BITOPOLOGICAL DUALITY FOR DISTRIBUTIVE LATTICES AND HEYTING ALGEBRAS

GURAM BEZHANISHVILI, NICK BEZHANISHVILI, DAVID GABELAIA, ALEXANDER KURZ

Abstract. We introduce pairwise Stone spaces as a natural bitopological generalization of Stone spaces—the duals of Boolean algebras—and show that they are exactly the bitopological duals of bounded distributive lattices. The category **PStone** of pairwise Stone spaces is isomorphic to the category **Spec** of spectral spaces and to the category **Pries** of Priestley spaces. In fact, the isomorphism of **Spec** and **Pries** is most naturally seen through **PStone** by first establishing that **Pries** is isomorphic to **PStone**, and then showing that **PStone** is isomorphic to **Spec**. We provide the bitopological and spectral descriptions of many algebraic concepts important for the study of distributive lattices. We also give new bitopological and spectral dualities for Heyting algebras, co-Heyting algebras, and bi-Heyting algebras, thus providing two new alternatives of Esakia's duality.

1. Introduction

It is widely considered that the beginning of duality theory was Stone's groundbreaking work in the mid 30ies on the dual equivalence of the category **Bool** of Boolean algebras and Boolean algebra homomorphism and the category **Stone** of compact Hausdorff zero-dimensional spaces, which became known as Stone spaces, and continuous functions. In 1937 Stone [28] extended this to the dual equivalence of the category **DLat** of bounded distributive lattices and bounded lattice homomorphisms and the category **Spec** of what later became known as spectral spaces and spectral maps. Spectral spaces provide a generalization of Stone spaces. Unlike Stone spaces, spectral spaces are not Hausdorff (not even T_1)[1], and as a result, are more difficult to work with. In 1970 Priestley [20] described another dual category of **DLat** by means of special ordered Stone spaces, which became known as Priestley spaces, thus establishing that **DLat** is also dually equivalent to the category **Pries** of Priestley spaces and continuous order-preserving maps. Since **DLat** is dually equivalent to both **Spec** and **Pries**, it follows that the categories **Spec** and **Pries** are equivalent. In fact, more is true: as shown by Cornish [4] (see also Fleisher [8]), **Spec** is actually isomorphic to **Pries**. The advantage of Priestley spaces is that they are easier to work with than spectral spaces. As a result, Priestley's duality became rather popular, and most dualities for distributive lattices with operators have been performed in terms of Priestley spaces. Here we only mention Esakia's duality for Heyting algebras, co-Heyting algebras, and bi-Heyting algebras [5, 6], which is a restricted version of Priestley's duality.[2] On the other hand, the advantage of spectral spaces is that they only have a topological structure, while Priestley spaces also

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[1]In fact, a spectral space X is a Stone space iff X is T_1.

[2]We note that Esakia's work was independent of Priestley's; a proof that Esakia spaces are Priestley spaces can be found in [7, p. 62].

have an order structure on top of topology, thus their signature is more complicated than that of spectral spaces.

Another way to represent distributive lattices is by means of bitopological spaces, as demonstrated by Jung and Moshier [15]. In fact, bitopological spaces provide a natural medium in establishing the isomorphism between **Pries** and **Spec**: with each Priestley space $(X, \tau, \leq)$, there are two natural topologies associated with it; the upper topology τ_1 consisting of open upsets of $(X, \tau, \leq)$, and the lower topology τ_2 consisting of open downsets of $(X, \tau, \leq)$. Then (X, τ_1, τ_2) is a bitopological space, and the spectral space associated with $(X, \tau, \leq)$ is obtained from (X, τ_1, τ_2) by forgetting τ_2. In this paper we provide an explicit axiomatization of the class of bitopological spaces obtained this way. We call these spaces *pairwise Stone spaces*. On the one hand, pairwise Stone spaces provide a natural generalization of Stone spaces as each of the three conditions defining a Stone space naturally generalizes to the bitopological setting: compact becomes pairwise compact, Hausdorff – pairwise Hausdorff, and zero-dimensional – pairwise zero-dimensional. On the other hand, pairwise Stone spaces provide a natural medium in moving from Priestley spaces to spectral spaces and backwards, thus Cornish's isomorphism of **Pries** and **Spec** can be established more naturally by first showing that **Pries** is isomorphic to the category **PStone** of pairwise Stone spaces and bicontinuous maps, and then showing that **PStone** is isomorphic to **Spec**. Thirdly, the signature of pairwise Stone spaces naturally carries the symmetry present in Priestley spaces (and distributive lattices), but hidden in spectral spaces. Moreover, the proof that **DLat** is dually equivalent to **PStone** is simpler and more natural than the existing proofs of the dual equivalence of **DLat** with **Spec** and **Pries**. Lastly, the isomorphism of **Pries**, **PStone**, and **Spec** fits nicely in a more general isomorphism of the categories of compact order-Hausdorff spaces, pairwise compact pairwise regular bitopological spaces, and stably compact spaces described in [10, Ch. VI-6] (see also [25] and [19]). For a variety of applications of these results we refer to the work of Jung, Moshier, and their collaborators [13, 14, 2, 15]. Here we only mention that there is a dual equivalence between these categories and the category of proximity lattices [27, 16], which are a generalization of distributive lattices, thus providing an interesting generalization of the duality for distributive lattices. We view our pairwise Stone spaces as a particular case of pairwise compact pairwise regular bitopological spaces, and our isomorphism of the categories of Priestley spaces, pairwise Stone spaces, and spectral spaces as a particular case of the isomorphism of the categories of compact order-Hausdorff spaces, pairwise compact pairwise regular bitopological spaces, and stably compact spaces.

