Weakening the Axiom of Overlap in Infinitary Lambda Calculus

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Abstract

In this paper we present a set of necessary and sufficient conditions on a set of lambda terms to serve as the set of meaningless terms in an infinitary bottom extension of lambda calculus. So far only a set of sufficient conditions was known for choosing a suitable set of meaningless terms to make this construction produce confluent extensions. The conditions covered the three main known examples of sets of meaningless terms. However, the much later construction of many more examples of sets of meaningless terms satisfying the sufficient conditions renewed the interest in the necessity question and led us to reconsider the old conditions.

The key idea in this paper is an alternative solution for solving the overlap between beta reduction and bottom reduction. This allows us to reformulate the Axiom of Overlap, which now determines together with the other conditions a larger class of sets of meaningless terms. We show that the reformulated conditions are not only sufficient but also necessary for obtaining a confluent and normalizing infinitary lambda beta bottom calculus. As an interesting consequence of the necessity proof we obtain for infinitary lambda calculus with beta and bot reduction that confluence implies normalization.

1998 ACM Subject Classification F.4.1 Mathematical Logic (F.1.1, I.2.2, I.2.3, I.2.4)

Keywords and phrases Infinitary Lambda Calculus, Confluence, Normalization

Digital Object Identifier 10.4230/LIPIcs.RTA.2011.313

Category Regular Research Paper

1 Introduction

In Lambda Calculus there exists a perhaps surprising number of different formalisations of the idea of undefined or meaningless term [3, 4, 1, 13, 6, 11]. The rough intuition is that such terms cannot contribute information to any context in which they are placed, and may be mapped to the bottom element of the semantic domain of a denotational semantics. In this paper we are interested in the sets of meaningless terms that arise when one tries to extend lambda calculus with infinite terms and infinite strongly converging reductions in such a way that the confluence property is preserved.

The first attempt to characterise *sets of meaningless terms* axiomatically was made for first order term rewriting [2]. These axioms were revised and further extended to lambda calculus in [11, 7], and recently to combinatory reduction systems [12]. The axioms are general assumptions for ensuring confluence and normalization of infinitary lambda calculi $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$ with a $\bot_{\mathcal{U}}$-rule that rewrites the terms of the set $\mathcal{U}$ of meaningless terms to $\bot$. This general notion of set of meaningless terms captures two well-known examples from lambda calculus: the set $\overline{\mathcal{HN}}$ of terms without head normal form and the set $\overline{\mathcal{WN}}$ of terms without

22nd International Conference on Rewriting Techniques and Applications.
Editor: M. Schmidt-Schauß; pp. 313–328
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

weak head normal form. The initial papers on infinitary lambda calculus revealed a third main example: the set $\overline{\mathcal{TN}}$ of terms without top normal form [3, 6, 9]. Only later in [15, 16] we realised that there are far more sets of meaningless lambda terms which give rise to an ample collection of models of the finitary and the infinitary lambda calculus.

It is now natural to ask: for which sets $\mathcal{U}$ of meaningless terms is the corresponding infinitary lambda calculus $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$ confluent? The confluence proofs in [11, 7] show that the axioms in the notion of set of meaningless term are a sufficient condition, but are they also *necessary*?

In this paper we will show sufficient and necessary conditions for having a confluent and normalizing infinitary lambda calculus $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$. The key idea can be found in Section 3 where we present a new solution for solving the overlap between beta reduction and bottom reduction. This allows us to give a reformulation of the Axiom of Overlap from [11, 7] which we call *Axiom of Weak Overlap.* If we replace Overlap by Weak Overlap in the definition of set of meaningless terms, then we obtain the larger class of *sets of weak meaningless terms.* In Section 4 we give many new examples of sets of weak meaningless sets. In Section 5, we prove that the infinitary lambda beta bottom calculus $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$ is confluent for any set of weak meaningless terms $\mathcal{U}$. In Section 8, we prove the converse: whenever an infinitary lambda beta bottom calculus $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$ is confluent, there exists a set $\mathcal{U}'$ of weak meaningless terms that defines the same reduction as $\mathcal{U}$. As an unexpected result in Section 7 we obtain that confluence implies normalization for infinitary lambda beta bottom calculi $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$.

2 Infinitary Lambda Calculus

We will now briefly recall some notions and facts of infinitary lambda calculus from our earlier work [8, 9, 7, 14, 17]. We assume familiarity with basic notions and notations from [3]. Let Λ be the set of λ-terms and $\Lambda_\bot$ be the set of finite λ-terms with $\bot$.

▶ **Definition 2.1** (Finite and Infinite Lambda Terms). The set $\Lambda^{\infty}_{\bot}$ of finite and infinite λ-terms is defined by coinduction using the grammar:

$$M ::= \bot \mid x \mid (\lambda x.M) \mid (MM)$$

where x is a variable from some fixed, large enough set of variables $\mathcal{V}$. The set Λ^{∞} consists of the terms in $\Lambda^{\infty}_{\bot}$ which do not contain $\bot$. The set $(\Lambda^{\infty})^0$ consists of the terms in Λ^{∞} that are closed, i.e. without free variables.

Having defined the raw terms, we now follow the usual conventions on syntax of finitary and infinitary lambda calculus [3, 7]. As explained in the latter, many concepts from finitary lambda calculus generalise immediately to the infinitary setting, context, position, (head) redex, free and bound variables, (head) normal form and so on. As customary in finitary lambda calculus, we identify terms that are α-convertible and we use the variable convention (bound variables are implicitly renamed before a substitution is made) to avoid variable capture. We will use the notation M^σ to denote the simultaneous substitution of the free variables in M by substitution $\sigma : \mathcal{V} \to \Lambda^{\infty}_{\bot}$.

▶ Notation 2.2. We will use the following abbreviations of λ-terms:

$$\begin{array}{lllllllll}
\mathbf{I} & = & \lambda x.x & \mathbf{O} & = & \lambda x_1.\lambda x_2.\lambda x_3.\dots & \mathbf{\Omega} & = & (\lambda x.xx)\lambda x.xx \\
\mathbf{1} & = & \lambda xy.xy & M^\omega & = & M(M(M\dots)) & \mathbf{\Omega}_\eta & = & \lambda x_1.(\lambda x_2.(\lambda x_3.\dots x_3)x_2)x_1 \\
\mathbf{K} & = & \lambda xy.x & & & & & &
\end{array}$$

The point of the syntax of infinitary lambda calculus is to have one framework that captures both finite and infinite terms. This has the pleasant consequence that Böhm trees don't have to be defined in a separate formalism. Böhm trees are nothing else than normal forms under a particular notion of reduction. So, in the following we will freely identify trees with terms in $\Lambda^\infty_\perp$. In [9, 11, 7], an alternative definition of the set $\Lambda^\infty_\perp$ is given using a metric. The coinductive and metric definitions are equivalent [5]. Note that here we follow [7] and consider only one set of λ-terms, namely $\Lambda^\infty_\perp$, in contrast to the formulations in [9, 11] where several metric completions (all subsets of $\Lambda^\infty_\perp$) of the set of finite terms are considered.

We will consider infinitary lambda calculus with two reductions rules: the familiar beta rule and the $\perp_\mathcal{U}$-rule which is parametrised by some set $\mathcal{U}$ of terms. This $\perp_\mathcal{U}$-rule generalises the $\perp$-rule used to define Böhm trees in which terms without head normal form with are identified with bottom [11, 7].

▶ **Definition 2.3** (β-rule)**.** We consider the β-rule on $\Lambda^\infty_\perp$:

$$(\lambda x.M)N \to M[x := N] \quad (\beta)$$

The one step reduction $\to_\beta$ is the smallest binary relation containing β and closed under contexts.

▶ **Definition 2.4** ($\perp_\mathcal{U}$-rule)**.** Let $\mathcal{U} \subseteq \Lambda^\infty$. We define the $\perp_\mathcal{U}$-rule on $\Lambda^\infty_\perp$:

$$\frac{M[\perp := \mathbf{\Omega}] \in \mathcal{U} \quad M \neq \perp}{M \to \perp} \, (\perp_\mathcal{U})$$

Occasionally, we may denote $\perp_\mathcal{U}$ just by $\perp$. The one step reduction $\to_{\perp_\mathcal{U}}$ is the smallest binary relation containing $\perp_\mathcal{U}$ and closed under contexts. The reduction $\to_{\beta\perp_\mathcal{U}}$ is the smallest binary relation containing β and $\perp_\mathcal{U}$ closed under contexts.

We will consider calculi with various combinations of these rules: $\lambda^\infty_{\beta\perp_\mathcal{U}}$, λ^∞_β and $\lambda^\infty_{\perp_\mathcal{U}}$. We will use the notation λ^∞_ρ where ρ is a variable ranging over $\{\beta\perp_\mathcal{U},\ \beta,\ \perp_\mathcal{U}\}$.

▶ **Definition 2.5** (Subterm at a certain Position)**.** Positions are finite sequences of 0, 1 and 2's and include the empty sequence $\langle\rangle$. Provided it exists, the subterm $M|_p$ of a term $M \in \Lambda_\perp$ at position p is defined by induction as usual:

$$M|_{\langle\rangle} = M \quad (\lambda x M)|_{0p} = M|_p \quad (MN)|_{1p} = M|_p \quad (MN)|_{2p} = N|_p$$

The depth of a subterm N at position p occurs in M is the length of p.

▶ **Definition 2.6** (Truncation)**.** The truncation of M at depth n is obtained by replacing all subterms at depth n by $\perp$ and is denoted by M^n.

▶ **Definition 2.7** (Metric)**.** We define a metric $d : \Lambda_\perp \times \Lambda_\perp \to [0, 1]$ as follows: $d(M, N) = 0$, if $M = N$ and $d(M, N) = 2^{-m}$, where $m = max\{M^n = N^n \mid n \in \mathbb{N}\}$.

The metric will be used in the definition of a transfinite reduction sequence. Note that we will use customary notation like α, β, γ for arbitrary ordinals and λ for limit ordinals. The context will disambiguate the overloading.

▶ **Definition 2.8** (Strongly Converging Reductions [7])**.** Let $\lambda^\infty_\rho = (\Lambda^\infty_\perp, \to_\rho)$.

