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An Infinitary Church-Rosser Property for Non-collapsing Orthogonal Term Rewriting Systems

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Abstract. For non-collapsing, orthogonal, infinitary term rewriting systems we prove an unconditioned infinitary Church-Rosser property for strongly converging reductions. The proof method extends to orthogonal term rewrite systems that contain non-collapsing rules together with only one collaps rule of the form $I(x) \to x$.

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1. INTRODUCTORY REMARKS ON INFINITARY TERM REWRITING

There are at least two good reasons to study Infinitary Term Rewriting. First, we believe that Infinitary Term Rewriting is of interest for its own sake, as natural extension of Finitary Term Rewriting. Secondly, Infinitary Term Rewriting provides a sound and thorough basis for Graph Rewriting, the theoretical model for implementations of functional programming languages.

Term Rewriting is a general model of computation. Computations can be finite and infinite. The usual focus is on successful finite computations: finite derivations ending in finite normal form. However, infinite computations computing a possible infinite answer are of interest as well: recursive procedures enumerating some infinite set: e.g. the natural numbers or the Fibonacci numbers. Until recently, infinite computations have hardly seriously been considered in the theory of Term Rewriting.

In functional programming languages like Miranda™ or ML it is possible to manipulate with lazy expressions representing infinite objects, like lists. Graph Rewriting has been introduced as a theoretical framework to show the soundness of such computing. Infinitary Term Rewriting is a foundation for Graph Rewriting: some instances of graph rewriting on shared graphs actually represent infinite computations on infinite terms.

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At present the theory of infinitary rewriting for orthogonal term rewriting systems (OTRS) is rapidly emerging in a series of papers. Dershowitz, Kaplan and Plaisted have opened the series with [Der89a,b, Der90a] taking a rather topological approach, resulting in notions of (among others) Cauchy convergence and ω-normal form. Top-terminating OTRSs play a central role in their papers: for top-terminating OTRSs they prove properties implying an infinitary Church-Rosser property and they show transfinite long reductions are compressable in reductions of at most length ω. Farmer and Watro [Far89] observed the necessity of strong convergence for some instances of compressing and pointed out the link with Graph Rewriting.

In [Ken90a,b] the theory of infinite term rewriting has been given a firm base concentrating on strong convergence together with normal forms, but not ignoring Cauchy convergence and ω-normal forms. For the theory involving strong convergence the transfinite Parallel Moves Lemma, the Compressing Lemma and the Unique Normal Form Property are provable, whereas counterexamples for these results can be given in the case of Cauchy convergence and ω-normal forms. A general infinitary Church-Rosser property can not hold for either theory as counterexamples in [Ken90a,b] show. These counterexamples concern an OTRS with two collapsing rules containing one variable or an OTRS with one collaps rule containing two variables. However, also in in [Ken90a,b] for OTRSs that are non-unifiable an infinite Church-Rosser property is proved for Cauchy converging reductions. See Section 2 for some details.

It is the purpose of the present paper to show that for strongly converging reductions the unconditioned infinitary Church-Rosser property holds in non-collapsing OTRSs, improving the results in [Der90a] and [Ken90b]. The proof method extends to orthogonal term rewrite systems that contain non-collapsing rules together with only one collaps rule of the form $I(x) \rightarrow x$.

Overview of this paper: In the next section 2 we briefly introduce infinitary term rewriting. In section 3 we define depth preserving left-linear term rewrite systems and prove that the infinite Church-Rosser property for strongly converging sequences holds for such term rewrite systems. Using a variant of Park's notion of hiaton we show in section 4 that any OTRS can be transformed into an depth-preserving OTRS, the so called ε-completion. Exploiting the properties of the ε-completions of non-collapsing OTRSs we finally prove the unconditioned infinitary Church-Rosser property for non-collapsing OTRSs in section 5.

2. Infinitary Orthogonal Term Rewriting Systems

We briefly recall the definition of a finitary term rewriting system, before we define infinitary orthogonal term rewriting systems involving both finite and infinite terms. For more details the reader is referred to [Der90b], [Klo90] and [Klo91].

2.1. Finitary term rewriting systems

A *finitary term rewriting system* over a signature Σ is a pair $(Ter(\Sigma),R)$ consisting of the set $Ter(\Sigma)$ of finite terms over the signature Σ and a set of rewrite rules $R \subseteq Ter(\Sigma)\times Ter(\Sigma)$.

The *signature* Σ consists of a countably infinite set Var_Σ of variables (x,y,z,...) and a non-empty set of function symbols (A,B,C,...,F,G,...) of various finite arities ≥ 0. Constants are function symbols with arity 0. The set $Ter(\Sigma)$ of *finite* terms (t,s,...) over Σ can be defined as usual: the smallest set containing the variables and closed under function application.