One of the advantages of Priestley's duality is that many algebraic concepts important for the study of distributive lattices can be easily described by means of Priestley spaces. In addition, we show that they have a natural dual description by means of pairwise Stone spaces. We also give their dual description by means of spectral spaces, which at times is less transparent than the order topological and bitopological descriptions. We conclude the paper by introducing the subcategories of **PStone** and **Spec**, which are isomorphic to the category **Esa** of Esakia spaces and dually equivalent to the category **Heyt** of Heyting algebras. This provides an alternative of Esakia's duality in the setting of bitopological spaces and spectral spaces. In addition, we establish similar dual equivalences for the categories of co-Heyting algebras and bi-Heyting algebras.

The paper is organized as follows. In Section 2 we recall some basic facts about bitopological spaces, introduce pairwise Stone spaces, and study their basic properties. In Section 3 we prove that the category **PStone** of pairwise Stone spaces is isomorphic to the category

Pries of Priestley spaces. In Section 4 we prove that **PStone** is isomorphic to the category **Spec** of spectral spaces, thus establishing that all three categories are isomorphic to each other. In Section 5 we give a direct proof that the category **DLat** of distributive lattices is dually equivalent to **PStone**, thus providing an alternative of Stone's and Priestley's dualities. In Section 6 we give the dual description of many algebraic concepts important for the study of distributive lattices by means of Priestley spaces, pairwise Stone spaces, and spectral spaces. In particular, we give the dual description of filters, prime filters, maximal filters, ideals, prime ideals, maximal ideals, homomorphic images, sublattices, complete lattices, McNeille completions, and canonical completions. At the end of the section we list all the obtained results in one table, which can be viewed as a dictionary of duality theory for distributive lattices, complementing the dictionary given in [22]. Finally, in Section 7 we develop new bitopological and spectral dualities for Heyting algebras, co-Heyting algebras, and bi-Heyting algebras, thus providing an alternative of Esakia's duality.

2. Pairwise Stone spaces

We recall that a *bitopological space* is a triple (X, τ_1, τ_2), where X is a nonempty set and τ_1 and τ_2 are two topologies on X. Ever since Kelly [17] introduced them, bitopological spaces have been subject of intensive investigation of many topologists. In particular, there has been a lot of research on the "correct" generalization of the basic topological properties to the bitopological setting. For our purposes it is important to find the right generalization of the concept of a Stone space. Therefore, we are interested in the bitopological versions of compactness, Hausdorffness, and zero-dimensionality.

There are several ways to generalize a topological property to the bitopological setting. Let (X, τ_1, τ_2) be a bitopological space and let $\tau = \tau_1 \vee \tau_2$. For a topological property P, we say that (X, τ_1, τ_2) is *bi-P* if both (X, τ_1) and (X, τ_2) are P, and we say that (X, τ_1, τ_2) is *join P* if (X, τ) is P. For example, (X, τ_1, τ_2) is bi-T_0, bi-T_1, or bi-T_2 if both (X, τ_1) and (X, τ_2) are T_0, T_1, or T_2, respectively; and (X, τ_1, τ_2) is join T_0, join T_1, or join T_2 if (X, τ) is T_0, T_1, or T_2, respectively. However, for our purposes, neither bi-Stone nor join Stone turns out to be the right generalization of the concept of a Stone space to the bitopological setting.

Definition 2.1. *Let (X, τ_1, τ_2) be a bitopological space.*

1. [24, Def. 2.1.1] *We call (X, τ_1, τ_2)* pairwise T_0 *if for any two distinct points $x, y \in X$ there exists $U \in \tau_1 \cup \tau_2$ containing exactly one of x, y.*
2. [24, Def. 2.1.3] *We call (X, τ_1, τ_2)* pairwise T_1 *if for any two distinct points $x, y \in X$ there exists $U \in \tau_1 \cup \tau_2$ such that $x \in U$ and $y \notin U$.*
3. [24, Def. 2.1.8] *We call (X, τ_1, τ_2)* pairwise T_2 *or* pairwise Hausdorff *if for any two distinct points $x, y \in X$ there exist disjoint $U \in \tau_1$ and $V \in \tau_2$ such that $x \in U$ and $y \in V$ or there exist disjoint $U \in \tau_2$ and $V \in \tau_1$ with the same property.*

Remark 2.2. We have chosen [24] as our primary source of reference, although the concepts of a pairwise T_0 space and a pairwise T_1 space have appeared earlier in the literature.

Remark 2.3. It would be more in the vein of Definition 2.1.1 and 2.1.2 if we defined a pairwise T_2 space as a bitopological space satisfying the following condition: For any two distinct points $x, y \in X$ there exist disjoint $U, V \in \tau_1 \cup \tau_2$ such that $x \in U$ and $y \in V$. Obviously if (X, τ_1, τ_2) is pairwise T_2, then it satisfies the condition above, but the converse is not true in general. Nevertheless, we will show below that in the realm of pairwise zero-dimensional spaces the two conditions are equivalent.

It follows from [24, Prop. 2.1.2 and 2.1.5] that each pairwise T_i space is join T_i for $i = 0, 1$. However, not every pairwise T_2 space is join T_2. It is also obvious that bi-T_i implies pairwise T_i for $i = 0, 1, 2$, but there are pairwise T_2 spaces that are not even bi-T_0. As we will see shortly, the concepts of bi-T_0, pairwise T_2, and join T_2 coincide in the realm of pairwise zero-dimensional spaces.