1. A transfinite reduction sequence of length α in λ^∞_ρ, where α is any ordinal, is a sequence of reduction steps $(M_\beta \to_\rho M_{\beta+1})_{\beta<\alpha}$. In the step $M_\beta \to_\rho M_{\beta+1}$, we denote the position of the contracted redex in M_β by p_β and the depth of this redex by d_β.

2. We define that a sequence $(M_\beta \to_\rho M_{\beta+1})_{\beta<\alpha}$ is a Cauchy (converging) reduction sequence from M_0 to M_α if, for every limit ordinal $\lambda \leq \alpha$, the distance $d(M_\beta, M_\lambda)$ tends to 0 as β approaches λ from below.
3. We define that a sequence $(M_\beta \to_\rho M_{\beta+1})_{\beta<\alpha}$ is *strongly converging* if, it is Cauchy converging and if, for every limit ordinal $\lambda \leq \alpha$, the depth d_β of the contracted redex in $M_\beta \to_\rho M_{\beta+1}$ tends to infinity as β approaches λ from below.

In contrast to strongly converging reductions Cauchy converging reductions don't project well. Hence strongly converging reduction is the natural notion of reduction to study. This preference is reflected in the next notation.

▶ Notation 2.9. Let $\lambda^\infty_\rho = (\Lambda^\infty_\bot, \to_\rho)$.
1. $M \to_\rho N$ denotes a one step reduction from M to N;
2. $M \twoheadrightarrow_\rho N$ denotes a finite reduction from M to N;
3. $M \twoheadrightarrow\!\!\!\!\rightarrow_\rho N$ denotes a strongly converging reduction from M to N.

▶ **Definition 2.10.** Let $\lambda^\infty_\rho = (\Lambda^\infty_\bot, \to_\rho)$.
1. λ^∞_ρ is *confluent*, if ${}_\rho\!\twoheadleftarrow\!\!\!\!\leftarrow \circ \twoheadrightarrow\!\!\!\!\rightarrow_\rho \subseteq \twoheadrightarrow\!\!\!\!\rightarrow_\rho \circ {}_\rho\!\twoheadleftarrow\!\!\!\!\leftarrow$.
2. A term M in λ^∞_ρ is in *ρ-normal form*, if there is no N in λ^∞_ρ such that $M \to_\rho N$.
3. λ^∞_ρ is *normalizing*, if for all $M \in \Lambda^\infty_\bot$ there is an N in ρ-normal form such that $M \twoheadrightarrow\!\!\!\!\rightarrow_\rho N$.

If $\lambda^\infty_{\beta\bot_\mathcal{U}}$ is confluent and normalizing, the normal form of a term M is unique and denoted by $\mathsf{nf}_\mathcal{U}(M)$.

▶ **Definition 2.11** (Rootactive)**.** Let $M \in \Lambda^\infty_\bot$. We say that M is *rootactive*, if for any $N \in \Lambda^\infty_\bot$, if $M \twoheadrightarrow\!\!\!\!\rightarrow_\beta N$ then $N \twoheadrightarrow_\beta (\lambda x.P)Q$ for some $P, Q \in \Lambda^\infty_\bot$. Let $\mathcal{R}$ denote the set $\{M \in \Lambda^\infty \mid M \text{ is rootactive}\}$ of bottom free rootactive terms.

▶ **Definition 2.12.** Let $M, N \in \Lambda^\infty$. We write $M \xleftrightarrow{\mathcal{U}} N$, if N can be obtained from M by replacing some (possibly infinitely many) subterms in $\mathcal{U}$ by other terms in $\mathcal{U}$.

In the next definition, we follow the axiomatisation of [11] which is equivalent to the one in [7] which combines Closure under β-reduction and Closure under substitutions in one Descendants axiom.

▶ **Definition 2.13** ([7])**.** We give names to the following properties that a set $\mathcal{U} \subseteq \Lambda^\infty$ may satisfy:
1. **Axiom of Rootactiveness:** $\mathcal{R} \subseteq \mathcal{U}$.
2. **Axiom of Closure under β-reduction:** $M \twoheadrightarrow\!\!\!\!\rightarrow_\beta N$ implies $N \in \mathcal{U}$ for all $M, N \in \mathcal{U}$.
3. **Axiom of Closure under Substitution:** $M^\sigma \in \mathcal{U}$ for all $M \in \mathcal{U}$ and substitutions σ.
4. **Axiom of Overlap:** for all $M \in \mathcal{U}$, if $M = \lambda x.P$ then $(\lambda x.P)Q \in \mathcal{U}$ for all $Q \in \Lambda^\infty$.
5. **Axiom of Indiscernibility:** for all $M, N \in \Lambda^\infty$ such that $M \xleftrightarrow{\mathcal{U}} N$, $M \in \mathcal{U}$ if and only if $N \in \mathcal{U}$.

In order to guarantee confluence of the Infinitary Lambda Calculi, we define the notion of sets of meaningless terms [11, 7].

▶ **Definition 2.14** (Meaningless Set)**.** 1. A set $\mathcal{U} \subseteq \Lambda^\infty$ is called a *set of meaningless terms* (meaningless set for short), if it satisfies the Axioms (1-5). These axioms are called *the axioms of meaningless terms.*
2. $\mathbb{M} = \{\mathcal{U} \subseteq \Lambda^\infty \mid \mathcal{U} \text{ is a set of meaningless terms}\}$.

In Section 4, we will show many examples of meaningless sets other than $\mathcal{R}$.

▶ **Theorem 2.15** (Sufficiency of Rootactiveness for Normalization [11, 7]). *Let $\mathcal{U} \subseteq \Lambda^\infty$. If $\mathcal{U}$ satisfies Rootactiveness, then $\lambda^\infty_{\beta\bot_\mathcal{U}}$ is normalizing.*

▶ **Theorem 2.16** (Sufficiency of Meaninglessness for Confluence and Normalization [11, 7]). *Let $\mathcal{U} \subseteq \Lambda^\infty$. If $\mathcal{U}$ is a set of meaningless terms, then $\lambda^\infty_{\beta\bot_\mathcal{U}}$ is confluent and normalizing.*

The following theorem relates the infinitary lambda calculus with models of the finite lambda calculus (see Definitions 5.2.7 and 5.3.1 in [3]).

▶ **Theorem 2.17** (λ-model $\mathfrak{M}_\mathcal{U}$). *Each set $\mathcal{U}$ such that $\lambda^\infty_{\beta\bot_\mathcal{U}}$ is confluent and normalizing gives rise to a λ-model denoted by $\mathfrak{M}_\mathcal{U}$.*

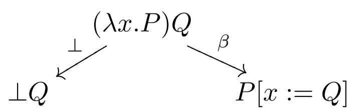
Proof. The domain of $\mathfrak{M}_\mathcal{U}$ is the set $\mathsf{nf}_\mathcal{U}(\Lambda^\infty_\bot)$ of normal forms of $\lambda^\infty_{\beta\bot_\mathcal{U}}$. We interpret a lambda term M by its normal form $\mathsf{nf}_\mathcal{U}(M)$ and we define application simply by $\mathsf{nf}_\mathcal{U}(M) \bullet \mathsf{nf}_\mathcal{U}(N) = \mathsf{nf}_\mathcal{U}(MN)$. ◀

We denote by $\mathbb{MOD}(\lambda) = \{\mathfrak{M}_\mathcal{U} \mid \mathcal{U}$ defines a confluent and normalizing $\lambda^\infty_{\beta\bot_\mathcal{U}}\}$, the class of models induced by the confluent and normalizing infinitary lambda calculi.

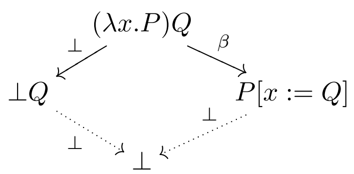
3 Reconsidering the Axiom of Overlap

In [11] it was noted that there is the possibility of overlap between the beta and the bottom rule for meaningles terms of the form $\lambda x.M$. This was resolved by the Axiom of Overlap. That was a satisfactory solution, as it covered the examples of meaningless terms that were known at the time. However, as we will show here, it is not the only way.

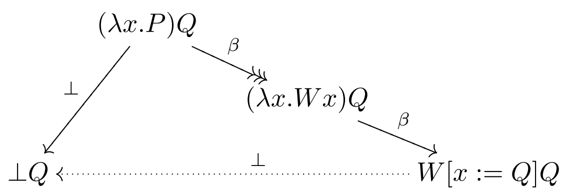
Let us first re-examine the rationale behind the Axiom of Overlap in detail. Let $\mathcal{U}$ be a set of meaningless terms. Overlap between $\bot$-reduction and β-reduction occurs when the $\bot$-redex is of the form $\lambda x.P$. This gives a divergence



The Axiom of Overlap solves this divergence in combination with Rootactiveness, Closure under β-reduction and Indiscernibility. When $\lambda x.P \in \mathcal{U}$ then by Overlap we have that $(\lambda x.P)Q \in \mathcal{U}$. By Rootactiveness and Indiscernibility also $\mathbf{\Omega} Q \in \mathcal{U}$. On one hand we have $\bot Q \to_\bot \bot$ since $(\bot Q)[\bot := \mathbf{\Omega}] = \mathbf{\Omega} Q \in \mathcal{U}$. On the other, by Closure under β-reduction, we have $P[x := Q] \in \mathcal{U}$ and thus also $P[x := Q] \to_\bot \bot$.



There is, however, another way of resolving this divergence, not considered in [11]. Suppose besides $\lambda x.P \in \mathcal{U}$ we also have $P \twoheadrightarrow_\beta Wx$ with $W \in \mathcal{U}$. Then if $\mathcal{U}$ satisfies Closure under substitution, we have that $W[x := Q] \in \mathcal{U}$ and then $(Wx)[x := Q] = W[x := Q]Q \to_\bot \bot Q$:



Thus we find an alternative axiom of overlap and an alternative notion of meaningless set:

▶ **Definition 3.1** (Axiom of Alternative Overlap, Set of Alternative Meaningless Terms)**.**

1. A set $\mathcal{U} \subseteq \Lambda^\infty$ is said to satisfy **Alternative Overlap**, if for each abstraction $\lambda x.P \in \mathcal{U}$, there is some $W \in \mathcal{U}$ such that $P \twoheadrightarrow_\beta Wx$.
2. A set $\mathcal{U} \subseteq \Lambda^\infty$ is called a *set of alternative meaningless terms*, if it satisfies Rootactiveness, Closure under β-reduction, Substitution, Alternative Overlap and Indiscernibility.
3. $\mathbb{AM} = \{\mathcal{U} \subseteq \Lambda^\infty \mid \mathcal{U}$ is a set of alternative meaningless terms$\}$.