The set O(t) of *occurrences* (or positions) in t is defined by induction to the structure of t as follows: $O(t) = \{< >\}$ if t is a variable and $O(t) = \{< >\} \cup \{<i,u> \mid 1 \leq i \leq n \text{ and } <u> \in O(t_i)\}$ if t is of the form $F(t_1,...,t_n)$. If $u \in O(t)$ then the subterm t/u at occurrence u is defined as follows: $t/< > = t$ and $F(t_1,...,t_n)/<i,u> = t_i/u$. The *depth* of a subterm of t at occurrence u is the length of u.

Contexts are terms in $Ter(\Sigma \cup \{\square\})$, in which the special constant $\square$, denoting an empty place, occurs exactly once. Contexts are denoted by C[ ] and the result of substituting a term t in place of $\square$ is $C[t] \in Ter(\Sigma)$. A *proper* context is a context not equal to $\square$.

Substitutions are maps $\sigma: Var_\Sigma \to Ter(\Sigma)$ satisfying $\sigma(F(t_1,...,t_n)) = F(\sigma(t_1),...,\sigma(t_n))$.

The set R of *rewrite rules* contains pairs (l,r) of terms in $Ter(\Sigma)$, written as $l \to r$, such that the left-hand side l is not a variable and the variables of the right-hand side r are contained in l. The result l^σ of the application of the substitution of σ to the term l is called an instance of l. A *redex* (reducible expression) is an instance of a left-hand side of a rewrite rule. A *reduction step* $t \to s$ is a pair of terms of the form $C[l^\sigma] \to C[r^\sigma]$, where $l \to r$ is a rewrite rule in R. Concatenating reduction steps we get a *finite reduction sequence* $t_0 \to t_1 \to ... \to t_n$, which we also denote by $t_0 \to_n t_n$, or an infinite reduction sequence $t_0 \to t_1 \to ...$.

2.2. Infinitary orthogonal term rewriting systems

An *infinitary term rewriting system* over a signature Σ is a pair $(Ter^\infty(\Sigma),R)$ consisting of the set $Ter^\infty(\Sigma)$ of finite and infinite terms over the signature Σ and a set of rewrite rules $R \subseteq Ter(\Sigma) \times Ter(\Sigma)$. It takes some elaboration to define the set $Ter^\infty(\Sigma)$ of *finite and infinite terms.* Finite terms may be represented as finite trees, well-labelled with variables and function symbols. Well-labelled means that a node with $n \geq 1$ successors is labelled with a function symbol of arity n and that a node with no successors is labelled either with a constant or a variable. Now *infinite terms* are infinite well-labelled trees with nodes at finite distance to the root. Substitutions, contexts and reduction steps generalize trivially to the set of infinitary terms $Ter^\infty(\Sigma)$.

To introduce the *prefix ordering* $\leq$ on terms we extend the signature Σ with a fresh symbol Ω. The prefix ordering $\leq$ on $Ter^\infty(\Sigma \cup \{\Omega\})$ is defined inductively: $x \leq x$ for any variable x, $\Omega \leq t$ for any term t and if $t_1 \leq s_1, ..., t_n \leq s_n$ then $F(t_1,...,t_n) \leq F(s_1,...,s_n)$.

If all function symbols of Σ occur in R we will write just R for $(Ter^\infty(\Sigma),R)$. The usual properties for finitary TRSs extend verbatim to infinitary TRSs:

2.2.1. DEFINITION. Let R be an infinitary TRS.

(i) R is *left-linear* if no variable occurs more than once in a left-hand side of R's rewrite rules;

(ii) (informally) R is *non-overlapping* (or non-ambiguous) if non-variable parts of different rewrite rules don't overlap and non-variable parts of the same rewrite rule overlap only entirely:

(ii') (formally) R is *non-overlapping* if for any two left hand sides s and t, any occurrence u in t, and any substitutions σ and $\tau: Var_\Sigma \to Ter(\Sigma)$ it holds that if $(t/u)^\sigma = s^\tau$ then either t/u is a variable or t and s are left hand sides of the same rewrite rule and u is the empty occurrence < >, the position of the root.

(iii) R is *orthogonal* if R is both left-linear and non-overlapping.

It is well-known (cf. [Ros73], [Klo91]) that finitary orthogonal TRSs satisfy the finitary Church-Rosser property, i.e., ${}^*\!\!\leftarrow \circ \rightarrow^* \subseteq \rightarrow^* \circ {}^*\!\!\leftarrow$, where $\rightarrow^*$ is the transitive, reflexive closure of the relation $\rightarrow$. It is obvious that infinitary orthogonal TRSs inherit this finitary property.