For a bitopological space (X, τ_1, τ_2), let δ_1 denote the collection of closed subsets of (X, τ_1) and δ_2 denote the collection of closed subsets of (X, τ_2). The next definition generalizes the notion of zero-dimensionality to bitopological spaces.

Definition 2.4. [23, p. 127] *We call a bitopological space* (X, τ_1, τ_2) pairwise zero-dimensional *if opens in* (X, τ_1) *closed in* (X, τ_2) *form a basis for* (X, τ_1) *and opens in* (X, τ_2) *closed in* (X, τ_1) *form a basis for* (X, τ_2)*; that is,* $\beta_1 = \tau_1 \cap \delta_2$ *is a basis for* τ_1 *and* $\beta_2 = \tau_2 \cap \delta_1$ *is a basis for* τ_2.

We point out that if (X, τ_1, τ_2) is pairwise zero-dimensional, then $\beta_2 = \{U^c \mid U \in \beta_1\}$ and $\beta_1 = \{V^c \mid V \in \beta_2\}$. Moreover, both β_1 and β_2 contain $\emptyset, X$ and are closed with respect to finite unions and intersections.

Lemma 2.5. *Suppose that* (X, τ_1, τ_2) *is pairwise zero-dimensional. Then the following conditions are equivalent:*

(1) (X, τ_1) *is* T_0.
(2) (X, τ_2) *is* T_0.
(3) (X, τ_1, τ_2) *is pairwise* T_2.
(4) *For any two distinct points* $x, y \in X$ *there exist disjoint* $U, V \in \tau_1 \cup \tau_2$ *such that* $x \in U$ *and* $y \in V$.
(5) (X, τ_1, τ_2) *is join* T_2.
(6) (X, τ_1, τ_2) *is bi-*T_0.

Proof. (1)⇒(2): Suppose that (X, τ_1) is T_0 and x, y are two distinct points of X. Then there exists $U \in \tau_1$ containing exactly one of x, y. Without loss of generality we may assume that $x \in U$ and $y \notin U$. Since (X, τ_1, τ_2) is pairwise zero-dimensional, there exists $V \in \beta_1$ such that $x \in V \subseteq U$. Therefore, $V^c \in \beta_2$, $y \in V^c$, and $x \notin V^c$. Thus, (X, τ_2) is T_0.

(2)⇒(3): Suppose that (X, τ_2) is T_0 and x, y are two distinct points of X. Then there exists $U \in \tau_2$ containing exactly one of x, y. Without loss of generality we may assume that $x \in U$ and $y \notin U$. Since (X, τ_1, τ_2) is pairwise zero-dimensional, there exists $V \in \beta_2$ such that $x \in V \subseteq U$. Then $x \in V \in \beta_2$, $y \in V^c \in \beta_1$, and V, V^c are disjoint. Thus, (X, τ_1, τ_2) is pairwise T_2.

(3)⇒(4) is obvious.

(4)⇒(5): Suppose that x, y are two distinct points of X. By (4), there exist disjoint $U, V \in \tau_1 \cup \tau_2$ such that $x \in U$ and $y \in V$. Without loss of generality we may assume that $U, V \in \tau_1$. Since (X, τ_1, τ_2) is pairwise zero-dimensional, there exists $U' \in \beta_1$ such that $x \in U' \subseteq U$. Let $V' = X - U'$. Then $V \subseteq V'$, so $y \in V' \in \beta_2$, and so there exist two disjoint τ-open sets U', V' such that $x \in U'$ and $y \in V'$. Thus, (X, τ_1, τ_2) is join T_2.

(5)⇒(6): Suppose that (X, τ_1, τ_2) is join T_2. We show that (X, τ_1) is T_0. Let x, y be two distinct points of X. Since (X, τ_1, τ_2) is pairwise zero-dimensional and join T_2, there exist $U_1, U_2 \in \beta_1$ and $V_1, V_2 \in \beta_2$ such that $x \in U_1 \cap V_1$, $y \in U_2 \cap V_2$, and $U_1 \cap V_1$ and $U_2 \cap V_2$ are disjoint. If $y \notin U_1$, then there is $U_1 \in \tau_1$ containing exactly one of x, y. If $y \in U_1$, then $y \notin V_1$. Therefore, $y \in U_2 \cap V_1^c$. Clearly $U_2 \cap V_1^c \in \beta_1$. Moreover, $x \notin U_2 \cap V_1^c$ as $x \notin V_1^c$. Thus, there exists $U_2 \cap V_1^c \in \tau_1$ containing exactly one of x, y. In either case, we separate x, y

by a τ_1-open set, and so (X, τ_1) is T_0. That (X, τ_2) is T_0 is proved similarly. Consequently, (X, τ_1, τ_2) is bi-T_0.

(6)⇒(1) is obvious. ⊣

On the other hand, (X, τ_1, τ_2) may be pairwise zero-dimensional and pairwise T_2 without either of τ_1, τ_2 being even T_1 as the following simple example shows.

Example 2.6. Let $X = \{0, 1\}$, $\tau_1 = \{\emptyset, \{1\}, X\}$ and $\tau_2 = \{\emptyset, \{0\}, X\}$. Then both τ_1 and τ_2 are the Sierpinski topologies on X, thus both are T_0, but not T_1. Nevertheless, (X, τ_1, τ_2) is pairwise zero-dimensional and pairwise T_2.

