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FROM FINITE TO INFINITE LAMBDA CALCULI

Abstract

In a previous paper we have established the theory of transfinite reduction for orthogonal term rewriting systems. In this paper we perform the same task for the lambda calculus. This results in several new Böhm models of the lambda calculus, and new unifying descriptions of existing models.

1. Introduction

In this note we outline our theory of infinite rewriting for lambda calculus. A more detailed version will appear in the forthcoming proceedings of RTA'95 [6].

Infinitely long rewrite sequences of possibly infinite terms are of interest for several reasons.

- First, it enriches finitary rewriting with the natural notion of computing towards some limit. If such a (possibly infinite) limit still contains redexes, then one can continue computing. The question of the computational meaning of such transfinite sequences will be dealt with below in the discussion of the Compressing Property.
- Secondly, in functional programming real computations with terms implemented as graphs allow the possibility of using cyclic graphs, which correspond in a natural way to infinite terms. Finite computations on cyclic graphs correspond to infinite computations on terms.
- Finally, the infinitary theory suggests new ways of dealing with some of the concepts that arise in the finitary theory, such as notions of undefinedness of terms. A very interesting example of such application can be found in recent work by Berarducci and Intrigila, who

independently define a notion of Böhm reduction based on so called *mute* terms to solve some long standing open problems about the *easy*-ness of particular lambda terms [4], [5].

Our papers [7], [1] deal with these issues for term rewriting. We assume familiarity with the lambda calculus, or as we shall refer to it there, the *finitary* lambda calculus. [3] is a standard reference.

2. Depth as a variable

We can think of lambda terms as trees. The various reasonable alternatives to assign "depth" to nodes in the lambda tree representation of a lambda term lead to different infinitary extensions of the finite lambda calculus.

DEFINITION 1. Given a term M and a position u of M, the depth of the subterm of M at u, denoted by $D^{abc}(M,u)$, if it exists, is defined by

$$\begin{aligned} D^{abc}(M,u) &= 0 \\ D^{abc}(\lambda x.M, 1\cdot u) &= a + D^{abc}(M,u) \\ D^{abc}(MN, 1\cdot u) &= b + D^{abc}(M,u) \\ D^{abc}(MN, 2\cdot u) &= c + D^{abc}(N,u) \end{aligned}$$

where $a, b, c \in \{0,1\}$.

Depending on the chosen concept D^{abc} of depth, distance between terms can now be defined: the distance $d^{abc}(s,t)$ of two terms t,s is 0 if t and s are identical, otherwise 2^{-d} where d is the largest depth such that s,t are identically labeled at all nodes at depth at most d. We will denote the completion of the set of finite lambda terms with the metric $d^{abc}(s,t)$ by Λ^{abc}. We will write Λ^{∞} and d when we do not specify the depth we base our notions. Clearly d^{000} is the discrete metric, hence Λ^{000} is the set of finite lambda terms itself.

We will say that a term $t \in \Lambda^{\infty}$ is a *zero-stable form*, for short *0-stable*, if t cannot reduce to a term with a redex at depth 0. For d^{111}, d^{101}, d^{001} the concept zero-stable form recaptures for finite terms: root-stable form (cf. [1] for our analogous concept in term rewriting, also called mute term by Berarducci and Intrigila [5]), whnf, hnf and nf, respectively.

THEOREM 1. *A term in Λ^∞ has no reduction to zero-stable form if and only if it has an infinite reduction which contains infinitely many reductions at depth 0.*

For finite terms the previous theorem has been proved by Wadsworth in case of hnfs (cf. [3]) versus head reductions, by Abramsky and Ong in case of whnfs versus lazy reduction in [2] and by Berarducci for rootstable terms in [4]. If we allow infinite terms, the above theorem remains true, provided we extend (finite) reduction to (possibly infinite) strongly convergent reduction, a concept which we defined in [7] for orthogonal term rewriting.

DEFINITION 2. A *pre-reduction* sequence of length α is a function ϕ from an ordinal α to reduction steps of Λ^∞, and a function τ from $\alpha + 1$ to terms of Λ^∞, such that if $\phi(\beta)$ is $a \to^r b$ then $a = \tau(\beta)$ and $b = \tau(\beta + 1)$. Note that in a pre-reduction sequence, there need be no relation between the term $\phi(\beta)$ and any of its predeccors when β is a limit ordinal.

It is a *strongly convergent reduction sequence* if it is Cauchy convergent and if, for every limit ordinal $\lambda \leq \alpha$, $\lim_{\beta \to \lambda} d_\beta = \infty$, where d_β is the depth of the redex reduces by the step $\phi(\beta)$.