In the present infinitary context it is natural to define that a term is a *normal form* if it contains no redexes, just like in the finitary context. A term t has a normal form s if there is a reduction t $\rightarrow_\alpha$ s. Dershowitz, Kaplan and Plaisted [Der89a, Der89b and Der90b] consider a weaker, more liberal notion of normal form: the *ω-normal forms*. An ω-normal form is a term such that if this term can reduce, then it reduces in one step to itself. One sees easily that restricted to finite terms normal forms and ω-normal forms are already different concepts: in the TRS with rule A $\rightarrow$ A the term A is an ω-normal form, but not a normal form.

2.3. Infinitary reductions and the infinitary Church-Rosser property

The set Ter(Σ) of finite terms for a signature Σ can be provided with an ultra-metric d: Ter(Σ)×Ter(Σ) $\rightarrow$ [0,1] (cf. e.g. [Arn80]). The distance d(t,s) of two terms t and s is 0 if t and s are equal, and otherwise 2^{-k}, where k∈ N is the largest number such that the labels of all nodes of s and t at depth less than or equal to k are equally labelled. The metric completion of Ter(Σ) is isomorphic to the set of infinitary terms $Ter^\infty(\Sigma)$ (cf. [Arn80])

In the complete metric space $Ter^\infty(\Sigma)$ all Cauchy sequences of ordinal length α have a limit. We will now recall the transfinite converging reductions by Dershowitz, Kaplan and Plaisted [Der90b].

2.3.1. DEFINITION. A *reduction of ordinal length* α is a set $(t_\beta)_{\beta<\alpha}$ of terms indexed by the ordinal α such that $t_\beta \rightarrow t_{\beta+1}$ for each β+1<α.

2.3.2. DEFINITION. By induction to the ordinal α we define when a reduction $(t_\beta)_{\beta\leq\alpha}$ is *converging* towards the limit t_α (notation: $t_0 \overset{c}{\rightarrow}_\alpha t_\alpha$):

(i) $t_0 \overset{c}{\rightarrow}_0 t_0$;

(ii) $t_0 \overset{c}{\rightarrow}_{\beta+1} t_{\beta+1}$ if $t_0 \overset{c}{\rightarrow}_\beta t_\beta$;

(iii) $t_0 \overset{c}{\rightarrow}_\lambda t_\lambda$ if $t_0 \overset{c}{\rightarrow}_\beta t_\beta$ for all β<λ and $\forall\varepsilon>0\, \exists\beta<\lambda\, \forall\gamma\, (\beta<\gamma<\lambda \rightarrow d(t_\gamma,t_\lambda) < \varepsilon)$.

By $t \overset{c}{\rightarrow}_{\leq\alpha} s$ we denote the existence of a converging reduction of length less than or equal to α.

This definition of transfinite convergence is an instance of so-called Moore-Smith convergence over nets (cf. for instance [Kel55]). If the topological space is a Hausdorff space like in the case of Ter(Σ) and $Ter^\infty(\Sigma)$ then each net in the space converges to atmost one point.

Converging reductions are not well behaved for even orthogonal TRSs, as shown in [Ken90a,b]: transfinite converging reductions resist against compression into converging reductions of length at most ω+1; the generalisation of the finite Parallel Moves Lemma fails (cf. 2.3.3); the infinite Church-Rosser property does not hold (cf. 2.3.8).

2.3.3. COUNTEREXAMPLE [Ken90a,b]. Against an infinite Parallel Moves Lemma for Cauchy convergence:

Rules: A(x,y) $\rightarrow$ A(y,x), C $\rightarrow$ D

Sequences: $A(C,C) \to A(C,C) \to A(C,C) \to A(C,C) \to \ldots \to_\omega A(C,C)$

$A(C,D) \to A(D,C) \to A(C,D) \to A(D,C) \to \ldots$

Clearly $A(C,C) \to^c_\omega A(C,C)$. The second infinite reduction obtained by standard projection over the one step reduction $C \to D$ is not a converging reduction, and hence has no limit.

The stronger notion of a *strongly converging reduction* which generalizes Farmer and Watro [Far89] has better properties: the full Compressing Lemma and infinite Parallel Moves Lemma hold for orthogonal TRSs as shown in [Ken90a,b]. In a strongly convergent reduction it holds that for every depth d, there is some point in the reduction after which all contractions are performed at depth larger than d. We present the definition of strongly convergence for transfinitely long reductions:

2.3.4. DEFINITION. By induction to the ordinal $\alpha \geq 1$ we will define that a converging reduction $(t_\beta)_{\beta<\alpha}$ is a transfinite *strongly converging reduction*. Let d_β denotes the *depth* of the contracted redex R_β in $t_\beta \to t_{\beta+1}$.