The next definition generalizes the notion of compactness to bitopological spaces.

Definition 2.7. [24, Def. 2.2.17] *We call a bitopological space* (X, τ_1, τ_2) pairwise compact *if for each cover* $\{U_i \mid i \in I\}$ *of* X *with* $U_i \in \tau_1 \cup \tau_2$*, there exists a finite subcover.*

Remark 2.8. In [24, Def. 2.2.17] Salbany defines a bitopological space (X, τ_1, τ_2) to be pairwise compact if (X, τ) is compact, where $\tau = \tau_1 \vee \tau_2$. In our terminology this means that (X, τ_1, τ_2) is join compact. But it is a consequence of Alexander's Lemma—a classical result in general topology—that the two notions of pairwise compact and join compact coincide.

It is obvious that if (X, τ_1, τ_2) is pairwise compact, then both (X, τ_1) and (X, τ_2) are compact; that is, (X, τ_1, τ_2) is bi-compact. On the other hand, it was observed by Salbany [24, p. 17] that the converse is not true in general. Let σ_1 and σ_2 denote the collections of compact subsets of (X, τ_1) and (X, τ_2), respectively.

Proposition 2.9. *A bitopological space* (X, τ_1, τ_2) *is pairwise compact iff* $\delta_1 \subseteq \sigma_2$ *and* $\delta_2 \subseteq \sigma_1$.

Proof. [⇒] Suppose that (X, τ_1, τ_2) is pairwise compact. We show that $\delta_1 \subseteq \sigma_2$. Let $A \in \delta_1$ and let $A \subseteq \bigcup\{U_i \mid i \in I\}$ with $\{U_i \mid i \in I\} \subseteq \tau_2$. Then the collection $\{U_i \mid i \in I\} \cup \{A^c\}$ is a cover of X. Since $A^c \in \tau_1$ and (X, τ_1, τ_2) is pairwise compact, there exist $i_1, \dots, i_n \in I$ such that $U_{i_1} \cup \dots \cup U_{i_n} \cup A^c = X$. It follows that $A \subseteq U_{i_1} \cup \dots \cup U_{i_n}$, and so $A \in \sigma_2$. Thus, $\delta_1 \subseteq \sigma_2$. That $\delta_2 \subseteq \sigma_1$ is proved similarly.

[⇐] Suppose that $\delta_1 \subseteq \sigma_2$ and $\delta_2 \subseteq \sigma_1$. To show that (X, τ_1, τ_2) is pairwise compact let $\{U_i \mid i \in I\} \subseteq \tau_1$ and $\{V_j \mid j \in J\} \subseteq \tau_2$ with $\bigcup\{U_i \mid i \in I\} \cup \bigcup\{V_j \mid j \in J\} = X$. We set $U = \bigcup\{U_i \mid i \in I\}$. Clearly $U \in \tau_1$ and $U \cup \bigcup\{V_j \mid j \in J\} = X$, so $U^c \subseteq \bigcup\{V_j \mid j \in J\}$. Since $U^c \in \delta_1$ and $\delta_1 \subseteq \sigma_2$, we have that $U^c \in \sigma_2$. Therefore, there exist $j_1, \dots, j_n \in J$ such that $U^c \subseteq V_{j_1} \cup \dots \cup V_{j_n}$. We set $V = V_{j_1} \cup \dots \cup V_{j_n}$. Then $U \cup V = X$, so $V^c \subseteq U = \bigcup\{U_i \mid i \in I\}$. Since $V^c \in \delta_2$ and $\delta_2 \subseteq \sigma_1$, we have that $V^c \in \sigma_1$. Therefore, there exist $i_1, \dots, i_m \in I$ such that $V^c \subseteq U_{i_1} \cup \dots \cup U_{i_m}$. Clearly the finite collection $\{V_{j_1}, \dots, V_{j_n}, U_{i_1}, \dots, U_{i_m}\}$ is a cover of X. Thus, X is pairwise compact. ⊣

Now we generalize the notion of a Stone space to that of a pairwise Stone space.

Definition 2.10. *We call* (X, τ_1, τ_2) *a* pairwise Stone space *if it is pairwise compact, pairwise Hausdorff, and pairwise zero-dimensional.*

We note that in the definition of a pairwise Stone space, pairwise Hausdorff can be replaced by any of the equivalent conditions of Lemma 2.5, and that pairwise compact can be replaced by $\delta_1 \subseteq \sigma_2$ and $\delta_2 \subseteq \sigma_1$, as follows from Proposition 2.9. Let **PStone** denote the category of pairwise Stone spaces and bi-continuous functions; that is functions which are continuous with respect to both topologies.

3. Priestley spaces and pairwise Stone spaces

Let $(X, \leq)$ be a poset. We recall that $A \subseteq X$ is an *upset* if $x \in A$ and $x \leq y$ imply $y \in A$, and that A is a *downset* if $x \in A$ and $y \leq x$ imply $y \in A$. For $Y \subseteq X$ let ${\uparrow}Y = \{x \mid \exists y \in Y \text{ with } y \leq x\}$ and ${\downarrow}Y = \{x \mid \exists y \in Y \text{ with } x \leq y\}$. Let $\mathsf{Up}(X)$ denote the set of upsets and $\mathsf{Do}(X)$ denote the set of downsets of $(X, \leq)$.