We can capture both axioms, Overlap and Alternative Overlap, in a single general axiom:

▶ **Definition 3.2** (Axiom of Weak Overlap, Set of Weak Meaningless Terms)**.**

1. A set $\mathcal{U} \subseteq \Lambda^\infty$ is said to satisfy the axiom of **Weak Overlap**, if for each abstraction $\lambda x.P \in \mathcal{U}$ there is some $W \in \mathcal{U}$ such that $P \twoheadrightarrow_\beta Wx$, or $(\lambda x.P)Q \in \mathcal{U}$ for all $Q \in \Lambda^\infty$.
2. A set $\mathcal{U} \subseteq \Lambda^\infty$ is called a *set of weak meaningless terms*, if it satisfies the Axioms of Closure under β-reduction, Substitution, Weak Overlap, Rootactiveness and Indiscernibility.
3. $\mathbb{WM} = \{\mathcal{U} \subseteq \Lambda^\infty \mid \mathcal{U}$ is a set of weak meaningless terms$\}$.

It is trivial to see that if $\mathcal{U}$ satisfies either Overlap or Alternative Overlap then it satisfies Weak Overlap. The converse is also true as proved in the following theorem.

▶ **Theorem 3.3.** *Let $\mathcal{U} \subseteq \Lambda^\infty$ satisfy the Axioms of Closure under β-reduction and Indiscernibility. Then, $\mathcal{U}$ satisfies the Axiom of Weak Overlap if and only if $\mathcal{U}$ either satisfies the Axiom of Overlap or the Axiom of Alternative Overlap. Moreover, if $\mathcal{U}$ contains an abstraction then $\mathcal{U}$ cannot satisfy both the Axioms of Overlap and Alternative Overlap simultaneously.*

Proof. $\Leftarrow$ is trivial. We prove $\Rightarrow$. Suppose $\lambda x.P_1 \in \mathcal{U}$ for which we have that $(\lambda x.P_1)Q \in \mathcal{U}$ for all $Q \in \Lambda^\infty$. Therefore for any other abstraction $\lambda x.P_2 \in \mathcal{U}$ we get by Indiscernibility $(\lambda x.P_2)Q \in \mathcal{U}$ for all $Q \in \Lambda^\infty$. That is, the axiom of Overlap holds.

If, however, for no abstraction $\lambda x.P_1 \in \Lambda^\infty$ we have that $(\lambda x.P_1)Q \in \mathcal{U}$ for all $Q \in \Lambda^\infty$, then by Weak Overlap it must be that for each abstraction $\lambda x.P \in \mathcal{U}$ we have that there is some $W \in \mathcal{U}$ such that $P \twoheadrightarrow_\beta Wx$. Hence the axiom of Alternative Overlap holds.

Assume $\lambda x.P \in \mathcal{U}$. Suppose $\mathcal{U}$ satisfies the Axiom of Overlap. Then by Overlap we have $(\lambda x.P)x \in \mathcal{U}$ and hence $P \in \mathcal{U}$ by Closure under β-reduction. By Indiscernibility we find $\lambda x.\mathbf{\Omega} \in \mathcal{U}$. But there is no W such that $\mathbf{\Omega}$ reduces to Wx. Therefore $\mathcal{U}$ does not satisfy Alternative Overlap. Hence $\mathcal{U}$ cannot satisfy both axioms simultaneously. ◀

▶ **Corollary 3.4.** *1.* $\mathbb{WM} = \mathbb{M} \cup \mathbb{AM}$

2. If $\mathcal{U} \in \mathbb{M} \cap \mathbb{AM}$, then $\mathcal{U}$ does not contain any abstraction.

4 Examples of Sets of Weak Meaningless Terms

In this section, we recall some examples of sets of meaningless terms from [11, 7, 16] and give new examples of sets of weak meaningless terms.

▶ **Definition 4.1.** Let $M \in \Lambda^\infty$. We say that

1. M is a *head normal form* (hnf) if $M = \lambda x_1 \dots x_n.yP_1 \dots P_k$. We define $\mathcal{HN} = \{M \in \Lambda^\infty \mid M \twoheadrightarrow_\beta N$ and N is a head normal form$\}$.
2. M is a *weak head normal form* (whnf) if M is a hnf or $M = \lambda x.N$. We define $\mathcal{WN} = \{M \in \Lambda^\infty \mid M \twoheadrightarrow_\beta N$ and N and N is a weak head normal form$\}$.

3. M is a *top normal form* (tnf) if it is either a whnf or an application (NP) where there is no Q such that $N \twoheadrightarrow_\beta \lambda x.Q$. We define $\mathcal{TN} = \{M \in \Lambda^\infty \mid M \twoheadrightarrow_\beta N$ and N is a top normal form$\}$.
4. M is a *strong active form* (saf) if $M = RP_1 \dots P_k$ and R is rootactive. We define $\mathcal{SA} = \{M \in \Lambda^\infty \mid M \twoheadrightarrow_\beta N$ and N is a strong active form$\}$.
5. M is a *strong active form relative to* X if $M = RP_1 \dots P_k$, R is rootactive and $P_1, \dots, P_k \in X$. We define $\mathcal{SA}_X = \{M \in \Lambda^\infty \mid M \twoheadrightarrow\!\!\!\!\twoheadrightarrow_\beta N$ and N is a strong active form relative to $X\}$.
6. M is a *strong infinite left spine form* (silsf) if $M = (\dots P_2)P_1$. We define $\mathcal{SIL} = \{M \in \Lambda^\infty \mid M \twoheadrightarrow\!\!\!\!\twoheadrightarrow_\beta N$ and N is a strong infinite left spine form$\}$.
7. M is a *head active form* (haf) if $M = \lambda x_1 \dots x_n.RP_1 \dots P_k$ and R is rootactive. We define $\mathcal{HA} = \{M \in \Lambda^\infty \mid M \twoheadrightarrow_\beta N$ and N is a head active form$\}$.
8. M is an *infinite left spine form* (ilsf) if $M = \lambda x_1 \dots x_n.(\dots P_2)P_1$. We define $\mathcal{IL} = \{M \in \Lambda^\infty \mid M \twoheadrightarrow\!\!\!\!\twoheadrightarrow_\beta N$ and N is an infinite left spine form$\}$.
9. We define $\mathcal{O} = \{M \in \Lambda^\infty \mid M \twoheadrightarrow\!\!\!\!\twoheadrightarrow_\beta \mathbf{O}\}$ where $\mathbf{O} = \lambda x_1.\lambda x_2.\lambda x_3.\dots$.

By $\overline{\mathcal{HN}}$, $\overline{\mathcal{WN}}$ and $\overline{\mathcal{TN}}$ we denote the complements in Λ^∞ of $\mathcal{HN}$, $\mathcal{WN}$ and $\mathcal{TN}$ respectively. Note that $\mathcal{R} = \overline{\mathcal{TN}}$, $\overline{\mathcal{WN}} = \mathcal{SA} \cup \mathcal{SIL}$ and $\overline{\mathcal{HN}} = \mathcal{HA} \cup \mathcal{IL} \cup \mathcal{O}$.

▶ **Theorem 4.2** ([11, 7])**.** *$\overline{\mathcal{TN}}$, $\overline{\mathcal{WN}}$ and $\overline{\mathcal{HN}}$ are meaningless sets.*

The Berarducci tree $\mathsf{BerT}(M)$ of a term M is its normal form in $\lambda^\infty_{\beta\bot_\mathcal{U}}$ where $\mathcal{U}$ is $\mathcal{R} = \overline{\mathcal{TN}}$ [6, 8]. The Lévy-Longo tree $\mathsf{LLT}(M)$ is the normal form of M in $\lambda^\infty_{\beta\bot_\mathcal{U}}$ where $\mathcal{U}$ is $\overline{\mathcal{WN}}$ [8]. The Böhm tree $\mathsf{BT}(M)$ is the normal form of M in $\lambda^\infty_{\beta\bot_\mathcal{U}}$ where $\mathcal{U}$ is $\overline{\mathcal{HN}}$ [3].

▶ **Theorem 4.3** ([16])**.** *The following are meaningless sets:*
1. *$\mathcal{SA}$, $\mathcal{HA}$, $\mathcal{HA} \cup \mathcal{IL}$ and $\mathcal{HA} \cup \mathcal{O}$.*
2. *$\mathcal{SA}_X$, provided $X \subseteq \mathsf{BerT}(\Lambda^\infty_\bot) \cap (\Lambda^\infty)^0$.*

Any set of meaningless terms that does not contain abstractions such as $\mathcal{R}$ or $\mathcal{SA}$ is trivially a set of alternative meaningless terms. We will give examples of sets of alternative meaningless terms which contain abstractions and which are not sets of meaningless terms.

▶ **Definition 4.4.** Define $\mathcal{U}^\eta = \mathcal{U} \cup \{M \mid M \twoheadrightarrow\!\!\!\!\twoheadrightarrow_\beta \lambda x.Nx$ and $N \in \mathcal{U}\}$ for $\mathcal{U} \subseteq \Lambda^\infty$.

If $\mathcal{U}$ is a set of meaningless terms, $\mathcal{U}^\eta$ does not have to be a set of weak meaningless terms. For example, $\mathcal{SA}^\eta$ is not a set of weak meaningless terms. It does not satisfy Indiscernibility since $\mathbf{\Omega}x \in \mathcal{SA}^\eta$ is a subterm of $\lambda x.\mathbf{\Omega}x \in \mathcal{SA}^\eta$ but $\lambda x.\mathbf{\Omega} \notin \mathcal{SA}^\eta$. Similarly, $(\mathcal{SA} \cup \mathcal{SIL})^\eta$ is not a set of weak meaningless terms. On the other hand, for $\mathcal{U} \in \{\mathcal{HA}, \mathcal{HA} \cup \mathcal{IL}, \mathcal{HA} \cup \mathcal{O}, \mathcal{HA} \cup \mathcal{IL} \cup \mathcal{O}, \Lambda^\infty\}$, the set $\mathcal{U}^\eta$ is a meaningless set because $\mathcal{U}^\eta = \mathcal{U}$; but $\mathcal{U}^\eta$ is not a set of alternative meaningless terms, as it cannot be both by Corollary 3.4.

