Logical Relations

Andrzej Murawski

University of Leicester

Part III

Midlands Graduate School 2012

Definition 1. *An n-ary* **logical relation** *is a family $\mathcal{R} = \{R_\theta\}_{\theta \in Types}$ of n-ary relations such that $R_\theta \subseteq \underbrace{[\![\theta]\!] \times \cdots \times [\![\theta]\!]}_{n}$ for any θ and*

$$R_{\theta_1 \to \theta_2}(f_1, \cdots, f_n)$$

$$\iff$$

for all $(d_1, \cdots, d_n) \in [\![\theta_1]\!]^n$,

if $R_{\theta_1}(d_1, \cdots, d_n)$ then $R_{\theta_2}(f_1(d_1), \cdots, f_n(d_n))$.

Theorem 2. *Let $\{R_\theta\}$ be a logical relation. For any closed λ-term $\vdash M : \theta$, $R_\theta([\![\vdash M : \theta]\!], \cdots, [\![\vdash M : \theta]\!])$.*

Types and Constants

Let us start with a collection B of base types. Let o range over B.

Types

$$\theta \quad ::= \quad o \quad | \quad \theta \to \theta$$

Let us also assume a set of typed constants C

Constants

$$c : \theta_c$$

Applicative Structures

A general setting for interpreting λ-terms.

Definition 3. *A (typed)* **applicative structure** $\mathcal{A}$ *is a triple*

$$\langle\, \{A_\theta\}, \{App_{\theta_1,\theta_2}\}, Const\, \rangle$$

such that

- A_θ *is a set,*
- App_{θ_1,θ_2} *is a function* $App_{\theta_1,\theta_2} : A_{\theta_1 \to \theta_2} \to A_{\theta_1} \to A_{\theta_2}$*,*
- $Const : C \to \bigcup_{\theta \in Types} A_\theta$ *satisfies* $Const(c) \in A_{\theta_c}$ *if* $c : \theta_c$*.*

Examples

- $\mathcal{A} = \langle \{A_\theta\}, \{App_{\theta_1,\theta_2}\}, Const \rangle$, where each A_θ is a set and

$$\begin{aligned} A_{\theta_1 \to \theta_2} &= \text{the set of functions from } A_{\theta_1} \text{ to } A_{\theta_2}, \\ App_{\theta_1,\theta_2} f x &= f(x). \end{aligned}$$

- $\mathcal{A} = \langle \{A_\theta\}, \{App_{\theta_1,\theta_2}\}, Const \rangle$, where each A_θ is a cpo and

$$\begin{aligned} A_{\theta_1 \to \theta_2} &= \text{the cpo of continuous functions from } A_{\theta_1} \text{ to } A_{\theta_2}, \\ App_{\theta_1,\theta_2} f x &= f(x). \end{aligned}$$

- $\mathcal{T} = \langle \{T_\theta\}, \{App_{\theta_1,\theta_2}\}, Const \rangle$, where T_θ is the set of simply-typed λ-terms M such that $\Gamma \vdash M : \theta$ for some finite $\Gamma \subseteq \mathcal{V}$, where $\mathcal{V}$ is a set of typed variables, and

$$\begin{aligned} App_{\theta_1,\theta_2} M N &= M N, \\ Const(c) &= c. \end{aligned}$$

Logical Relation

Definition 4. *Let*

$$\mathcal{A} = \langle \{A_\theta\}, \{App^{\mathcal{A}}_{\theta_1,\theta_2}\}, \{Const^{\mathcal{A}}\} \rangle,$$
$$\mathcal{B} = \langle \{B_\theta\}, \{App^{\mathcal{B}}_{\theta_1,\theta_2}\}, \{Const^{\mathcal{B}}\} \rangle$$

be applicative structures. A (binary) **logical relation** *over $\mathcal{A}$ and $\mathcal{B}$ is a family $\mathcal{R} = \{R_\theta\}$ such that*

- $R_\theta \subseteq A_\theta \times B_\theta$,
- $R_{\theta_1 \to \theta_2}(f, g)$ *iff, for all* $(x, y) \in A_{\theta_1} \times B_{\theta_1}$, *if* $R_{\theta_1}(x, y)$ *then* $R_{\theta_2}(App^{\mathcal{A}}_{\theta_1,\theta_2} f x, App^{\mathcal{A}}_{\theta_1,\theta_2} g y)$,
- $R_{\theta_c}(Const_{\mathcal{A}}(c), Const_{\mathcal{B}}(c))$ *for every constant* $c : \theta_c$.