Let $(X, \tau, \leq)$ be an ordered topological space. We denote by $\mathsf{OpUp}(X)$ the set of open upsets, by $\mathsf{ClUp}(X)$ the set of closed upsets, and by $\mathsf{CpUp}(X)$ the set of clopen upsets of $(X, \tau, \leq)$. Similarly, let $\mathsf{OpDo}(X)$ denote the set of open downsets, $\mathsf{ClDo}(X)$ denote the set of closed downsets, and $\mathsf{CpDo}(X)$ denote the set of clopen downsets of $(X, \tau, \leq)$. The next definition is well-known.

Definition 3.1. *An ordered topological space $(X, \tau, \leq)$ is a* Priestley space *if (X, τ) is compact and whenever $x \not\leq y$, there exists a clopen upset A such that $x \in A$ and $y \notin A$.*

The second condition in the above definition is known as the *Priestley separation axiom* (PSA for short). The next lemma is well-known.

Lemma 3.2. *Let $(X, \tau, \leq)$ be an ordered topological space.*

1. *If $(X, \tau, \leq)$ is a Priestley space, then (X, τ) is a Stone space.*
2. *If $(X, \tau, \leq)$ is a Priestley space, then ${\uparrow}F$ and ${\downarrow}F$ are closed for each closed subset F of X.*
3. *In a Priestley space, every open upset is the union of clopen upsets, every closed upset is the intersection of clopen upsets, every open downset is the union of clopen downsets, and every closed downset is the intersection of clopen downsets.*
4. *In a Priestley space, clopen upsets and clopen downsets form a subbasis for the topology.*
5. *$(X, \tau, \leq)$ is a Priestley space iff (X, τ) is compact and for closed subsets F and G of X, whenever ${\uparrow}F \cap {\downarrow}G = \emptyset$, there exists a clopen upset A of X such that $F \subseteq A$ and $G \subseteq A^c$.*

We will refer to condition (5) in the lemma as the *strong Priestley separation axiom* (SPSA for short). Let **Pries** denote the category of Priestley spaces and continuous order-preserving maps. We show that the categories **Pries** and **PStone** are isomorphic. To this end, we will define two functors $\Phi : \mathbf{PStone} \to \mathbf{Pries}$ and $\Psi : \mathbf{Pries} \to \mathbf{PStone}$ which will set the required isomorphism.

For a topological space (X, τ), let $\leq$ denote the *specialization order* of (X, τ); that is,

$$x \leq y \text{ iff } x \in \mathrm{Cl}(y) \text{ iff } (\forall U \in \tau)(x \in U \text{ implies } y \in U).$$

It is well-known that $\leq$ is reflexive and transitive, and that $\leq$ is antisymmetric iff (X, τ) is T_0.

Lemma 3.3. *Let (X, τ_1, τ_2) be a bitopological space, $\leq_1$ be the specialization order of (X, τ_1), and $\leq_2$ be the specialization order of (X, τ_2). If (X, τ_1, τ_2) is pairwise zero-dimensional, then $\leq_1 = \geq_2$.*

Proof. Let (X, τ_1, τ_2) be pairwise zero-dimensional; that is, $\beta_1 = \tau_1 \cap \delta_2$ is a basis for τ_1 and $\beta_2 = \tau_2 \cap \delta_1$ is a basis for τ_2. Then, for each $x, y \in X$, we have:

$$\begin{array}{ll} x \leq_1 y & \text{iff} \\ (\forall U \in \tau_1)(x \in U \text{ implies } y \in U) & \text{iff} \\ (\forall U \in \beta_1)(x \in U \text{ implies } y \in U) & \text{iff} \\ (\forall U \in \beta_1)(y \in U^c \text{ implies } x \in U^c) & \text{iff} \\ (\forall V \in \beta_2)(y \in V \text{ implies } x \in V) & \text{iff} \\ (\forall V \in \tau_2)(y \in V \text{ implies } x \in V) & \text{iff} \\ y \leq_2 x. & \end{array}$$

⊣

For a pairwise Stone space (X, τ_1, τ_2), let $\tau = \tau_1 \vee \tau_2$, and let $\leq = \leq_1$ be the specialization order of (X, τ_1).

Proposition 3.4. *If (X, τ_1, τ_2) is a pairwise Stone space, then $(X, \tau, \leq)$ is a Priestley space. Moreover:*

(i) $\mathsf{CpUp}(X, \tau, \leq) = \beta_1$.
(ii) $\mathsf{OpUp}(X, \tau, \leq) = \tau_1$.
(iii) $\mathsf{ClUp}(X, \tau, \leq) = \delta_2$.
(iv) $\mathsf{CpDo}(X, \tau, \leq) = \beta_2$.
(v) $\mathsf{OpDo}(X, \tau, \leq) = \tau_2$.
(vi) $\mathsf{ClDo}(X, \tau, \leq) = \delta_1$.

Proof. Since (X, τ_1, τ_2) is pairwise compact, (X, τ_1, τ_2) is join compact, and so (X, τ) is compact. Also, as (X, τ_1, τ_2) is pairwise Hausdorff, it follows from Lemma 2.5 that (X, τ_1) is T_0. Therefore, $\leq = \leq_1$ is a partial order. We show that $(X, \tau, \leq)$ satisfies PSA. If $x \not\leq y$, then $x \not\leq_1 y$, so there exists $U \in \beta_1$ such that $x \in U$ and $y \notin U$. Since $\leq_1$ is the specialization order of (X, τ_1), U is an $\leq_1$-upset. From $U \in \beta_1$ it follows that $U^c \in \beta_2 \subseteq \tau$. So both U and U^c are open in (X, τ), and so U is clopen in (X, τ). Therefore, U is a clopen upset of $(X, \tau, \leq)$, implying that $(X, \tau, \leq)$ satisfies PSA. Thus, $(X, \tau, \leq)$ is a Priestley space.