▶ **Theorem 4.5.** *1. $\mathcal{R}^\eta$ is a set of alternative meaningless terms.*
2. $\mathcal{SA}^\eta_X$ is a set of alternative meaningless terms, provided $X \subseteq \mathsf{BerT}(\Lambda^\infty_\bot) \cap (\Lambda^\infty)^0$.

We skip the proof as it follows the same pattern as the proofs for meaningless sets presented in [16]. By Corollary 3.4, the sets in the above theorem are not sets of meaningless terms because they contain abstractions. Since $\{\mathcal{SA}^\eta_X \mid X \subseteq \mathsf{BerT}(\Lambda^\infty_\bot) \cap (\Lambda^\infty)^0\}$ has the cardinality $2^\mathfrak{c}$ of the continuum, we have that:

▶ **Corollary 4.6.** *Let $\mathfrak{c}$ be the cardinality of the continuum. There are $2^\mathfrak{c}$ sets of alternative meaningless terms which are not meaningless sets.*

▶ Remark. The set $\mathbb{WM}$ of all sets of weak meaningless terms forms a poset with a top, $\mathcal{R}$, and a bottom, Λ^∞. In Figure 1 we depict the relative order of the sets mentioned in this section. The notation $\mathcal{U}_1 \to \mathcal{U}_2$ indicates that $\mathcal{U}_1 \supset \mathcal{U}_2$.

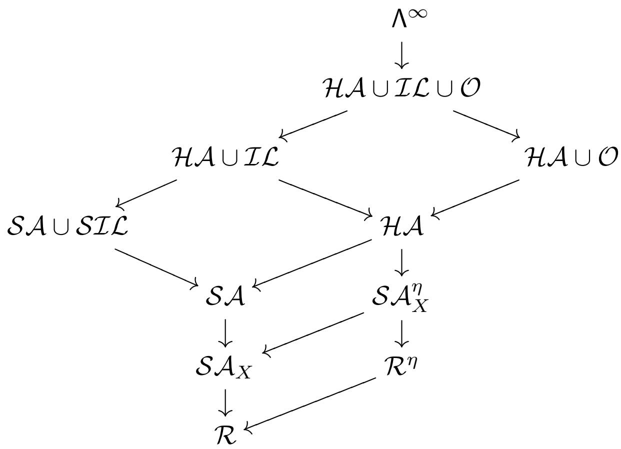


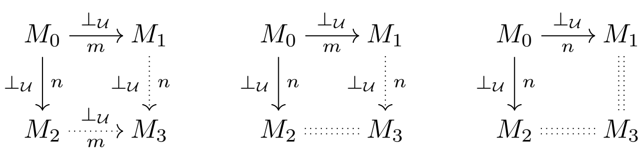
Figure 1 A poset of sets of weak meaningless terms

5 Weak Meaninglessness implies Confluence and Normalization

In this section, we prove confluence of $\lambda^\infty_{\beta\bot_\mathcal{U}}$ when $\mathcal{U}$ is a set of weak meaningless terms. This extends the result in [11, 7] where confluence of $\lambda^\infty_{\beta\bot_\mathcal{U}}$ is shown under the provision that $\mathcal{U}$ is a set of meaningless terms. First we need some auxiliary results.

▶ **Proposition 5.1.** *Let $\mathcal{U} \subseteq \Lambda^\infty$ satisfy the Axioms of Rootactiveness and of Indiscernibility. Then the calculus $\lambda^\infty_{\bot_\mathcal{U}} = (\Lambda^\infty_\bot, \to_{\bot_\mathcal{U}})$ is confluent.*

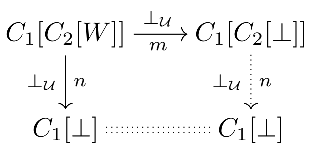
Proof. We sketch a standard transfinite inductive tiling diagram proof. The basic information that this proof uses are the following elementary tiling diagrams for one-set coinitial $\bot_\mathcal{U}$-reductions.



The labels n, m used in the diagrams indicate the depth at which the $\bot_\mathcal{U}$ reduction takes place. The important thing to note is that the depth of a $\bot_\mathcal{U}$-redex in a term does not change when it is not erased in the contraction of another $\bot_\mathcal{U}$-redex elsewhere in the term.

The diagrams reflect three possibilities. The redexes in the two coinitial reductions are either disjoint, properly nested or identical. In each case the respective diagram show how to complete confluence with two cofinal $\bot_\mathcal{U}$-reductions, which are either one-step or empty.

The middle diagram requires Indiscernibility. Suppose M_1 is of the form $C_1[C_2[W]]$ for contexts $C_1[\ ], C_2[\ ]$. And suppose W and $C_2[W]$ belong to $\mathcal{U}$. Then by Indiscernibility we get $C_2[\mathbf{\Omega}] \in \mathcal{U}$ and so $C_1[C_2[\bot]] \to_\bot C_1[\bot]$. This completes the diagram:



Given two transfinite coinitial $\bot_{\mathcal{U}}$-reductions one now constructs the following tiling diagram [7] inductively in which all vertical and horizontal reductions are strongly converging.

$$\begin{array}{ccccccc}
M_{0,0} & \xrightarrow[m_0]{\bot_{\mathcal{U}}} & M_{0,1} & \xrightarrow[m_1]{\bot_{\mathcal{U}}} & M_{0,2} & \cdots\cdots & M_{0,\beta} \\
\bot_{\mathcal{U}}\downarrow n_0 & & \bot_{\mathcal{U}}\downarrow n_0 & & \bot_{\mathcal{U}}\downarrow n_0 & & \bot_{\mathcal{U}}\downarrow n_0 \\
M_{1,0} & \xrightarrow[m_0]{\bot_{\mathcal{U}}} & M_{1,1} & \xrightarrow[m_1]{\bot_{\mathcal{U}}} & M_{1,2} & \cdots\cdots & M_{1,\beta} \\
\bot_{\mathcal{U}}\downarrow n_1 & & \bot_{\mathcal{U}}\downarrow n_1 & & \bot_{\mathcal{U}}\downarrow n_1 & & \bot_{\mathcal{U}}\downarrow n_1 \\
M_{2,0} & \xrightarrow[m_0]{\bot_{\mathcal{U}}} & M_{2,1} & \xrightarrow[m_1]{\bot_{\mathcal{U}}} & M_{2,2} & \cdots\cdots & M_{2,\beta} \\
\vdots & & \vdots & & \vdots & & \vdots \\
M_{\alpha,0} & \xrightarrow[m_0]{\bot_{\mathcal{U}}} & M_{\alpha,1} & \xrightarrow[m_1]{\bot_{\mathcal{U}}} & M_{\alpha,2} & \cdots\cdots & M_{\alpha,\beta}
\end{array}$$

We skip the proof, which is similar to confluence proof of $\lambda_\eta^{h\infty}$ in [14], because the elementary tiles that load the induction are similar for $\bot_{\mathcal{U}}$ and η. The simplicity of these rules makes it unnecessary to specify the positions; the information of the depth in each step of the reduction sequence suffices. ◀

▶ **Lemma 5.2.** [11, Lemma 27] *Let $\mathcal{U} \subseteq \Lambda^\infty$ satisfy the Axiom of Closure under Substitution. If $M \twoheadrightarrow\!\!\!\!\!\rightarrow_{\beta\bot_{\mathcal{U}}} N$, then $M \twoheadrightarrow\!\!\!\!\!\rightarrow_{\beta} L \twoheadrightarrow\!\!\!\!\!\rightarrow_{\bot_{\mathcal{U}}} N$ for some $L \in \Lambda_\bot^\infty$.*

We need somre terminology and notation: An outermost $\bot_{\mathcal{U}}$-redex of M is a maximal subterm N of M such that $N[\bot := \mathbf{\Omega}] \in \mathcal{U}$. We denote by $M \xrightarrow{out}_{\bot_{\mathcal{U}}} N$ if the contracted redex in $M \to_{\bot_{\mathcal{U}}} N$ is an outermost $\bot_{\mathcal{U}}$-redex.

The information stored at the root of a term M is denoted by $\mathsf{root}(M)$ and defined by cases: $\mathsf{root}(x) = x$, $\mathsf{root}(\lambda x.M) = \lambda x$ and $\mathsf{root}(MN) = @$. We denote $M \sim_{\mathsf{root}} N$ if $\mathsf{root}(M) = \mathsf{root}(N)$.

▶ **Lemma 5.3.** *Let $\mathcal{U} \subseteq \Lambda^\infty$ satisfy the Axioms of Rootactiveness and Indiscernibility. If $M \twoheadrightarrow\!\!\!\!\!\rightarrow_{\bot_{\mathcal{U}}} N$ and N is in $\beta\bot_{\mathcal{U}}$-normal form then $M \overset{out}{\twoheadrightarrow\!\!\!\!\!\rightarrow}_{\bot_{\mathcal{U}}} N$.*

Proof. Suppose $M = M_0 \twoheadrightarrow\!\!\!\!\!\rightarrow_{\bot_{\mathcal{U}}} M_\alpha = N$ is a reduction of length α and N is a $\beta\bot_{\mathcal{U}}$-normal form. We define a new reduction $N_0 \overset{out}{\twoheadrightarrow\!\!\!\!\!\rightarrow}_{\bot_{\mathcal{U}}} N_\alpha$ by induction on α satisfying the property $\Phi(\beta)$ for all $0 \leq \beta \leq \alpha$ where $\Phi(\beta)$ is defined as follows. If $(M_\beta)|_p = \bot$, then $(N_\beta)|_p[\bot := \mathbf{\Omega}] \in \mathcal{U}$. Otherwise, if $(M_\beta)|_p \neq \bot$, then $(M_\beta)|_p \sim_{\mathsf{root}} (N_\beta)|_p$.

Base Case. Let N_0 be equal to M.