Environments

Definition 5. *Let $\mathcal{V}$ be the set of variables.*

- *An environment is a function $\rho : \mathcal{V} \to \bigcup_\theta A_\theta$.*
- *If Γ is a context (finite type assignment), we say that ρ satisfies Γ (written $\rho \models \Gamma$) if $\rho(x) \in A_\theta$ whenever $(x : \theta) \in \Gamma$.*
- *$\rho[x \mapsto d]$ stands for the environment mapping x to d, and y to $\rho(y)$ for y different from x.*

Definition 6. *Let $\mathcal{A}, \mathcal{B}$ be applicative structures, Γ a context and $\rho_\mathcal{A}, \rho_\mathcal{B}$ be environments satisfying Γ. Let $\mathcal{R}$ be a logical relation over $\mathcal{A}$ and $\mathcal{B}$. $\rho_\mathcal{A}, \rho_\mathcal{B}$ are* **related** *if $R_\theta(\rho_\mathcal{A}(x), \rho_\mathcal{B}(x))$ for all $(x : \theta) \in \Gamma$.*

Interpretation

Definition 7. *A partial mapping* $[\![\cdots]\!]_{\mathcal{A}}$ *from terms and environments* *(*$[\![\Gamma \vdash M : \theta]\!](\rho)$*) is an* **acceptable meaning function** *if*

$$[\![\Gamma \vdash M : \theta]\!](\rho) \in A_\theta \text{ whenever } \rho \models \Gamma$$

and the following conditions are satisfied.

$$\begin{aligned}
[\![\Gamma \vdash x : \theta]\!]_{\mathcal{A}}(\rho) &= \rho(x) \\
[\![\Gamma \vdash c : \theta_c]\!]_{\mathcal{A}}(\rho) &= Const(c) \\
[\![\Gamma \vdash MN : \theta_2]\!]_{\mathcal{A}}(\rho) &= (App_{\theta_1,\theta_2}([\![\Gamma \vdash M]\!]_{\mathcal{A}}))([\![\Gamma \vdash N]\!]_{\mathcal{A}})
\end{aligned}$$

Examples

- Interpretations using sets/functions and cpo's/continuous functions.
- Recall the applicative structure $\mathcal{T}$ based on λ-terms. T_θ is the set of simply-typed λ-terms M such that $\Gamma \vdash M : \theta$ for some finite $\Gamma \subseteq \mathcal{V}$.

$$\begin{aligned} App_{\theta_1,\theta_2} M N &= M N \\ Const(c) &= c \end{aligned}$$

Consider

$$[\![\Gamma \vdash M]\!]_{\mathcal{A}}(\rho) = M[\overline{\rho(x)/x}].$$

Note that

$$(M N)[Q/x] \equiv M[Q/x]\, N[Q/x].$$

Fundamental Theorem

Theorem 8 (Mitchell)**.** *Let* $\mathcal{A}, \mathcal{B}$ *be applicative structures, let* $[\![\cdots]\!]_{\mathcal{A}}, [\![\cdots]\!]_{\mathcal{B}}$ *be acceptable meaning functions, let* $\mathcal{R}$ *be a logical relation over* $\mathcal{A}$ *and* $\mathcal{B}$*. Suppose* $\rho_{\mathcal{A}}, \rho_{\mathcal{B}}$ *are related environments satisfying* Γ*. Then*

$$R_\theta([\![\Gamma \vdash M]\!]_{\mathcal{A}}(\rho_{\mathcal{A}}), [\![\Gamma \vdash M]\!]_{\mathcal{B}}(\rho_{\mathcal{B}}))$$

for every $\Gamma \vdash M : \theta$*.*

Proof by structural induction. Not yet! More constraints are needed.

