

Theorem 9 (Plotkin, Sieber)**.** *Let θ be a type of order at most two. Then $d \in [\![\theta]\!]$ is definable if and only if d is invariant under all logical relations.*

Plotkin showed the result for infinite X, Sieber for finite X.
Note that if X is finite then invariance under all logical relations is decidable (the arity can then be restricted and there are finitely many relations of given arity).

Theorem 10 (Loader)**.** *There exists a finite set X such that the associated definability problem for $[\![\theta]\!]$ is undecidable for third-order types.*

Hence, invariance under logical relations does not characterize λ-definability.

Intuitionistic Logic

$$\phi \to (\theta \to \phi) \qquad (\phi \to (\theta \to \xi)) \to ((\phi \to \theta) \to (\phi \to \xi)) \qquad \frac{\phi \to \theta \quad \phi}{\theta}$$

Definition 11. *A* Kripke model *is a triple* $\langle W, \leq, \Vdash \rangle$ *such that* $(W, \leq)$ *is a preorder,* $\Vdash \subseteq W \times Types$ *and*

- $w \Vdash o$ *implies* $w' \Vdash o$ *for any* $w' \geq w$,
- $w \Vdash \theta_1 \to \theta_2$ *if and only if, for all* $w' \geq w$, *if* $w' \Vdash \theta_1$ *then* $w' \Vdash \theta_2$.

θ *holds in* $\langle W, \leq, \Vdash \rangle$ *if* $w \Vdash \theta$ *for all* $w \in W$.

Kripke models provide a sound and complete semantics for intuitionistic logic: θ is provable if and only if θ holds in all models. Curry-Howard Isomorphism: θ is provable if and only if there exists a λ-term M such that $\vdash M : \theta$.

Kripke Logical Relations

Transfer of Kripke semantics to logical relations.

Definition 12. *Let* $\langle W, \leq \rangle$ *be a preorder. An* n*-ary* **Kripke logical relation** *is a family of* n*-ary relations* $\mathcal{R}^W = \{R^w_\theta\}_{w,\theta}$ *such that*

- *if* $w \leq w'$ *then* $R^w_\theta \subseteq R^{w'}_\theta$,
- $R^w_{\theta_1 \to \theta_2}(f_1, \cdots, f_n)$ *if and only if, for all* $w' \geq w$ *and* $(d_1, \cdots, d_n) \in [\![\theta_1]\!]^n$, *if* $R^{w'}_{\theta_1}(d_1, \cdots, d_n)$ *then* $R^{w'}_{\theta_2}(f_1(d_1), \cdots, f_n(d_n))$.

$d \in [\![\theta]\!]$ *satisfies* $\mathcal{R}^W$ *provided* $R^w(d, \cdots, d)$ *for all* $w \in W$.

Note that the framework reduces to the previous one for $W = \{\star\}$.

Definability Revisited

Theorem 13 (Plotkin)**.** *For any λ-term $\vdash M$, $[\![\vdash M : \theta]\!]$ satisfies any Kripke logical relation.*

Theorem 14 (Plotkin)**.** *Let X be infinite. Then $d \in [\![\theta]\!]$ is definable if and only if d satisfies any (ternary) Kripke logical relation.*

By Loader's result, in the finitary case (X finite) Kripke logical relations do not capture definability either.

Thus far all constituent relations of a logical relation had the same fixed (finite) arity. It is also possible to consider fixed infinite arities. In contrast, the next definition will allow arities to vary.

Kripke Logical Relations with Changing Arity

Definition 15. *Let $\mathcal{C}$ be a small category of sets and functions. $\{R_\theta^w\}_{\theta \in Types}^{w \in Ob\,\mathcal{C}}$ is a* **Kripke logical relation with varying arity** *if*

- $R_\theta^w \subseteq [\![\theta]\!]^{|w|}$ *(i.e. the arity of R_θ^w is $|w|$),*
- *for all $f : v \to w$, if $R_o^w(\langle\, d_j \,\rangle_{j\in w})$ then $R_o^v(\langle\, d_{f(i)} \,\rangle_{i \in v})$,*
- $R_{\theta_1 \to \theta_2}^w(\langle\, f_j \,\rangle_{j \in w})$ *if and only if, for all*

$$g : v \to w \quad \text{and} \quad \langle\, d_i \,\rangle_{i\in v} \in [\![\theta_1]\!]^{|v|},$$

$R_{\theta_1}^v(\langle\, d_i \,\rangle_{i \in v})$ *implies* $R_{\theta_2}^v(\langle\, f_{g(i)}(d_i) \,\rangle_{i\in v})$.

Observe that this generalises the previous frameworks.

Fundamental Theorem

Exercise: Let $\{R^w_\theta\}$ be a Kripke logical relation with varying arity. Then, for all $f : v \to w$, if $R^w_\theta(\langle\, d_j \,\rangle_{j\in w})$ then $R^v_\theta(\langle\, d_{f(i)} \,\rangle_{i\in v})$.

Theorem 16 (Jung,Tiuryn)**.** *For every Kripke logical relation with varying arity* $\{R^w_\theta\}^w_\theta$*, any* $w \in Ob(\mathcal{C})$ *and any closed term* $\vdash M : \theta$*,*

$$R^w_\theta(\langle\, [\![\vdash M]\!] \,\rangle_{j\in w}).$$

This begs the question whether $d \in [\![\theta]\!]$ is definable if and only if it is invariant under all Kripke logical relations with varying arity for any $\mathcal{C}$.

A special category $\mathcal{C}$

- Objects: $\prod_{i=1}^{k} [\![\theta_i]\!]$ (special case $\{\star\}$ for $k = 0$)
- Morphisms: projections $\prod_{i=1}^{k} [\![\theta_i]\!] \times \prod_{i=1}^{k'} [\![\theta'_i]\!] \to \prod_{i=1}^{k} [\![\theta_i]\!]$

Define $\{T_\theta^w\}$ as follows.

- Let $w = \prod_{i=1}^{k} [\![\theta_i]\!]$. Recall that T_o^w must have arity $|w|$.
- Set

$$T_o^w(\langle\, r_{(d_1,\cdots,d_k)} \,\rangle_{(d_1,\cdots,d_k)\in w})$$

$$\Longleftrightarrow$$

there exists $\vdash\ M : \theta_1 \to \ldots \to \theta_k \to o$ such that

$$r_{(d_1,\cdots,d_k)} = [\![\vdash\ M]\!](d_1)\cdots(d_k).$$

Definability

Theorem 17 (Jung, Tiuryn). $d \in [\![\theta]\!]$ *is definable if and only if* $T_\theta^{\{\star\}}(d)$.

This result preceded Loader's undecidability result for λ-definability and raised hopes that the problem might be decidable. In more recent work, Joly gave a new proof of the result and a classification of decidable cases.

- The problem is known to be undecidable if $[\![o]\!]$ is a two-element set.
- It is also known at which types the problem is undecidable ($|[\![o]\!]|$ is part of the input in this case).

Theme: Use of logical relations to capture λ-definability.

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