Successor Case. Suppose we have constructed $N_0 \twoheadrightarrow\!\!\!\!\!\rightarrow_{\bot_{\mathcal{U}}} N_\beta$ and $\Phi(\beta)$ holds. And suppose $M_\beta \to_{\bot_{\mathcal{U}}} M_{\beta+1}$ by contraction of a term of $\mathcal{U}$ at position p in M_β. If the subterm at p in M_β is an outermost $\bot_{\mathcal{U}}$-redex, we construct $N_\beta \xrightarrow{out}_{\bot_{\mathcal{U}}} N_{\beta+1}$ by reducing the corresponding term at p in N_β to $\bot$. And if it is not an outermost $\bot_{\mathcal{U}}$-redex, we put $N_{\beta+1} = N_\beta$. It is not difficult to prove $\Phi(\beta+1)$ using Indiscernibility.

Note that the constructed reduction sequence is strongly convergent because the original sequence $M = M_0 \twoheadrightarrow\!\!\!\!\!\rightarrow_{\bot_{\mathcal{U}}} M_\alpha = N$ is strongly convergent.

Limit Case. Since the constructed reduction sequences $N_0 \overset{out}{\twoheadrightarrow\!\!\!\!\!\rightarrow}_{\bot_{\mathcal{U}}} N_\beta$ are strongly convergent, the limit λ always exists. It is not difficult to prove $\Phi(\lambda)$ using strong convergence of the reduction and induction hypothesis, i.e. $\Phi(\beta)$ holds for all $\beta < \lambda$.

Thus we have constructed $M_0 = N_0 \overset{out}{\twoheadrightarrow\!\!\!\!\!\rightarrow}_{\bot_{\mathcal{U}}} N_\alpha$ satisfying Φ. Since M_α is a $\bot_{\mathcal{U}}$-normal form, the $\bot$'s remaining in M_α have been introduced by an outermost $\bot$-reduction. Hence we find $\bot$'s at the same location in N_α. By $\Phi(\alpha)$ we get that at the other positions p, $(M_\alpha)|_p \sim_{\mathsf{root}} (N_\alpha)|_p$. That is $M_\alpha = N_\alpha$. Hence $M_0 = N_0 \overset{out}{\twoheadrightarrow\!\!\!\!\!\rightarrow}_{\bot_{\mathcal{U}}} N_\alpha = M_\alpha$. ◀

▶ **Lemma 5.4.** *Let $\mathcal{U} \subseteq \Lambda^\infty$ satisfy the Axiom of Rootactiveness, Closure under Substitution, β-reduction and Indiscernibility. If $W \in \mathcal{U}$ then $\mathsf{nf}_{\mathcal{R}}(W)[\bot := \mathbf{\Omega}] \in \mathcal{U}$.*

Proof. We have $W \twoheadrightarrow_{\beta\bot_{\mathcal{R}}} \mathsf{nf}_{\mathcal{R}}(W)$. By Lemma 5.2 and Closure under Substitution, there exists W_0 such that $W \twoheadrightarrow_{\beta} W_0 \twoheadrightarrow_{\bot_{\mathcal{R}}} \mathsf{nf}_{\mathcal{R}}(W)$. By Closure under β-reduction, $W_0 \in \mathcal{U}$. We can assume that the $\bot$-steps in $W_0 \twoheadrightarrow_{\bot_{\mathcal{R}}} \mathsf{nf}_{\mathcal{R}}(W_0)$ all contract outermost redexes by Lemma 5.3. Then W_0 is obtained from $\mathsf{nf}_{\mathcal{R}}(W_0)$ by replacing rootactive subterms by $\bot$. Since $\mathcal{R} \subseteq \mathcal{U}$, we have that $W_0 \stackrel{\mathcal{U}}{\longleftrightarrow} \mathsf{nf}_{\mathcal{R}}(W_0)[\bot := \mathbf{\Omega}]$. By Indiscernibility, $\mathsf{nf}_{\mathcal{R}}(W_0)[\bot := \mathbf{\Omega}] \in \mathcal{U}$. ◀

▶ **Proposition 5.5.** *Let $\mathcal{U} \subseteq \Lambda^\infty$ be a set of alternative meaningless terms. If $L \twoheadrightarrow_{\bot_{\mathcal{U}}} N$ and N is a $\beta\bot_{\mathcal{U}}$-normal form, then $\mathsf{nf}_{\mathcal{R}}(L) \twoheadrightarrow_{\bot_{\mathcal{U}}} N$.*

Proof. If $L \twoheadrightarrow_{\bot_{\mathcal{U}}} N$ and N is a $\beta\bot_{\mathcal{U}}$ normal form. By Lemma 5.3 we may assume that $L \stackrel{out}{\twoheadrightarrow}_{\bot_{\mathcal{U}}} N$. Since N is a $\beta\bot_{\mathcal{U}}$ normal form, if a β-redex $(\lambda x.P)Q$ occurs in L, then either $(\lambda x.P)[\bot := \mathbf{\Omega}] \in \mathcal{U}$ or $(\lambda x.P)Q$ is contained in some subterm W of L such that $W[\bot := \mathbf{\Omega}] \in \mathcal{U}$. We consider the following set:

$$\mathcal{W} = \{W \mid W \text{ is a maximal subterm of } L \text{ such that } W[\bot := \mathbf{\Omega}] \in \mathcal{U}\}$$

We enumerate $\mathcal{W}$ in the order of the increasing depth of its subterms, and subterms with the same depth we order them from left to right. We obtain either $\mathcal{W} = \{W_n \mid n \leq k\}$ if $\mathcal{W}$ is finite or $\mathcal{W} = \{W_n \mid n \in \mathbb{N}\}$ if $\mathcal{W}$ is infinite. We will define inductively a $\beta\bot_{\mathcal{R}}$ reduction $L = L_0 \twoheadrightarrow_{\beta\bot_{\mathcal{R}}} L_1 \twoheadrightarrow_{\beta\bot_{\mathcal{R}}} L_2 \twoheadrightarrow_{\beta\bot_{\mathcal{R}}} \dots$ with a segment $L_{n-1} \twoheadrightarrow_{\beta\bot_{\mathcal{R}}} L_n$ for each $W_n \in \mathcal{W}$. We will show by induction on n that the following properties hold for all n,

(A) Let $L = C[W_1, \dots, W_n]$. Then $L_n = C[W'_1, \dots, W'_n]$ and $W'_m[\bot := \mathbf{\Omega}] \in \mathcal{U}$ for all $m \leq n$.

(B) The new terms $W'_1, \dots W'_n$ of L_n are in $\beta\bot_{\mathcal{R}}$-normal form. If for some i, W'_i is an abstraction, then it does not overlap a β-redex, i.e. W'_i does not occur in an application $W'_i Q$ of L_n.

Suppose we have constructed $L = L_0 \twoheadrightarrow_{\beta\bot_{\mathcal{R}}} L_{n-1}$ and W_n is the next maximal meaningless subterm of $\mathcal{W}$ to be considered. Let $L = C[W_1, \dots, W_{n-1}, W_n]$. The context obtained from C instantiating the last hole with W_n has $n-1$ holes. By Induction Hypothesis, $L_{n-1} = C[W'_1, \dots, W'_{n-1}, W_n]$ and $W'_m[\bot := \mathbf{\Omega}] \in \mathcal{U}$ for all $m \leq n-1$. In particular, W_n is a subterm of L_{n-1}. We have two possibilities:

(1) $\mathsf{nf}_{\mathcal{R}}(W_n) = \lambda x.P$ and W_n occurs in an application $W_n Q$ of L. By Lemma 5.4, $\lambda x.P[\bot := \mathbf{\Omega}] \in \mathcal{U}$ because $W_n \in \mathcal{U}$. By Alternative Overlap $P[\bot := \mathbf{\Omega}] = P_0 x$ with $P_0 \in \mathcal{U}$. Since W_n is maximal and it occurs in L_{n-1}, we have $L_{n-1} = C_n[W_n Q_n]$. We extend $L = L_0 \twoheadrightarrow_{\beta\bot_{\mathcal{R}}} L_{n-1}$ with

$$L_{n-1} = C_n[W_n Q_n]] \twoheadrightarrow_{\beta\bot_{\mathcal{R}}} C_n[\mathsf{nf}_{\mathcal{R}}(P_0)Q_n] = L_n$$

Proof of (A). Now, $L_n = C[W'_1, \dots, W'_{n-1}, W'_n]$ where $W'_n[\bot := \mathbf{\Omega}] = P_0 \in \mathcal{U}$.
Proof of (B). Since P is in $\beta\bot_{\mathcal{R}}$-normal form and $P[\bot := \mathbf{\Omega}] = P_0 x$, we have $P = \mathsf{nf}_{\mathcal{R}}(P) = \mathsf{nf}_{\mathcal{R}}(P_0)x$. Hence $\mathsf{nf}_{\mathcal{R}}(P_0)$ cannot be an abstraction and $\mathsf{nf}_{\mathcal{R}}(P_0)Q_n$ is not a β-redex and there are no β-redexes in $W'_n = \mathsf{nf}_{\mathcal{R}}(P_0)$ either.

(2) Otherwise, i.e. either $\mathsf{nf}_{\mathcal{R}}(W_n)$ is not an abstraction or it is an abstraction and it does not occur in an application $\mathsf{nf}_{\mathcal{R}}(W_n)Q$ in L. Since W_n is a subterm of L_{n-1}, we have $L_{n-1} = C_n[W_n]$. Then, we extend $L = L_0 \twoheadrightarrow_{\beta\bot_{\mathcal{R}}} L_{n-1}$ with

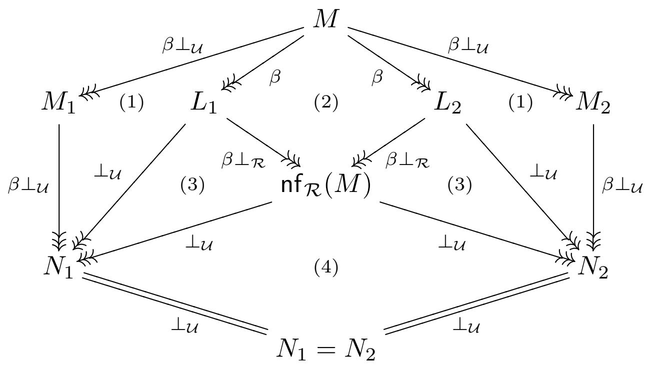
$$L_{n-1} = C_n[W_n] \twoheadrightarrow_{\beta\bot_{\mathcal{R}}} C_n[\mathsf{nf}_{\mathcal{R}}(W_n)] = L_n$$

Proof of A). Now, $L_n = C[W'_1, \ldots, W'_{n-1}, W'_n]$ where $W'_n[\bot := \mathbf{\Omega}] = \mathsf{nf}_{\mathcal{R}}(W_n)[\bot := \mathbf{\Omega}] \in \mathcal{U}$ by Lemma 5.4.
Proof of (B). By Induction Hypothesis, if a β-redex occurs in L_{n-1} then it should occur inside or overlap some of its subterms in $\{W_m \mid m > n-1\} \subseteq \mathcal{W}$. We replaced W_n by $\mathsf{nf}_{\mathcal{R}}(W_n)$ which does not have any β-redex and it cannot overlap a β-redex in L_n. If a β-redex occurs in L_n then it should occur inside or overlap some of its subterms in $\{W_m \mid m > n\} \subseteq \mathcal{W}$.