(i) We already showed that $\beta_1 \subseteq \mathsf{CpUp}(X, \tau, \leq)$. Let $A \in \mathsf{CpUp}(X, \tau, \leq)$. We show that $A = \bigcup\{U \in \beta_1 \mid U \subseteq A\}$. That $\bigcup\{U \in \beta_1 \mid U \subseteq A\} \subseteq A$ is obvious. Let $x \in A$. Since A is an upset, for each $y \in A^c$ we have $x \not\leq y$. Therefore, $x \not\leq_1 y$, and as β_1 is a basis for (X, τ_1), there exists $U_y \in \beta_1$ such that $x \in U_y$ and $y \notin U_y$. It follows that $A^c \cap \bigcap\{U_y \mid y \in A^c\} = \emptyset$. Thus, $\{A^c\} \cup \{U_y \mid y \in A^c\}$ is a family of closed subsets of (X, τ) with the empty intersection, and as (X, τ) is compact, there are $U_1, \dots, U_n \in \beta_1$ with $A^c \cap U_1 \cap \dots \cap U_n = \emptyset$. Therefore, $x \in U_1 \cap \dots \cap U_n \subseteq A$. Since β_1 is closed under finite intersections, we obtain that there is $U \in \beta_1$ such that $x \in U \subseteq A$. Thus, $A = \bigcup\{U \in \beta_1 \mid U \subseteq A\}$. Now since A is a closed subset of a compact space, A is compact, so it is a finite union of elements of β_1, thus $A \in \beta_1$.

(ii) Since every open upset is the union of clopen upsets of $(X, \tau, \leq)$ and β_1 is a basis for (X, τ_1), the result follows from (i).

(iv) and (v) are proved similar to (i) and (ii).

(iii) Since closed upsets are intersections of clopen upsets of $(X, \tau, \leq)$, and clopen upsets are elements of β_1, closed upsets are intersections of elements of β_1. Because $\beta_1 = \{U^c \mid U \in \beta_2\}$, intersections of elements of β_1 are intersections of complements of elements of β_2, so are complements of unions of elements of β_2. As unions of elements of β_2 are elements of τ_2, we obtain that closed upsets are complements of elements of τ_2, so are elements of δ_2. Consequently, $\mathsf{ClUp}(X, \tau, \leq) = \delta_2$.

(vi) is proved similar to (iii). ⊣

Proposition 3.5. *Let (X, τ_1, τ_2) and (X', τ_1', τ_2') be pairwise Stone spaces. If $f : (X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ is bi-continuous, then $f : (X, \tau, \leq) \to (X', \tau', \leq')$ is continuous and order-preserving.*

Proof. Since f is bi-continuous, the f inverse image of every element of $\tau_1' \cup \tau_2'$ is an element of $\tau_1 \cup \tau_2$. As $\tau_1' \cup \tau_2'$ is a subbasis for (X, τ'), it follows that $f : (X, \tau) \to (X', \tau')$ is continuous. Also, since the f inverse image of an element of τ_1' is an element of τ_1 and $\leq' = \leq_1'$, it follows that $f : (X, \leq) \to (X', \leq')$ is order-preserving. Thus, $f : (X, \tau, \leq) \to (X', \tau', \leq')$ is continuous and order-preserving. ⊣

We define the functor $\Phi : \mathbf{PStone} \to \mathbf{Pries}$ as follows. For (X, τ_1, τ_2) a pairwise Stone space, we put $\Phi(X, \tau_1, \tau_2) = (X, \tau, \leq)$, and for $f : (X, \tau, \leq) \to (X', \tau', \leq')$ a bi-continuous map, we put $\Phi(f) = f$. It follows from Propositions 3.4 and 3.5 that Φ is well-defined.

For $(X, \tau, \leq)$ a Priestley space, let $\tau_1 = \mathsf{OpUp}(X, \tau, \leq)$ and $\tau_2 = \mathsf{OpDo}(X, \tau, \leq)$. Clearly τ_1 and τ_2 are topologies on X.

Proposition 3.6. *If $(X, \tau, \leq)$ is a Priestley space, then (X, τ_1, τ_2) is a pairwise Stone space. Moreover:*

(i) $\beta_1 = \mathsf{CpUp}(X, \tau, \leq)$.
(ii) $\beta_2 = \mathsf{CpDo}(X, \tau, \leq)$.
(iii) $\leq = \leq_1 = \geq_2$.

Proof. Since (X, τ) is compact and $\tau_1 \cup \tau_2 \subseteq \tau$, it follows that (X, τ_1, τ_2) is pairwise compact. To show that (X, τ_1, τ_2) is pairwise Hausdorff, let x, y be two distinct points of X. Since $\leq$ is a partial order, we have $x \not\leq y$ or $y \not\leq x$. In either case, by PSA, one of the points has a clopen upset neighborhood U not containing the other. Clearly U^c is a clopen downset. Therefore, $U \in \tau_1$ and $U^c \in \tau_2$ separate x and y. Thus, (X, τ_1, τ_2) is pairwise Hausdorff. That (X, τ_1, τ_2) is pairwise zero-dimensional follows from (i), (ii), and the fact that open upsets are unions of clopen upsets and open downsets are unions of clopen downsets (see Lemma 3.2.3). Consequently, (X, τ_1, τ_2) is a pairwise Stone space.