(i) $(t_0)_{0<1} = (t_\beta)_{\beta<1}$ is strongly converging;

(ii) if $(t_\gamma)_{\gamma<\beta}$ is strongly converging and $t_\beta \to t_{\beta+1}$, then $(t_\gamma)_{\gamma<\beta+1}$ is strongly converging;

(iii) if λ is a limit ordinal, $(t_\gamma)_{\gamma<\beta}$ is strongly converging for all $\beta<\lambda$ and $\forall d>0\ \exists \beta<\lambda\ \forall \gamma\ (\beta<\gamma<\lambda \to d_\gamma>d)$, then $(t_\gamma)_{\gamma<\lambda}$ is strongly converging.

If $(t_\beta)_{\beta<\alpha+1}$ is a transfinite strongly converging reduction, then we say that t_α is the limit of $(t_\beta)_{\beta<\alpha}$; notational short hand: $t_0 \to_\alpha t_\alpha$. By $t \to_{\leq\alpha} s$ we denote the existence of a strongly converging reduction of length less than or equal to α.

The following lemma of Farmer and Watro will be useful. It provides a sufficient and necessary condition when an infinite sequence of strongly converging reductions of length $\omega+1$ itself is strongly converging.

2.3.5. LEMMA [Far89]. *Let* $t_{n,0} \to_{\leq\omega} t_{n,\omega} = t_{n+1,0}$ *be strongly converging for all* $n \in \mathbb{N}$. *Let* $d_{n,k}$ *denotes the depth of the contracted redex* $R_{n,k}$ *in* $t_{n,k} \to t_{n,k+1}$. *If for all* n *there is a* d_n *such that for all* k *it holds that* $d_{n,k} > d_n$, *and* $\lim d_k = \infty$, *then there exists a term* $t_{\omega,\omega}$ *such that* $t_{0,0} \to_{\omega\omega} t_{\omega,\omega}$ *via the strongly converging reduction* $t_{0,0} \to_{\leq\omega} t_{0,\omega} = t_{1,0} \to_{\leq\omega} t_{1,\omega} = t_{2,0} \to_{\leq\omega} \ldots \to_{\omega\omega} t_{\omega,\omega}$.

In order to state the infinite Parallel Moves Lemma for strongly convergent reductions as proved in [Ken90a,b] we recall the notion of descendant.

2.3.6. DEFINITION. The set of occurrences $v\backslash R$ that consists of the *descendants* of S after contraction of R is defined relative to position v of S with respect to R.

(i) S and R are disjoint. Then $v\backslash R = \{v\}$;

(ii) S and R are identical or S is above R. Then $v\backslash R = \varnothing$;

(iii) R is above S, i.e. $v = uwv'$ for some $w \in O(R)$ and v'.

Then $v\backslash R = \{uw'v' \mid$ variable in l at w and variable in r at w' are the same$\}$.

2.3.7. INFINITE PARALLEL MOVES LEMMA for strongly convergent reductions [Ken90a,b].
Let $(t_n)_{n\in\mathbb{N}}$ *be a stronglyy converging reduction of* t_0 *with limit* t_ω *and let* $t_0 \to s_0$ *be a reduction of*

a redex R *of* t_0. *Then there is a strongly converging reduction* $(s_n)_{n\in A}$ *with limit* s_ω, *where for all* $n\in \mathbb{N}\cup\{\omega\}$, s_n *is obtained by contraction of all descendants of* R *in* t_n.

Strongly converging (and hence converging) reductions generally don't satisfy the infinite Church-Rosser property for orthogonal TRSs, despite the infinite Parallel Moves Lemma for strongly converging reductions. To be precise a TRS has the *infinite Church-Rosser* property w.r.t. strong convergence if ${}_\omega\!\leftarrow \circ \rightarrow_\omega \subseteq \rightarrow_{\leq\omega} \circ {}_{\leq\omega}\!\leftarrow$. The following counterexample is taken from [Ken90a,b]:

2.3.8. COUNTEREXAMPLE I. Against the infinite Church-Rosser property for strong convergence:

Rules: $A(x) \rightarrow x$, $B(x) \rightarrow x$, $C \rightarrow A(B(x))$

Sequences: $C \rightarrow A(B(C)) \rightarrow A(C) \rightarrow A(A(B(C))) \rightarrow A(A(C)) \rightarrow_\omega A^\omega$

$C \rightarrow A(B(C)) \rightarrow B(C) \rightarrow B(A(B(C))) \rightarrow B(B(C)) \rightarrow_\omega B^\omega$

Hence $C \rightarrow_{\leq\omega} A^\omega$ as well as $C \rightarrow_{\leq\omega} B^\omega$. But there is no term t such that $A^\omega \rightarrow_{\leq\omega} t \leftarrow_{\leq\omega} B^\omega$ be it Cauchy or strongly converging.