The concatenation of all these strongly converging reductions is strongly converging [9]. Let K the last term L_k or (if $\mathcal{W}$ is finite) the limit L_ω of $L = L_0 \twoheadrightarrow_{\beta\bot_{\mathcal{R}}} L_1 \twoheadrightarrow_{\beta\bot_{\mathcal{R}}} L_2 \twoheadrightarrow_{\beta\bot_{\mathcal{R}}} \ldots$. It follows from (A) and strong convergence that K is obtained from L by replacing its subterms in $\mathcal{W} \subseteq \mathcal{U}$ by other subterms in $\mathcal{U}$, and hence, we have $K \twoheadrightarrow_{\bot} N$. It follows from (A,B) and strong convergence that K is obtained from L by replacing all its maximal subterms of $\mathcal{U}$ by $\beta\bot_{\mathcal{R}}$-normal forms and if some subterm $W \in \mathcal{W}$ is an abstraction then it does not overlap with a β-redex. Therefore K is a $\beta\bot_{\mathcal{R}}$ normal form and by Confluence of $\lambda^{\infty}_{\beta\bot_{\mathcal{R}}}$ (Theorems 2.16 and 4.2), we have $K = \mathsf{nf}_{\mathcal{R}}(L)$. ◀

▶ **Theorem 5.6** (Sufficiency of Alternative Meaninglessness for Confluence). *Let $\mathcal{U}$ be a set of alternative meaningless terms. Then, $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$ is confluent.*

Proof. The proof is sketched in the following diagram.



Suppose we have a divergence $M_1 \ {}_{\beta\bot}\!\twoheadleftarrow M \twoheadrightarrow_{\beta\bot} M_2$. By Rootactiveness for $\mathcal{U}$, we can reduce M_1 and M_2 further to their respective $\beta\bot_{\mathcal{U}}$-normal forms N_1 and N_2 by Theorem 2.15. (1) By Closure under substitution for $\mathcal{U}$ and Lemma 5.2 we find L_1 and L_2 such that $M \twoheadrightarrow_{\beta} L_1 \twoheadrightarrow_{\bot_{\mathcal{U}}} N_1$ and $M \twoheadrightarrow_{\beta} L_2 \twoheadrightarrow_{\bot_{\mathcal{U}}} N_2$. (2) By normalization and confluence of $\lambda^{\infty}_{\beta\bot_{\mathcal{R}}}$ we construct the reductions $L_1 \twoheadrightarrow_{\beta\bot_{\mathcal{U}}} \mathsf{nf}_{\mathcal{R}}(M)$ and $L_2 \twoheadrightarrow_{\beta\bot_{\mathcal{U}}} \mathsf{nf}_{\mathcal{R}}(M)$. (3) By Proposition 5.5 we then find the reductions $\mathsf{nf}_{\mathcal{R}}(L_1) \twoheadrightarrow_{\bot_{\mathcal{U}}} N_1$ and $\mathsf{nf}_{\mathcal{R}}(L_2) \twoheadrightarrow_{\bot_{\mathcal{U}}} N_2$. By normalization and confluence of $\lambda^{\infty}_{\beta\bot_{\mathcal{R}}}$, we have $\mathsf{nf}_{\mathcal{R}}(M) = \mathsf{nf}_{\mathcal{R}}(L_1) = \mathsf{nf}_{\mathcal{R}}(L_2)$. (4) Finally Proposition 5.1 on confluence of $\bot_{\mathcal{U}}$ and the fact that N_1 and N_2 are by construction normal forms for $\bot_{\mathcal{U}}$-reduction implies that N_1 and N_2 are identical. ◀

▶ **Corollary 5.7** (Sufficiency of Weak Meaninglessness for Confluence and Normalization). *Let $\mathcal{U}$ be a set of weak meaningless terms. Then, $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$ is confluent and normalizing.*

Proof. Immediate from Theorems 5.6 and 2.16, and Corollary 3.4. ◀

By Theorem 4.5 and Corollary 5.7, the Infinitary lambda calculi $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$ where $\mathcal{U} \in \{\mathcal{SA}^{\eta}_{X} \mid X \subseteq \mathsf{BerT}(\Lambda) \cap (\Lambda^{\infty})^{0}\}$ are confluent and normalizing. By Theorem 2.17, they all induce different models of the finite lambda calculus. Since $\{\mathcal{SA}^{\eta}_{X} \mid X \subseteq \mathsf{BerT}(\Lambda) \cap (\Lambda^{\infty})^{0}\}$ has cardinality 2^{ω}, we have that:

▶ **Corollary 5.8.** *There are 2^ω different models of the finite lambda calculus such that:*

1. $\lambda x.\mathbf{\Omega} x = \mathbf{\Omega}$ *but* $\mathbf{I} \neq \mathbf{1}$,
2. $M = N$ *if* $\mathsf{BerT}(M) = \mathsf{BerT}(N)$ *and*
3. $\mathbf{\Omega} M = \mathbf{\Omega}$ *for some* $M \in \Lambda$ *such that* $\mathsf{BerT}(M) \in (\Lambda^\infty)^0$.

6 Axioms of Closure under Expansion

In this section, we define two axioms: Closure under β-expansions [10] and Closure under $\beta\bot$-expansion from $\bot$. In the cited paper we introduced Closure under β-expansion to obtain ω-compression. In this paper we use the axiom to show that an arbitrary weak meaningless set $\mathcal{U} \subseteq \Lambda^\infty$ and its β-expansion $\overline{\mathcal{U}}$ determine the same lambda models.

▶ **Definition 6.1.** We define the following axioms on a set $\mathcal{U} \subseteq \Lambda^\infty$.

1. We say that $\mathcal{U}$ satisfies the Axiom of Closure under β-expansion if for all $N \in \mathcal{U}$, if $M \twoheadrightarrow_\beta N$ then $M \in \mathcal{U}$.
2. We say that $\mathcal{U}$ satisfies the Axiom of Closure under $\beta\bot$-expansion from $\bot$ if for all $M \in \Lambda^\infty$, if $M \twoheadrightarrow_{\beta\bot_\mathcal{U}} \bot$ then $M \in \mathcal{U}$.
3. $\mathbb{B} = \{\mathcal{U} \subseteq \Lambda^\infty \mid \mathcal{U}$ satisfies the Axiom of Closure under $\beta\bot$-expansion from $\bot\}$.

▶ Remark. Note that if $\mathcal{U}$ satisfies Axiom of Closure under $\beta\bot$-expansion from $\bot$ then $\mathcal{U} = \{M \in \Lambda^\infty \mid M \twoheadrightarrow_{\beta\bot_\mathcal{U}} \bot\}$, i.e. $\mathcal{U}$ is the set of $\beta\bot$-expansions of $\bot$. We also have that $\mathcal{U}$ satisfies Closure under β-expansion.

▶ Remark. All sets of weak meaningless terms of Figure 1, have been defined to satisfy Closure under β-expansion to facilitate the proof of Indiscernibility. If a set $\mathcal{U}$ satisfies Rootactiveness and Indiscernibility, then $\mathcal{U}$ is closed under certain β-expansions. Since the set $\mathcal{R}$ is closed under β-expansions, we have that, for example, $\mathbf{I\Omega} \in \mathcal{R}$. By Indiscernibility, if $M \in \mathcal{U}$ then $\mathbf{I}M$ should also belong to $\mathcal{U}$.

▶ Remark. All examples of sets of (weak) meaningless terms given in Section 4 also satisfy Closure under $\beta\bot$-expansion from $\bot$. Suppose $M \twoheadrightarrow_{\beta\bot_\mathcal{U}} \bot$. By Closure under Substitution, by Lemma 5.2, there is an N with $M \twoheadrightarrow_\beta N \twoheadrightarrow_{\bot_\mathcal{U}} \bot$. Hence $N \overset{out}{\twoheadrightarrow}_{\bot_\mathcal{U}} \bot$ by Lemma 5.3. Then N reduces in one step to $\bot$ so that $N \in \mathcal{U}$. Closure under β-expansion implies $M \in \mathcal{U}$.

Given a set $\mathcal{U}$, we can always extend it to a set $\overline{\mathcal{U}}$ that satisfies Closure under $\beta\bot$-expansion from $\bot$ by taking: $\overline{\mathcal{U}} = \{M \in \Lambda^\infty \mid M \twoheadrightarrow_{\beta\bot_\mathcal{U}} \bot\}$. We have that $\mathcal{U}$ and $\overline{\mathcal{U}}$ define the same reduction:

▶ **Theorem 6.2** (Same Reduction). *Let $M, N \in \Lambda^\infty_\bot$ and $\mathcal{U} \subseteq \Lambda^\infty$. Then, $M \twoheadrightarrow_{\beta\bot_\mathcal{U}} N$ if and only if $M \twoheadrightarrow_{\beta\bot_{\overline{\mathcal{U}}}} N$.*

Proof. Let $M = C[P] \to_{\bot_{\overline{\mathcal{U}}}} C[\bot] = N$ where $P \in \overline{\mathcal{U}}$. Then $P \twoheadrightarrow_{\beta\bot_\mathcal{U}} \bot$. Hence, $M \twoheadrightarrow_{\beta\bot_\mathcal{U}} C[\bot] = N$. The converse is trivial. ◀

We define the equivalence relation $\mathcal{U} \sim \mathcal{U}'$ if $\twoheadrightarrow_{\beta\bot_\mathcal{U}}$ *and* $\twoheadrightarrow_{\beta\bot'_\mathcal{U}}$ are equal. Then, every $\sim$-equivalence class $[\mathcal{U}]$ has a unique canonical representative obtained by taking the union of all the members of the class, i.e. $\overline{\mathcal{U}} = \bigcup[U]$.