(i) For $U \subseteq X$ we have:

$$\begin{array}{ll} A \in \beta_1 & \text{iff} \\ A \in \tau_1 \text{ and } A^c \in \tau_2 & \text{iff} \\ A \in \mathsf{OpUp}(X, \tau, \leq) \text{ and } A^c \in \mathsf{OpDo}(X, \tau, \leq) & \text{iff} \\ A \in \mathsf{CpUp}(X, \leq). & \end{array}$$

Thus, $\beta_1 = \mathsf{CpUp}(X, \leq)$.

(ii) is proved similar to (i).

(iii) For $x, y \in X$, by PSA, we have:

$$\begin{array}{ll} x \leq y & \text{iff} \\ (\forall U \in \mathsf{OpUp}(X, \tau, \leq))(x \in U \Rightarrow y \in U) & \text{iff} \\ (\forall U \in \tau_1)(x \in U \Rightarrow y \in U) & \text{iff} \\ x \leq_1 y. & \end{array}$$

Thus, $\leq = \leq_1$. That $\leq = \geq_2$ is proved similarly. ⊣

Proposition 3.7. *If $f : (X, \tau, \leq) \to (X', \tau', \leq')$ is continuous and order-preserving, then $f : (X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ is bi-continuous.*

Proof. Since f is continuous and order-preserving, $U \in \mathsf{OpUp}(X', \tau', \leq')$ implies $f^{-1}(U) \in \mathsf{OpUp}(X, \tau, \leq)$ and $U \in \mathsf{OpDo}(X', \tau', \leq')$ implies $f^{-1}(U) \in \mathsf{OpDo}(X, \tau, \leq)$. By the definition of the topologies, $\mathsf{OpUp}(X,\tau,\leq) = \tau_1$, $\mathsf{OpUp}(X',\tau',\leq') = \tau_1'$, $\mathsf{OpDo}(X,\tau,\leq) = \tau_2$, and $\mathsf{OpDo}(X',\tau',\leq') = \tau_2'$. Thus, $f : (X,\tau_1,\tau_2) \to (X',\tau_1',\tau_2')$ is bi-continuous. $\dashv$

Now we define Ψ : **Pries** $\to$ **PStone** as follows. For $(X,\tau,\leq)$ a Priestley space, we put $\Psi(X,\tau,\leq) = (X,\tau_1,\tau_2)$, and for $f : (X,\tau,\leq) \to (X',\tau',\leq')$ continuous and order-preserving, we put $\Psi(f) = f$. It follows from Propositions 3.6 and 3.7 that Ψ is well-defined.

Theorem 3.8. *The functors Φ and Ψ establish isomorphism of the categories* **PStone** *and* **Pries**.

Proof. We already verified that Φ and Ψ are well-defined. That they are natural is easy to see. Moreover, for each pairwise Stone space (X,τ_1,τ_2), by Proposition 3.4, we have $\Psi\Phi(X,\tau_1,\tau_2) = \Psi(X,\tau,\leq) = (X,\mathsf{OpUp}(X,\tau,\leq),\mathsf{OpDo}(X,\tau,\leq)) = (X,\tau_1,\tau_2)$. Also, for each Priestley space $(X,\tau,\leq)$, by Lemma 3.2.4 and Proposition 3.6, we have $\Phi\Psi(X,\tau,\leq) = \Phi(X,\tau_1,\tau_2) = (X,\tau_1\vee\tau_2,\leq_1) = (X,\tau,\leq)$. Thus, Φ and Ψ establish isomorphism of **PStone** and **Priest**. $\dashv$

4. Pairwise Stone spaces and spectral spaces

For a topological space (X,τ), let $\mathcal{E}(X,\tau)$ denote the set of *compact open* subsets of (X,τ). We recall that (X,τ) is *coherent* if $\mathcal{E}(X,\tau)$ is closed under finite intersections and forms a basis for the topology. We also recall that a subset A of X is *irreducible* if $A = F \cup G$, with F, G closed, implies that $A = F$ or $A = G$, and that (X,τ) is *sober* if every irreducible closed subset of (X,τ) is the closure of a point. Clearly a closed subset of X is irreducible iff it is a join-prime element in the lattice of closed subsets of (X,τ). We will use this fact in the proof of Proposition 4.2.

Definition 4.1. [12, p. 43] *A topological space (X,τ) is called a* spectral space *if (X,τ) is compact, T_0, coherent, and sober.*

Let (X,τ) and (X',τ') be two spectral spaces. We recall [12, p. 43] that a map $f : (X,\tau) \to (X',\tau')$ is a *spectral map* if $U \in \mathcal{E}(X',\tau')$ implies $f^{-1}(U) \in \mathcal{E}(X,\tau)$. Clearly every spectral map is continuous.

Let **Spec** denote the category of spectral spaces and spectral maps. It follows from [4] that **Spec** is isomorphic to **Pries**. Thus, by Theorem 3.8, **Spec** is isomorphic to **PStone**. Nevertheless, we give a direct proof of this result. On the one hand, it will underline the utility of sobriety in the definition of a spectral space; on the other hand, it will provide a more natural proof of Cornish's result that **Pries** and **Spec** are isomorphic, by first establishing the intermediate isomorphisms of **Pries** and **PStone** and **PStone** and **Spec**.