Admissible Relations

Definition 9. *Let* $\mathcal{A}, \mathcal{B}$ *be applicative structures, let* $[\![\cdots]\!]_{\mathcal{A}}, [\![\cdots]\!]_{\mathcal{B}}$ *be acceptable meaning functions, let* $\mathcal{R}$ *be a logical relation over* $\mathcal{A}$ *and* $\mathcal{B}$*. Suppose* $\rho_{\mathcal{A}}, \rho_{\mathcal{B}}$ *are related environments satisfying* Γ*.* $\mathcal{R}$ *is called* ***admissible*** *if, for all* $\Gamma, x : \tau \vdash M : \theta$ *and* $\Gamma, x : \tau \vdash N : \theta$*.*

$$\begin{array}{ll} \forall_{a,b} & \text{if } R_\tau(a,b) \text{ then} \\ & R_\theta(\quad [\![\Gamma, x \vdash M]\!]_{\mathcal{A}}\,(\rho_{\mathcal{A}}[x \mapsto a]), \\ & \qquad\quad [\![\Gamma, x \vdash N]\!]_{\mathcal{B}}\,(\rho_{\mathcal{B}}[x \mapsto b])\ ) \end{array}$$

implies

$$\begin{array}{ll} \forall_{a,b} & \text{if } R_\tau(a,b) \text{ then} \\ & R_\theta(\quad App^{\mathcal{A}}\,([\![\Gamma \vdash \lambda x.M]\!]_{\mathcal{A}}\,(\rho_{\mathcal{A}}))\,a, \\ & \qquad\quad App^{\mathcal{B}}\,([\![\Gamma \vdash \lambda x.N]\!]_{\mathcal{B}}\,(\rho_{\mathcal{B}}))\,b\ ). \end{array}$$

Logical predicates

Definition 10. *Let* $\mathcal{A} = \langle \{A_\theta\}, \{App^{\mathcal{A}}_{\theta_1,\theta_2}\}, \{Const^{\mathcal{A}}\} \rangle$ *be an applicative structure. A* **logical predicate** *over* $\mathcal{A}$ *is a family* $\mathcal{R} = \{R_\theta\}$ *such that*

- $R_\theta \subseteq A_\theta$,
- $R_{\theta_1 \to \theta_2}(f)$ *iff, for all* $x \in A_{\theta_1}$*, if* $R_{\theta_1}(x)$ *then* $R_{\theta_2}(App^{\mathcal{A}}_{\theta_1,\theta_2} f x)$,
- $R_{\theta_c}(Const_{\mathcal{A}}(c))$ *for every constant* $c : \theta_c$.

Theorem 11 (Mitchell)**.** *Let* $\mathcal{A}$ *be an applicative structure, let* $[\![\cdots]\!]_{\mathcal{A}}$ *be an acceptable meaning function, let* $\mathcal{R}$ *be a logical predicate over* $\mathcal{A}$*. Suppose* $\rho_{\mathcal{A}}$ *satisfies* Γ*. Then* $R_\theta([\![\Gamma \vdash M]\!]_{\mathcal{A}}(\rho_{\mathcal{A}}))$ *for every* $\Gamma \vdash M : \theta$.

Strong Normalisability

Let us write $SN(M)$ for "M is strongly normalising".

Theorem 12 (Tait)**.** *Every typable λ-term is strongly normalising.*

Let us prove the result through the Fundamental Theorem.

1. Define a logical predicate $\mathcal{P} = \{P_\theta\}$ on $\mathcal{T}$.
2. Show that $P_\theta(M)$ implies $SN(M)$.
3. Show that $\mathcal{P}$ is admissible.

By 1. and 3. we can apply the Fundamental Theorem to deduce that $P_\theta(M)$ for any $M \in T_\theta$. By 2. $SN(M)$ holds for any M.

Finding $\mathcal{P}$

$$
\begin{aligned}
P_o(M) &\iff SN(M) \\
P_{\theta_1 \to \theta_2}(M) &\iff P_{\theta_2}(MN) \text{ for all } N \in T_{\theta_1} \text{ such that } P_{\theta_1}(N)
\end{aligned}
$$

Strong normalisability is a consequence (point 2.)

Lemma 13.

(i) If $xM_1 \cdots M_k \in T_\theta$ *and* $SN(M_1), \cdots, SN(M_k)$ *then* $P_\theta(xM_1 \cdots M_k)$.