If α is a limit ordinal, then an *open* pre-reduction sequence is defined as above, except that the domain of τ is α. If τ is continuous, the sequence is *Cauchy continuous*, and if the condition of strong convergence is satisfied at each limit ordinal less than α, it is *strongly continuous*.

When we speak of a reduction sequence, we will mean a strongly convergent reduction sequence unless otherwise stated.

LEMMA 1. *Let t be a term in Λ^∞. Any Cauchy converging reduction from t to zero-stable form is a strongly converging reduction.*

Another computional relevant reason to prefer strongly converging reductions is that the Compression Property holds for strongly convergent reductions. The Compression Property says that for every reduction sequence from a term s to a term t, there is a reduction sequence from s to t of length at most ω. It is not difficult to find one counterexample for all Λ^∞ for Cauchy converging reductions.

A third reason is that in the limit of a sequence loses the descendantship relation between subterms of different terms in the sequence.

Consider infinite reduction sequence: $I^\omega \to I^\omega \to I^\omega \to \ldots$ which at each stage reduces the outermost redex. The limit is I^ω. It is not possible to say that any redex in the limit term arises from any of the redexes in the previous terms in the sequence.

LEMMA 2. *Λ^{abc} is the subset of Λ^{111} consisting exactly of those terms which do not contain an infinite sequence of nodes in which each node is at the same abc-depth as its parent.*

Λ^{111} contains the term $((((\ldots)I)I)I)I$. This term has a combination of properties which is rather strange from the point of view of finitary lambda calculus. By the usual definition of head normal form — being of the form $\lambda x_1 \ldots \lambda x_n.yt_1 \ldots t_m$ — it is not in head normal form. By an alternative (trivially equivalent in the finitary case) formulation —it is in head normal form — it is in head normal form — it has no head redex. It is also a normal form, yet it is unsolvable (that is, there are no terms $N_1, \ldots, N_n$ such that $MN_1 \ldots N_n$ reduces to $\lambda x.x$). The problem is that application is strict in its first argument, and so an infinitely left-branching chain of applications has no obvious meaning. We can say much the same for an infinite chain of abstractions $\lambda x_1.\lambda x_2.\lambda x_3.\ldots$ Therefore, Λ^{001} seems to be the natural infinite extension of usual *eager* finite lambda calculus. Λ^{101} is an infinite extension associated with the lazy lambda calculus [2] in which abstraction is considered lazy — $\lambda x.M$ is meaningful even when M is not. Both Λ^{111} and Λ^{101} contain unsolvable normal forms, such as $\lambda x_1.\lambda x_2.\lambda x_3 \ldots$ In Λ^{001} every normal form is solvable. We will later find that Λ^{010} and Λ^{011} — have unsatisfactory technical properties.

3. Results

In this section we will consider an arbitrary infinite lambda calculus Λ^∞, unless otherwise specified. For strongly convergent reductions we can generalize the concepts of descendants and contribution (which occurrences in the initial term contribute to a redex in the final term of a reduction).

THEOREM 2.

1. *For any strongly convergent sequence $t_0 \to^\alpha t_\alpha$ and any position u of t_α, the set of all positions of all terms in the sequence which contribute*

to u is finite, and the set of all reduction steps contributing to u is finite.

2. *If $t \to^\infty s$ and s' is a finite prefix of s, then t is reducible in finitely many steps to a term having s' as a prefix. In particular, if t is reducible to a finite term, it is reducible to that term in finitely many steps.*
3. *If a finite term is reducible to a finite normal form, it is reducible to that normal form in the finitary lambda calculus.*

Complete developments can also be naturally extended to the extended concept of reduction. However, in the Λ^∞ other than Λ^{000} not every set of redexes has a complete development. E.g. an example is provided by the term $I^\omega = I(I(I(\ldots)))$ in Λ^{ab1} Every attempt to reduce all the redexes in this term must give a reduction sequence containing infinitely many reduction steps at the root of the term, which, by every notion of $ab1$-depth, is not strongly convergent. Yet, we can prove:

THEOREM 3. *Complete developments of the same set of redexes end at the same term.*

THEOREM 4. *(Compressing Property[1]) In Λ^∞, for every strongly convergent sequence there is a strongly convergent sequence with the same endpoints whose length is at most ω.*

In the context of infinitary lambda calculus, the Böhm tree of a term can be seen as being simply its normal form with respect to transfinite reduction with respect to the β rule together with an additional rule for erasing subterms having no head normal form. More generally, we find that with each notion of depth there can naturally be associated a notion of head normal form.