2.3.9. COUNTEREXAMPLE II. Against the infinite Church-Rosser property for strong convergence:

Rules: $D(x,y) \rightarrow x$, $C \rightarrow D(A,D(B,C))$

Sequences: $C \rightarrow D(A,D(B,C)) \rightarrow D(A,C) \rightarrow^* D(A,D(A,C)) \rightarrow^* D(A,D(A,D(A,C))) \rightarrow ...$

$C \rightarrow D(A,D(B,C)) \rightarrow D(B,C) \rightarrow^* D(B,D(B,C)) \rightarrow^* D(B,D(B,D(B,C))) \rightarrow ...$

Clearly the limits of both reductions cannot be joint by either Cauchy converging or strong converging reductions.

From the work of Dershowitz, Plaisted and Kaplan on convergent reductions it follows that any left-linear, top-terminating and semi-ω-confluent TRS satisfies the infinite Church-Rosser property:

$$ {}_\omega\!\overset{c}{\leftarrow} \circ \overset{c}{\rightarrow}_\omega \subseteq \overset{c}{\rightarrow}_{\leq\omega} \circ {}_{\leq\omega}\!\overset{c}{\leftarrow} $$

(cf. [Der90a]: combine Theorem 1, Proposition 2 with Theorem 9.). A TRS is *top-terminating* if there are no top-terminating reductions of length ω, that is reductions with infinitely many rewrites at the root of the initial term of the reduction. Semi-ω-confluency, that is

$$ {}^*\!\leftarrow \circ \overset{c}{\rightarrow}_\omega \subseteq \overset{c}{\rightarrow}_{\leq\omega} \circ {}_{\leq\omega}\!\overset{c}{\leftarrow}, $$

holds if the Transfinite Parallel Moves Lemma holds for converging reductions. On the assumption that we are in a orthogonal TRS in which all convergent reductions are strong the infinite Church-Rosser Property holds for this TRS. Top-termination implies this assumption.

Hence in top-terminating orthogonal TRSs the infinite Church-Rosser Property holds. In [Ken90a,b] this result has been a bit improved using the following syntactic equivalent of the previous assumption. We recall without proof:

2.3.10. DEFINITION [Ken90a,b]. A TRS is called *unifiable* if it contains a *unifiable* rule, that is a rule $l \to r$ such that for some substitution σ with finite and infinite terms for variables $l^\sigma = r^\sigma$.

Note that unifiability in the space of finite and infinite terms means unifiability "without the occurs check": the terms I(x) and x are unifiable in this setting, and their most general unifier is the infinite term I^ω. Collapsing rules, i.e. rules which right hand side is a variable are unifiable.

2.3.11. LEMMA [Ken90a,b]. *The following are equivalent for an orthogonal TRS:*

(i) *the TRS is non-unifiable,*

(ii) *all convergent reductions of the TRS are strong convergent,*

(iii) *all convergent reductions are top-terminating.*

2.3.12. THEOREM [Ken90a,b]. *Any non-unifiable orthogonal TRS has the infinite Church-Rosser Property for (strongly) converging reductions.*

3. Depth preserving orthogonal term rewriting systems

In [Ken90a,b] it has been shown that for orthogonal TRSs the infinite Church-Rosser property holds when restricted to terms that can be reduced to a normal form, i.e., in the infinitary setting via a possibly infinite reduction to a possibly infinite term having no redexes. In the present section and the next we consider two natural classes of orthogonal TRSs for which the infinite Church-Rosser property hold without extra conditions for strongly convergent sequences.

3.1. DEFINITION. A *depth preserving* TRS is a left linear TRS such that for all rules the depth of any variable in a right-hand side is greater than or equal to the depth of the same variable in the corresponding left-hand side.