▶ **Corollary 6.3** (Same Normal Form). *Let $\mathcal{U} \subseteq \Lambda^\infty$.*

1. *$\lambda^\infty_{\beta\bot_\mathcal{U}}$ is confluent (normalizing) if and only if $\lambda^\infty_{\beta\bot_{\overline{\mathcal{U}}}}$ is confluent (normalizing).*
2. *Let $\lambda^\infty_{\beta\bot_\mathcal{U}}$ be confluent and normalizing. Then, $\mathsf{nf}_\mathcal{U} = \mathsf{nf}_{\overline{\mathcal{U}}}$.*

3. *Let* $\lambda^{\infty}_{\beta\perp_{\mathcal{U}_1}}$ *and* $\lambda^{\infty}_{\beta\perp_{\mathcal{U}_2}}$ *be confluent and normalizing. Then,* $\overline{\mathcal{U}_1} = \overline{\mathcal{U}_2}$ *iff* $\mathsf{nf}_{\mathcal{U}_1} = \mathsf{nf}_{\mathcal{U}_2}$.

We say that two models are equal, i.e. $\mathfrak{M}_{\mathcal{U}_1} = \mathfrak{M}_{\mathcal{U}_2}$, if they have the same domain and their interpretation functions are equal. As an immediate consequence of the previous corollary and Theorem 2.17, we have that $\mathcal{U}$ and $\overline{\mathcal{U}}$ define the same model:

▶ **Corollary 6.4** (Same Model). *Let* $\mathfrak{M}_{\mathcal{U}_1}, \mathfrak{M}_{\mathcal{U}_1} \in \mathbb{MOD}(\lambda)$. *Then,* $\overline{\mathcal{U}_1} = \overline{\mathcal{U}_2}$ *iff* $\mathfrak{M}_{\mathcal{U}_1} = \mathfrak{M}_{\mathcal{U}_2}$.

7 Confluence implies Normalization

In this section, we prove that if $\lambda^{\infty}_{\beta\perp_{\mathcal{U}}}$ is confluent then $\mathcal{U}$ satisfies the Axiom of Rootactiveness provided that $\mathcal{U}$ is the set of expansions of $\perp$. As a corollary, we conclude that confluence of $\lambda^{\infty}_{\beta\perp_{\mathcal{U}}}$ implies normalization of $\lambda^{\infty}_{\beta\perp_{\mathcal{U}}}$.

▶ **Definition 7.1.** For any $M \in \Lambda^{\infty}_{\perp}$, let $M^{\mathbf{I}}$ be the result of replacing every application PQ in M by $\mathbf{I}(PQ)$.

For example, $\mathbf{\Omega}^{\mathbf{I}} = \mathbf{I}((\lambda x.\mathbf{I}(xx))(\lambda x.\mathbf{I}(xx)))$.

▶ **Lemma 7.2.** *1.* $(P[x := Q])^{\mathbf{I}} = P^{\mathbf{I}}[x := Q^{\mathbf{I}}]$.
2. *If* $M \to_\beta N$ *then* $M^{\mathbf{I}} \twoheadrightarrow_\beta N^{\mathbf{I}}$.
3. *If* $M \twoheadrightarrow_\beta (\lambda x.P)Q$ *then* $M^{\mathbf{I}} \twoheadrightarrow_\beta \mathbf{I}(P[x := Q])^{\mathbf{I}}$.

Proof. We prove Part 2. Suppose $M = (\lambda x.P)Q \to_\beta P[x := Q]$. Using Part 1, we have that $M^{\mathbf{I}} = \mathbf{I}((\lambda x.P^{\mathbf{I}})Q^{\mathbf{I}}) \to_\beta \mathbf{I}(P^{\mathbf{I}}[x := Q^{\mathbf{I}}]) = \mathbf{I}(P[x := Q])^{\mathbf{I}} \to_\beta (P[x := Q])^{\mathbf{I}}$.

We prove Part 3. Suppose $M \twoheadrightarrow_\beta (\lambda x.P)Q$. Using Parts 1 and 2, we have $M^{\mathbf{I}} \twoheadrightarrow_\beta ((\lambda x.P)Q)^{\mathbf{I}} = \mathbf{I}((\lambda x.P^{\mathbf{I}})Q^{\mathbf{I}}) \to_\beta \mathbf{I}(P^{\mathbf{I}}[x := Q^{\mathbf{I}}]) = \mathbf{I}(P[x := Q])^{\mathbf{I}}$. ◀

▶ **Lemma 7.3.** *For any* $M \in \mathcal{R}$, $M^{\mathbf{I}}$ *reduces both to* M *and* $\mathbf{I}^{\omega}$.

Proof. It is easy to show that $M^{\mathbf{I}} \twoheadrightarrow\!\!\!\!\!\rightarrow_\beta M$ for all $M \in \Lambda^{\infty}_{\perp}$. Since M is rootactive, there is an infinite reduction starting for M containing infinitely many root reduction steps, i.e. $M \twoheadrightarrow_\beta (\lambda x.P_0)Q_0 \to_\beta P_0[x := Q_0] \twoheadrightarrow_\beta (\lambda x.P_1)Q_1 \to_\beta P_1[x := Q_2] \ldots$. Applying Lemma 7.2 Parts 2 and 3, we can construct the following reduction sequence.

$$M^{\mathbf{I}} \twoheadrightarrow_\beta ((\lambda x.P_0)Q_0)^{\mathbf{I}} \to_\beta \mathbf{I}(P_0[x := Q_0])^{\mathbf{I}} \twoheadrightarrow_\beta \mathbf{I}((\lambda x.P_1)Q_1)^{\mathbf{I}} \to_\beta \mathbf{I}(\mathbf{I}(P_1^{\mathbf{I}}[x := Q_1^{\mathbf{I}}]))\ldots$$

The limit of the above sequence is $\mathbf{I}^{\omega}$. ◀

▶ **Theorem 7.4** (Necessity of Rootactiveness for Confluence). *Let* $\mathcal{U} \subseteq \Lambda^{\infty}$ *satisfy Closure under* $\beta\perp$*-expansion from* $\perp$. *If* $\lambda^{\infty}_{\beta\perp_{\mathcal{U}}}$ *is confluent then* $\mathcal{U}$ *satisfies Rootactiveness.*

Proof. We prove that $\mathcal{U}$ satisfies the Axiom of Rootactiveness. By Lemma 7.3, $\mathbf{\Omega}^{\mathbf{I}} \twoheadrightarrow\!\!\!\!\!\rightarrow_\beta \mathbf{I}^{\omega}$ and $\mathbf{\Omega}^{\mathbf{I}} \twoheadrightarrow_\beta \mathbf{\Omega}$. Since $\lambda^{\infty}_{\beta\perp_{\mathcal{U}}}$ is confluent, there exists P such that $\mathbf{I}^{\omega} \twoheadrightarrow\!\!\!\!\!\rightarrow_{\beta\perp} P$ and $\mathbf{\Omega} \twoheadrightarrow\!\!\!\!\!\rightarrow_{\beta\perp} P$. Since $\mathbf{\Omega}$ only β-reduces to itself, we have that $\mathbf{\Omega} \to_\perp Q \twoheadrightarrow\!\!\!\!\!\rightarrow_{\beta\perp} P$. Hence, $\mathbf{\Omega} = C[M] \to_\perp C[\perp] = Q$ for $M \in \mathcal{U}$. Suppose M is a proper subterm of Ω. We have the following cases.

1. Case $M = x$. Then $x[x := P] \to_\perp \perp$ and $P \in \mathcal{U}$ for all $P \in \Lambda^{\infty}$. In particular, $\mathbf{\Omega} \in \mathcal{U}$.
2. Case $M = xx$. Then $xx[x := \lambda x.xx] \to_\perp \perp$. Hence, $\mathbf{\Omega} \in \mathcal{U}$.
3. Case $M = \lambda x.xx$. Hence $\mathbf{\Omega} \twoheadrightarrow_\perp \perp\perp$ and also $\mathbf{I}^{\omega} \twoheadrightarrow\!\!\!\!\!\rightarrow_{\beta\perp} \perp\perp$. Since $\mathbf{I}^{\omega}$ can only β-reduce to itself, $\mathbf{I}^{\omega} = C'[N] \to_\perp C[\perp] \twoheadrightarrow\!\!\!\!\!\rightarrow_{\beta\perp} \perp\perp$. Suppose N is a proper subterm of $\mathbf{I}^{\omega}$. There are two possibilities:
   a. Case $N = \mathbf{I}^{\omega}$. Then $\mathbf{\Omega} \twoheadrightarrow\!\!\!\!\!\rightarrow_{\beta\perp} \perp$ and $\mathbf{\Omega} \in \mathcal{U}$.

b. Case $N = \mathbf{I}$. Then, $\mathbf{I}^\omega \twoheadrightarrow_\bot \bot^\omega = \bot(\bot(\bot\ldots))$. On the other hand, $\mathbf{I}^\omega \twoheadrightarrow_{\beta\bot} \bot\bot$. This is possible only if $\bot^\omega \to_\bot \bot$. Hence, $\mathbf{\Omega} \to_\bot \bot$ and $\mathbf{\Omega} \in \mathcal{U}$.