DEFINITION 3. A term of Λ^∞ is *potentially* 0-stable if it can be reduced to a 0-stable term. It is *0-active* if it is not potentially 0-stable.

We shall demonstrate that for each notion of depth, the class of 0-active terms satisfies the axioms of [1] for a set of undefined terms. These axioms are (1) both the set and its complement are closed under reduction,

[1]The Compressing Property is false for $\beta\eta$-reduction.

and (2) the set includes all the terms which cannot be reduced to root-stable form (i.e. to a term which cannot be reduced to a redex). This immediately gives rise to models of lambda-calculus.

The second of the axioms is immediate from the definition. If a term cannot be reduced to root-stable form, then it cannot be reduced to a 0-stable form, since a redex at the root of a term is at depth 0 for every notion of depth.

Half of the first axiom is immediate: the set of 0-active terms is certainly closed under reduction. It only remains to prove that the set of potentially 0-stable terms is also closed. To do this we must develop some theory of Böhm reduction. For this it is convenient to extend the various calculi with a fresh symbol $\bot$.

Definition 4.

1. A $\Lambda_\bot$ term is a term of the version of lambda calculus obtained by adding $\bot$ as a new symbol. $\Lambda^\infty_\bot$ is defined from $\Lambda_\bot$ as Λ^∞ is from Λ.
2. *Böhm reduction* is reduction in $\Lambda^\infty_\bot$ by the β rule and the $\bot$ rule, viz. $M \to \bot$ if M is 0-active and not $\bot$. We write $\to_{\mathcal{B}}$ for Böhm reduction and $\to_\bot$ for reduction by the $\bot$-rule alone. A *Böhm tree* is a normal form of $\Lambda^\infty_\bot$ with respect to Böhm reduction.

We extend the notions of 0-stability etc. to terms containing $\bot$ thus. A term of $\Lambda^\infty_\bot$ is 0-stable if it cannot be reduced to a term containing a Böhm redex at depth 0 or $\bot$. Potential 0-stability and 0-activeness are similarly extended.

0-stability and 0-activeness were defined in terms of reduction, but now we have defined a new notion of reduction in terms of these concepts, which in turn gives us new notions of 0-stability and 0-activeness. It is important to check that the new notions agree with the old on terms of Λ^∞. This turns out not to be the case for two of the eight possible notions of depth, which we regard as sufficient grounds for excluding them. The rest of this section deals only with depth measures to which Theorem 5 applies.

Theorem 5.

1. *A term of Λ^∞ is 0-stable with respect to beta reduction if and only if it is 0-stable with respect to Böhm reduction.*
2. *Except in Λ^{010} and Λ^{011}, a term of Λ^∞ is potentially 0-stable with*

respect to beta reduction if and only if it is potentially 0-stable with respect to Böhm reduction.

These theorems allow us to speak of (potential) 0-stability and 0-activeness w.r.t. to beta reduction or Böhm reduction interchangeably.

LEMMA 3.

1. *In $\Lambda^\infty_\perp$, for any Böhm reduction sequence $t \to^\infty_{\mathcal{B}} t'$, there are sequences $t' \to^\infty_\perp t''$ and $t \to^\infty_{\mathcal{B}} t''$, such that the latter sequence consists of alternating segments in which first a complete development is performed of a set of beta redexes, none of which is contained in any 0-active subterm.*
2. *For any Böhm reduction sequence in $\Lambda^\infty_\perp$, $t \to^\infty_{\mathcal{B}} t'$, there is a sequence $t \to^\infty_\beta t'' \to^\infty_\perp t'$.*

THEOREM 6. *In $\Lambda^\infty_\perp$, Böhm reduction is Church-Rosser.*

PROOF (outline). Given two coinitial Böhm reduction sequences, we transform them as described by Lemma 3. For sequences of that form, the Church-Rosser property can be proved by a tiling argument analogous to that commonly used in proving the finitary Church-Rosser property. From this the Church-Rosser property for arbitrary Böhm reductions follows.

COROLLARY. *The set of potentially 0-stable terms in Λ^∞,is closed under reduction.*

THEOREM 7.

1. *In $\Lambda^\infty_\perp$, every term has exactly one Böhm normal form.*
2. *In $\Lambda^\infty_\perp$, beta reduction is Church-Rosser up to identification of 0-active terms.*
3. *Λ^∞ has the unique normal form property*

We thus have a model of lambda calculus, where the objects are the Böhm normal forms. The usual Böhm model is the model associated with applicative depth. The larger model described by Berarducci in [4] is the one associated with d^{111}. The model for d^{001} is the usual one based on solvable terms, and the model for d^{101} is related to the Böhm model for the lazy lambda calculus [2].

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