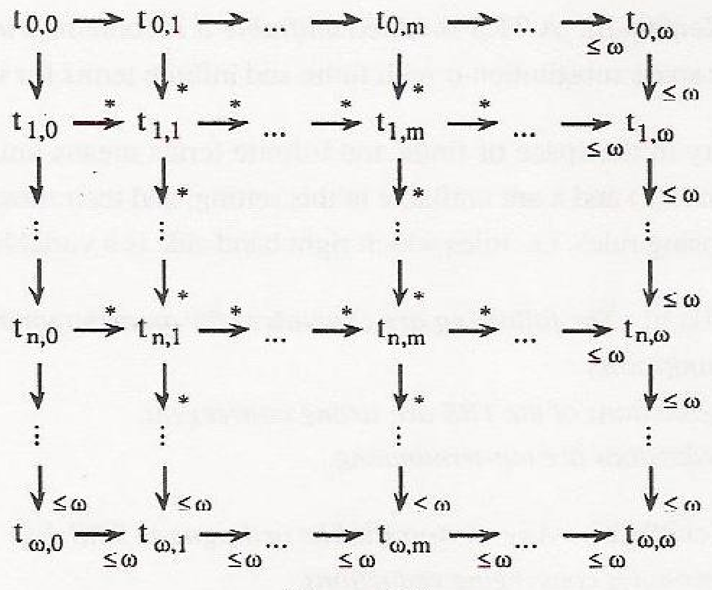
3.2. LEMMA. *Depth preserving TRS are distance preserving in the following sense: Let* $l \to r$ *be a depth-preserving rule. Then for all contexts* C[ ], *all* $t_1,\dots,t_n$ *and* $s_1,\dots,s_n$ *it holds that*

$$d(C[l(t_1,\dots,t_n)], C[l(s_1,\dots,s_n)]) \geq d(C[r(t_1,\dots,t_n)], C[r(s_1,\dots,s_n)]). \quad \square$$

3.3. THEOREM. *Any depth preserving orthogonal TRS has the infinite Church Rosser Property for strongly converging sequences.*

PROOF. Let $t_{0,0} \to t_{0,1} \to \dots \to_{\leq\omega} t_{0,\omega}$ and $t_{0,0} \to t_{1,0} \to \dots \to_{\leq\omega} t_{\omega,0}$ be strongly convergent.

(i) Using the infinite Parallel Moves Lemma for strongly convergent reductions we construct the horizontal strongly converging sequences $t_{n,0} \to^* t_{n,1} \to^* \dots \to_{\leq\omega} t_{n,\omega}$ as depicted in Figure 3.1. The vertical reductions are constructed similarly.



(figure 3.1)

(ii) The construction of the infinite Parallel Moves Lemma also implies that the reduction $t_{n,\omega} \to_{\leq\omega} t_{n+1,\omega}$ is strongly converging.

(iii) By the depth preserving property it holds for all $m,n \in \mathrm{N} \cup \{\omega\}$ the depth of the reduced redexes in $t_{n,m} \to^* t_{n,m+1}$, which are all descendants of the redex $R_{0,m}$ in $t_{0,m} \to t_{0,m+1}$, is at least the depth of $R_{0,m}$ itself. Because $t_{0,0} \to t_{0,1} \to \ldots \to_{\leq\omega} t_{0,\omega}$ is strongly convergent we find by Lemma 2.3.5 that $t_{\omega,0} \to_{\leq\omega} t_{\omega,1} \to_{\leq\omega} t_{\omega,2}$ ... is strongly converging. Let us call its limit $t_{\omega,\omega}$.

(iv) In the same way the terms $t_{n,\omega}$ are part of a strongly converging sequence. The limit of this sequence is also equal to $t_{\omega,\omega}$, as can be seen with the following argument.

Let $\varepsilon > 0$. There is N_1 such that for all $m \geq N_1$ we have $d(t_{\omega,m},t_{\omega,\omega}) < \frac{1}{3}\varepsilon$.

Because of the strong convergence of $t_{0,0} \to t_{1,0} \to \ldots \to_{\leq\omega} t_{\omega,0}$ there is an N_2 such that for $n \geq N_2$ we have that $2^{-d_n} < \frac{1}{3}\varepsilon$ where d_n is the depth of the redex R_n reduced at step $t_{n,0} \to t_{n+1,0}$. Since the depth of the descendants of this redex R_n occur at least at the same depth, and since the TRS is the depth preserving we get $d(t_{\omega,m},t_{n,m}) < \frac{1}{3}\varepsilon$ for all $m \in \mathrm{N} \cup \{\omega\}$ and all $n \geq N_2$.

For similar reasons there is N_3 such that for all $n \in \mathrm{N} \cup \{\omega\}$ and all $m \geq N_3$ we have that $d(t_{n,\omega},t_{n,m}) < \frac{1}{3}\varepsilon$.

Let N be the maximum of N_1, N_2 and N_3. Then for $n \geq N$ we find

$$
\begin{aligned}
d(t_{n,\omega},t_{\omega,\omega}) &\leq d(t_{n,\omega},t_{n,N})+d(t_{n,N},t_{\omega,\omega}) \text{ for any } m \geq N \\
&\leq d(t_{n,\omega},t_{n,N})+d(t_{n,N},t_{\omega,N})+d(t_{\omega,N},t_{\omega,\omega}) \\
&\leq \tfrac{1}{3}\varepsilon+\tfrac{1}{3}\varepsilon+\tfrac{1}{3}\varepsilon \\
&\leq \varepsilon.
\end{aligned}
$$

□

3.4. REMARK. Observe that in this proof there are two places where it is essential that the reduction are strong convergent. The first is the appeal to the infinite Parallel Moves Lemma. The second is in the argument that the sequences $(t_{\omega,n})$ and $(t_{n,\omega})$ have the same limit.