Hence, $Q = \bot = P$ and also $\mathbf{I}^\omega \twoheadrightarrow_{\beta\bot} \bot$. By Lemma 7.3, for any $M \in \mathcal{R}$, $M^{\mathbf{I}} \twoheadrightarrow_\beta M$ and $M \twoheadrightarrow_\beta \mathbf{I}^\omega$. Since $\lambda^\infty_{\beta\bot_\mathcal{U}}$ is confluent and $\mathbf{I}^\omega \to_\bot \bot$ we have $M \twoheadrightarrow_{\beta\bot} \bot$. Since $\mathcal{U}$ is the set of expansions of $\bot$, we have $M \in \mathcal{U}$. ◀

▶ **Corollary 7.5** (Confluence implies Normalization). *If $\lambda^\infty_{\beta\bot_\mathcal{U}}$ is confluent then $\lambda^\infty_{\beta\bot_\mathcal{U}}$ is normalizing.*

Proof. Let $\lambda^\infty_{\beta\bot_\mathcal{U}}$ be confluent. By Corollary 6.3, $\lambda^\infty_{\beta\bot_{\overline{\mathcal{U}}}}$ is confluent. By Theorem 7.4, $\overline{\mathcal{U}}$ satisfies Rootactiveness. By Theorem 2.15 and Corollary 6.3 $\lambda^\infty_{\beta\bot_{\overline{\mathcal{U}}}}$ and $\lambda^\infty_{\beta\bot_\mathcal{U}}$ are normalizing. ◀

As a consequence of the previous corollary, if the infinitary lambda calculus $\lambda^\infty_{\beta\bot_\mathcal{U}}$ is confluent then it induces a λ-model (see Theorem 2.17).

8 Confluence implies Weak Meaninglessness

In Section 5, we proved that if $\mathcal{U}$ is a set of weak meaningless terms then $\lambda^\infty_{\beta\bot}$ is confluent (Theorem 5.6). In this section, we study whether the converse holds. We will prove that confluence of $\lambda^\infty_{\beta\bot_\mathcal{U}}$ implies that $\overline{\mathcal{U}}$ is a set of weak meaningless terms. In other words, if $\lambda^\infty_{\beta\bot_\mathcal{U}}$ is confluent then there exists a set $\mathcal{U}'$ of weak meaningless terms that defines the same reduction as $\mathcal{U}$.

▶ **Theorem 8.1** (Necessity of Weak Meaninglessness for Confluence I). *Let $\mathcal{U} \subseteq \Lambda^\infty$ satisfy Closure under $\beta\bot$-expansion from $\bot$. If $\lambda^\infty_{\beta\bot_\mathcal{U}}$ is confluent then $\mathcal{U}$ is a set of weak meaningless terms.*

Proof. Suppose $\lambda^\infty_{\beta\bot}$ is confluent. Rootactiveness of $\mathcal{U}$ follows from Theorem 7.4. We prove that $\mathcal{U}$ satisfies the remaining axioms:

We prove that $\mathcal{U}$ satisfies Indiscernibility. Suppose $M \overset{\mathcal{U}}{\longleftrightarrow} N$. It is not difficult to show that there exists P such that $M \twoheadrightarrow_\bot P$ and $N \twoheadrightarrow_\bot P$. If $M \in \mathcal{U}$ then $M \to_\bot \bot$. Since $\lambda^\infty_\bot$ is confluent, we have $N \twoheadrightarrow_{\beta\bot} \bot$. By Closure under $\beta\bot$-expansion from $\bot$, we get $N \in \mathcal{U}$.

We prove that $\mathcal{U}$ satisfies Closure under Substitution. Let $P \in \mathcal{U}$ and $Q \in \Lambda^\infty$. We will prove $P[x := Q] \in \mathcal{U}$. Since $P \in \mathcal{U}$, we have $(\lambda x.P)Q \to_\bot (\lambda x.\bot)Q \to_\beta \bot$. We also have $(\lambda x.P)Q \to_\beta P[x := Q]$. Since $\lambda^\infty_{\beta\bot_\mathcal{U}}$ is confluent, $P[x := Q] \twoheadrightarrow_{\beta\bot} \bot$. By Closure under $\beta\bot$-expansion from $\bot$, we have $P[x := Q] \in \mathcal{U}$.

We prove that $\mathcal{U}$ satisfies Closure under β-reduction. If $M \twoheadrightarrow_\beta N$ and $M \in \mathcal{U}$ then $M \to_\bot \bot$. By Confluence, $N \twoheadrightarrow_{\beta\bot} \bot$. By Closure under $\beta\bot$-expansion from $\bot$, we find $N \in \mathcal{U}$.

Finally, we prove that $\mathcal{U}$ satisfies Weak Overlap. If $\lambda x.P \in \mathcal{U}$ then $(\lambda x.P)x \to_\bot \bot x$ and $(\lambda x.P)x \to_\beta P$. Since $\lambda^\infty_{\beta\bot_\mathcal{U}}$ is confluent, there exists N such that $P \twoheadrightarrow_{\beta\bot} N$ and $\bot x \twoheadrightarrow_{\beta\bot} N$. We have two possibilities:

1. $N = \bot x$. Then $P \twoheadrightarrow_{\beta\bot} \bot x$. By Theorem 5.2, we have that $P \twoheadrightarrow_\beta P' \twoheadrightarrow_\bot \bot x$ for some $P' \in \Lambda^\infty_\bot$. Then, $P' = Wx$ and $W \twoheadrightarrow_\bot \bot$. By Closure under $\beta\bot$-expansion from $\bot$, we have $W \in \mathcal{U}$. Trivially, $W \in \Lambda^\infty$ because $P \in \Lambda^\infty$ and $P \twoheadrightarrow_\beta Wx$.
2. $N = \bot$. Then, $(\lambda x.P)x \to_\bot P \twoheadrightarrow_{\beta\bot} \bot$. By Closure under $\beta\bot$-expansion from $\bot$, we have that $(\lambda x.P)x \in \mathcal{U}$. By Closure under Substitutions, $(\lambda x.P)Q \in \mathcal{U}$ for all $Q \in \Lambda^\infty$. ◀

▶ **Corollary 8.2** (Necessity of Weak Meaninglessness for Confluence II)**.** *If* $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$ *is confluent then there exists a set* $\mathcal{U}'$ *of weak meaningless terms that defines the same reduction as* $\mathcal{U}$*.*

Proof. By Theorem 6.2, $\mathcal{U}$ and $\overline{\mathcal{U}}$ define the same reduction. By Corollary 6.3, $\lambda^{\infty}_{\beta\bot_{\overline{\mathcal{U}}}}$ is confluent. By Theorem 8.1, $\overline{\mathcal{U}}$ is a set of weak meaningless terms. ◀

The following corollary can also be proved directly.

▶ **Corollary 8.3.** *If* $\mathcal{U}$ *is a set of weak meaningless terms then so is* $\overline{\mathcal{U}}$*.*

Proof. Let $\mathcal{U}$ be a set of weak meaningless terms. By Corollary 5.7, $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$ is confluent and normalizing. By Corollary 6.3, we have that $\lambda^{\infty}_{\beta\bot_{\overline{\mathcal{U}}}}$ is confluent and normalizing. By Theorem 8.1, $\overline{\mathcal{U}}$ is a set of weak meaningless terms. ◀

▶ **Corollary 8.4.** $\mathbb{MOD}(\lambda) = \{\mathfrak{M}_{\mathcal{U}} \mid \mathcal{U} \in \mathbb{WM} \cap \mathbb{B}\} = \{\mathfrak{M}_{\mathcal{U}} \mid \mathcal{U} \in \mathbb{WM}\}$.

Proof. We first prove $\mathbb{MOD}(\lambda) \subseteq \{\mathfrak{M}_{\mathcal{U}} \mid \mathcal{U} \in \mathbb{WM} \cap \mathbb{B}\}$. Let $\mathfrak{M}_{\mathcal{U}} \in \mathbb{MOD}(\lambda)$. By Corollary 6.3, if $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$ is confluent and normalizing, so is $\lambda^{\infty}_{\beta\bot_{\overline{\mathcal{U}}}}$. By Corollary 6.4, $\mathfrak{M}_{\mathcal{U}} = \mathfrak{M}_{\overline{\mathcal{U}}}$. By Theorem 8.1, $\overline{\mathcal{U}} \in \mathbb{WM} \cap \mathbb{B}$. Hence, $\mathfrak{M}_{\mathcal{U}} = \mathfrak{M}_{\overline{\mathcal{U}}} \in \{\mathfrak{M}_{\mathcal{U}} \mid \mathcal{U} \in \mathbb{WM} \cap \mathbb{B}\}$.

It is trivial to see that $\{\mathfrak{M}_{\mathcal{U}} \mid \mathcal{U} \in \mathbb{WM} \cap \mathbb{B}\} \subseteq \{\mathfrak{M}_{\mathcal{U}} \mid \mathcal{U} \in \mathbb{WM}\}$. The inclusion $\{\mathfrak{M}_{\mathcal{U}} \mid \mathcal{U} \in \mathbb{WM}\} \subseteq \mathbb{MOD}(\lambda)$ follows from Corollary 5.7. ◀

▶ **Corollary 8.5.** *There is a bijection from the set* $\mathbb{WM} \cap \mathbb{B}$ *to* $\mathbb{MOD}(\lambda)$*.*

Proof. Let $\mathcal{U} \in \mathbb{WM} \cap \mathbb{B}$. By Corollary 5.7, the infinitary lambda calculus $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$ is confluent and normalizing. Hence, we can consider the mapping that given $\mathcal{U} \in \mathbb{WM} \cap \mathbb{B}$ yields $\mathfrak{M}_{\mathcal{U}}$. This mapping is surjective by Corollary 8.4 and it is injective by Corollary 6.4. ◀

9 Conclusions and Future Research

In this paper, we have weakend the Axiom of Overlap in order to find an axiomatization that is both necessary and sufficient for having confluent and normalizing infinitary lambda calculi $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$.

In a natural sequel to this paper we plan to study the same question for first order term rewriting. Afteral the axioms of meaningless sets (minus substitution) were first formulated for such systems [2, 11]. If successful a generalisation to combinatory reduction systems (extending [12]) may then well be possible.

The sets shown in Figure 1 are not the only sets of weak meaningless terms. We also plan to study the structure of the set $\mathbb{WM}$ of sets of weak meaningless terms closed under β expansion and provide an exhaustive classification if possible.

One reason that this set is of interest is that each such weak meaningless set gives rise to its own model of the infinitary lambda calculus, which in turn defines a finitary lambda theory. We are hopeful that the set of weakly meaningless sets is in fact a lattice. And it is of interest to explore the relation with the well-studied lattice of lambda theories.

Finally it is of interest to see how other denotational semantics can model the infinitary lambda caluli. Or to see whether each of the infinite lambda calculi $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$ can be provided with an intersection type discipline sucht that two terms have the same normal form if and only if the have the same type.

Acknowledgements

We would like to thank the reviewers for their detailed and helpful comments and suggestions that they provided.

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