(ii) If $P_\theta(M)$ *then* $SN(M)$.

Proving 2.

(i) If $xM_1 \cdots M_k \in T_\theta$ and $SN(M_i)$ then $P_\theta(xM_1 \cdots M_k)$.

(ii) If $P_\theta(M)$ then $SN(M)$.

Case $\theta \equiv o$.

(i) Because $SN(M_i)$ for $1 \leq i \leq k$, we also have $SN(xM_1 \cdots M_k)$. Hence, $P_o(xM_1 \cdots M_k)$ by definition of P_o.

(ii) Follows from the definition of P_o.

Proving 2. (ii)

(i) If $xM_1 \cdots M_k \in T_\theta$ and $SN(M_i)$ then $P_\theta(xM_1 \cdots M_k)$.

(ii) If $P_\theta(M)$ then $SN(M)$.

Case $\theta \equiv \theta_1 \to \theta_2$.

(i) Take $N \in T_{\theta_1}$ such that $P_{\theta_1}(N)$. By (ii) for θ_1 we have $SN(N)$ and, by (i) for θ_2, we get $P_{\theta_2}(xM_1 \cdots M_k N)$. So $P_{\theta_1 \to \theta_2}(xM_1 \cdots M_k)$, as required.

(ii) Suppose $P_{\theta_1 \to \theta_2}(M)$. By (i) for $x : \theta_1$, $P_{\theta_1}(x)$, so $P_{\theta_2}(Mx)$. By (ii) for θ_2, $SN(Mx)$. If $SN(Mx)$ then $SN(M)$.

Admissibility

Lemma 14. *Suppose* $M[N/x]N_1 \cdots N_k \in T_o$. *If* $SN(N)$ *and* $SN(M[N/x]N_1 \cdots N_k)$ *then* $SN((\lambda x.M)NN_1 \cdots N_k)$.

Since $SN(M[N/x]N_1 \cdots N_k)$, we have $SN(M), SN(N_1), \cdots, SN(N_k)$. Suppose $(\lambda x.M)NN_1 \cdots N_k$ is not strongly normalising. Then the λ must be reduced at some point.

$$(\lambda x.M)NN_1 \cdots N_k \rightarrow^*_{\beta\eta} (\lambda x.M')N'N'_1 \cdots N'_k \rightarrow_{\beta\eta} Q$$

- β-reduction: $Q \equiv M'[N'/x]N'_1 \cdots N'_k$. Then we also have $M[N/x]N_1 \cdots N_k \rightarrow^*_{\beta\eta} M'[N'/x]N'_1 \cdots N'_k$, which contradicts $SN(M[N/x]N_1 \cdots N_k)$.
- η-reduction: $M' \equiv M''x$ (x does not occur in M'') and $Q \equiv M'N'N'_1 \cdots N'_k$. Then Q can also be reached via a β-reduction, so this case reduces to the one above.

Adequacy for PCF and the cpo interpretation

For $\vdash M : \mathbf{nat}$, if $[\![M]\!] = n$ then $M \Downarrow n$.

$$R_\theta \subseteq [\![\theta]\!] \times \text{PCF}_\theta$$

$$\begin{array}{rcl} R_{\mathbf{nat}}(d, M) & \Longleftrightarrow & \text{if } d = n \text{ then } M \Downarrow n \\ R_{\theta_1 \to \theta_2}(d, M) & \Longleftrightarrow & \forall_{d_1, M_1} (R_{\theta_1}(d_1, M_1) \Rightarrow R_{\theta_2}(d d_1, M M_1)) \end{array}$$

The Fundamental Theorem yields $R_\theta([\![M]\!]_{\mathcal{A}}, [\![M]\!]_{\mathcal{B}})$ for closed terms.
This amounts to $R_\theta([\![M]\!], M)$, which is exactly the Adequacy result.

Admissibility needs to proved, but it is not too difficult.

References

More details can be found in [1, 2].

[1] C. A. Gunter. *Semantics of Programming Languages: Structures and Techniques*. MIT Press, 1992.

[2] J. C. Mitchell. *Foundations for Programming Languages*. MIT Press, 2000.