4. Non-collapsing orthogonal term rewriting systems

4.1. DEFINITION. A TRS R is *non-collapsing* if there is no rewrite rule in R whose right-hand side is a single variable.

We will show that any non-collapsing orthogonal TRS satifies the infinitary Church-Rosser property with respect strong convergence. The proofs will use a variant of Park's notion of hiaton. The idea is to replace a depth losing rule like $A(x,B(y)) \to B(x)$ by a depth-preserving variant $A(x,B(y)) \to B(\varepsilon(x))$. In order to keep the rewrite rules applicable to terms involving hiatons, we also have to add many more variants: $A(x,\varepsilon^m(B(y))) \to_\varepsilon B(\varepsilon^{m+1}(y))$ for all $m > 0$. We will call the new TRS the ε-completion of the old one.

4.2. CONSTRUCTION. Let R be a left-linear TRS. The *ε-completion* R_ε is defined as the TRS $(\Sigma\cup\{\varepsilon\},R_\varepsilon)$, where ε is a fresh unary symbol with respect to R, and R_ε consists of all rewrite rules $l_\varepsilon \to r_\varepsilon$, where l_ε is obtained from a left-hand side of a rewrite rule $l \to r$ in R by substituting any proper subterm t (that is not a variable, or l itself) in l by $\varepsilon^n(t)$ for some $n\in \mathbb{N}$, and r_ε is obtained from the corresponding right-hand side r by replacing each occurrence of a variable, say x, by $\varepsilon^m(x)$, where m is the minimum of 0 and the depth of x in l_ε minus the depth of this particular variable x in r.

The proof of the following proposition is straightforward and omitted.

4.3. PROPOSITION. *The ε-completion of an orthogonal TRS is depth preserving and orthogonal.* □

4.4. LEMMA. *Let* R *be a non-collapsing TRS. If* $t \to^{\varepsilon}_{\omega} s$ *is an infinite, Cauchy converging* $\to_\varepsilon$*-reduction of length* ω, *where* t *is an* ε*-free term. Then*

(i) *there are no branches ending in an infinite string of* ε *in the tree representation of* s;

(ii) *the term* s/ε *obtained by erasing all* ε *in* s *is well formed term of the original TRS* R;

Let $t \to_\omega s/\varepsilon$ *be the reduction obtained from* $t \to^{\varepsilon}_{\omega} s$ *by erasing all* ε*'s,*

(iii) *if* $t \to^{\varepsilon}_{\omega} s$ *is Cauchy converging, then so is* $t \to_\omega s/\varepsilon$;

(iv) *if* $t \to^{\varepsilon}_{\omega} s$ *is strongly converging, then so is* $t \to_\omega s/\varepsilon$.

(v) *if* $t \to_\omega s$ *is strongly converging in* R, *then there exists a strongly converging reduction* $t \to^{\varepsilon}_{\omega} r$ *in* R_ε *such that erasure of all* ε*'s in* $t \to^{\varepsilon}_{\omega} r$ *results again in the sequence* $t \to_\omega s$.

PROOF. (i) In the limit term of a Cauchy converging reduction starting with an ε-free term one easily sees that an infinite string of ε's can only be produced by infinite applications of rules containing no function symbols in the right-hand side. However, by assumption we have excluded such collapsing rules.

(ii) Clearly the root itself is not equal to ε, and it is harmless to delete any finite string of unary ε's in a branch. (i) excludes the harmful situation of an infinite string of ε's on a branch, deletion of which would leave the last function symbol on the branch with a missing argument.

(iv) Suppose $t \to_\omega s$. represents a strongly convergent sequence $t = t_0 \to t_1 \to \ldots \to_\omega t_\omega = s$ in

R_ε. Let $p \in \mathbb{N}$. Let q be the minimal natural number below which depth at any branch a function symbol F can be found for which there are p function symbols not equal to ε on the branch in between F and the root. Such a number q has to exist, since by (i) all infinite branches contain infinitely many function symbols unequal to ε. (The construction actually involves König's Lemma. If we cut all infinite branches at the point where we count the p^{th} function symbol from the root, we end up with a finitely branching tree with finite branches. Then by the contraposition of König's Lemma there is an upperbound on the length of the branches in the truncated tree. Let q be this upperbound.)

Because $t \rightarrow_{\omega}^{\varepsilon} s$ is strongly converging we can find an $N \in \mathbb{N}$ such that $d_n > q$ for all $n \geq N$. Clearly, after deleting all ε in t_n and s we get as remaining depth $d_{n/\varepsilon} < 2^{-p}$. Hence $t \rightarrow_{\omega}^{\varepsilon} s/\varepsilon$ is strongly convergent.

(iii) A similar argument can be given as for (iv).

(v) Let $t \rightarrow_\omega s$ be strongly converging in R. Clearly, by imitating the steps we can construct a strongly converging reduction $t \rightarrow_{\omega}^{\varepsilon} r$ in R_ε, such that erasing of all ε's in $t \rightarrow_{\omega}^{\varepsilon} r$ results again in the sequence $t \rightarrow_\omega s$. □

5. Main Theorem

The main Theorem 5.1 and and its corollary 5.2 in this section are improvements of respectively [Der90a] and [Ken90a].

The results in [Der90a] imply that top-terminating OTRS, that is OTRS such that there are no derivations of length ω with infinitely many rewrites at topmost position, satisfy the infinite Church-Rosser property for Cauchy converging reductions: combine Theorem 1, Proposition 2, Theorem 10 (which is true under the condition of top-termination) with Theorem 9 in [Der90a].

We will strengthen this in 5.1 to: non-collapsing OTRSs satisfy the infinite Church-Rosser property for strongly converging reductions. This is a stronger result because (i) under the assumption of top-termination every Cauchy converging reduction is strongly converging and (ii) any top-terminating infinitary TRS is non-collapsing, as one easily sees. Actually it will follow from our construction that the Church-Rosser property holds also for OTRSs all which rules are non-collapsing but one, the exception being a collaps rule of the form $I(x) \rightarrow x$, i.e., a rule that contains only one variable in its left hand side (cf. 5.2).

5.1. THEOREM. *Any non-collapsing orthogonal TRS satisfies the infinite Church-Rosser Property for strongly converging reductions*

PROOF. Let R be an OTRS. Constructs its ε-completion R_ε. By Theorem 3.3 the depth-preserving OTRS R_ε satisfies the infinite Church-Rosser property. So if we start with two strongly convergent reductions $t \rightarrow_{\leq\omega} s_1$ and $t \rightarrow_{\leq\omega} s_2$, then by Lemma 4.4 (v) we can lift these to strongly converging reductions in R_ε, let us say $t \rightarrow_{\leq\omega}^{\varepsilon} r_1$ and $t \rightarrow_{\leq\omega}^{\varepsilon} r_2$. By Theorem 3.3 we find a join u for the two lifted reductions such that $r_1 \rightarrow_{\leq\omega}^{\varepsilon} u$ as well as $r_2 \rightarrow_{\leq\omega}^{\varepsilon} u$. Erasing all ε's we see that in R the term u/ε is the join of $t \rightarrow_{\leq\omega} s_1$ and $t \rightarrow_{\leq\omega} s_2$ because by Lemma 4.4 (iv) and (v) the reductions $s_1 = r_1/\varepsilon \rightarrow_{\leq\omega} u/\varepsilon$ and $s_2 = r_2/\varepsilon \rightarrow_{\leq\omega} u/\varepsilon$ are both strongly convergent in R.

□

5.2. COROLLARY. *Any OTRS defined by non-collapsing rules and possibly one collapsing rule involving only one variable satisfies the infinite Church Rosser Property for strongly converging reductions.*

PROOF. In an OTRS defined by non-collapsing rules and possibly one collapsing rule involving only one variable we can follow the same proof strategy as before. Suppose the OTRS has one collapsing rule I(x) → x. As hiaton we take the symbol I instead of ε. Clearly 4.4(i) does no longer hold. However, if in the erasing proces s/ε we erase only finite string of of ε's and no infinite strings, then we can prove and apply Lemma 4.4(iv) and (v) again. □

QUESTION. Do non-collapsing OTRSs satisfy the infinite Church-Rosser property for Cauchy converging reductions?

We feel the answer to this question is positive. We see two proof strategies: one could try to show that depth preserving OTRSs satisfy the infinite Church-Rosser property for Cauchy converging reductions. Although the proof of Theorem 3.3 seems to depend rather essentially on strong convergence, cf. the Remark 3.4, we didn't exploited Lemma 3.2. Observe that the Counterexample 2.3.3 against an infinitary parallel moves lemma for Cauchy converging sequences involves a depth preserving OTRS.

An different technique could be via an application of the generalized notion of Böhm normal form, that we will introduce in the extended version [Ken90b] of [Ken90a].

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