Proposition 4.2. *If (X,τ_1,τ_2) is a pairwise Stone space, then (X,τ_1) is a spectral space. Moreover, $\mathcal{E}(X,\tau_1) = \beta_1$.*

Proof. Since (X,τ_1,τ_2) is pairwise compact, it is immediate that (X,τ_1) is compact. It follows from Lemma 2.5 that (X,τ_1) is T_0. We show that $\mathcal{E}(X,\tau_1) = \beta_1$. By Proposition 2.9, $\beta_1 = \tau_1 \cap \delta_2 \subseteq \tau_1 \cap \sigma_1 = \mathcal{E}(X,\tau_1)$. Conversely, suppose that $U \in \mathcal{E}(X,\tau_1)$. Since β_1 is a basis for (X,τ_1), we have U is the union of elements of β_1. As U is compact, it is a finite union of elements of β_1, thus belongs to β_1 because β_1 is closed under finite unions. Therefore, $\mathcal{E}(X,\tau_1) = \beta_1$. It follows that $\mathcal{E}(X,\tau)$ is closed under finite intersections and forms a basis for the topology. Therefore, (X,τ) is coherent. To show that (X,τ_1) is sober, let F be a join-prime element in the lattice of closed subsets of (X,τ_1). We show that F is equal to the closure in (X, τ_1) of a point of F. If not, then for each $x \in F$ there exists $y \in F$ such that $y \notin \mathrm{Cl}_1(x)$. Therefore, there exists $U_y \in \beta_1$ such that $y \in U_y$ and $x \notin U_y$. Let $U_x = U_y^c$. Then $x \in U_x \in \beta_2$, $y \notin U_x$, and F is covered by the family $\{U_x \mid x \in F\}$. Since $F \in \delta_1 \subseteq \sigma_2$, there exist $x_1, \ldots, x_n \in F$ such that $F \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$. As F is join-prime in δ_1 and for each i we have $U_{x_i} \in \beta_2 \subseteq \delta_1$, there exists k such that $F \subseteq U_{x_k}$. On the other hand, the y_k corresponding to x_k belongs to F and does not belong to U_{x_k}, a contradiction. Thus, there is $x \in F$ such that $F = \mathrm{Cl}_1(x)$. Consequently, (X, τ_1) is sober, and so (X, τ_1) is a spectral space. $\dashv$

Proposition 4.3. *Let (X, τ_1, τ_2) and (X', τ_1', τ_2') be two pairwise Stone spaces. If $f : (X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ is bi-continuous, then $f : (X, \tau_1) \to (X', \tau_1')$ is spectral.*

Proof. Since f is bi-continuous, by Proposition 4.2, we have:

$$\begin{array}{ll} U \in \mathcal{E}(X', \tau_1') & \Rightarrow \\ U \in \beta_1' & \Rightarrow \\ U \in \tau_1' \cap \delta_2' & \Rightarrow \\ f^{-1}(U) \in \tau_1 \cap \delta_2 & \Rightarrow \\ f^{-1}(U) \in \beta_1 & \Rightarrow \\ f^{-1}(U) \in \mathcal{E}(X, \tau_1). & \end{array}$$

Thus, f is spectral. $\dashv$

We define the functor $\mathsf{F} : \mathbf{PStone} \to \mathbf{Spec}$ as follows. For a pairwise Stone space (X, τ_1, τ_2), we put $\mathsf{F}(X, \tau_1, \tau_2) = (X, \tau_1)$, and for $f : (X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ bi-continuous, we put $\mathsf{F}(f) = f$. It follows from Propositions 4.2 and 4.3 that F is well-defined. Note that F is a forgetful functor, forgetting the topology τ_2.

For (X, τ) a spectral space, let $\tau_1 = \tau$ and τ_2 be the topology generated by the basis $\Delta(X, \tau) = \{U^c \mid U \in \mathcal{E}(X, \tau)\}$.

Remark 4.4. Let (X, τ) be a topological space. We recall (see, e.g., [18, Def. 4.4]) that the *de Groot dual* of τ is the topology τ^* whose closed sets are generated by compact saturated sets of (X, τ). Since in a spectral space (X, τ) the compact saturated sets are exactly the intersections of compact open sets, we obtain that the topology generated by $\Delta(X, \tau)$ is exactly the de Groot dual τ^* of τ.

Proposition 4.5. *If (X, τ) is a spectral space, then (X, τ_1, τ_2) is a pairwise Stone space. Moreover:*

(i) $\beta_1 = \mathcal{E}(X, \tau)$.
(ii) $\beta_2 = \Delta(X, \tau)$.

Proof. First we show that (X, τ_1, τ_2) is pairwise compact. For this it suffices to show that any collection $K \subseteq \mathcal{E}(X, \tau) \cup \Delta(X, \tau)$ with the FIP (Finite Intersection Property) has a nonempty intersection. Let $\delta = \{F \mid F^c \in \tau\}$ denote the collection of closed subsets of (X, τ). Since $\Delta(X, \tau) \subseteq \delta$, we have that $K \subseteq \mathcal{E}(X, \tau) \cup \delta$. To show that $\bigcap K \neq \emptyset$, by Zorn's Lemma, we extend K to a maximal subset M of $\mathcal{E}(X, \tau) \cup \delta$ with the FIP. Let C denote the intersection of all τ-closed sets in M; that is, $C = \bigcap\{F \mid F \in M \cap \delta\}$. Since (X, τ) is compact, $C \in \delta$ is nonempty. Because $\mathcal{E}(X, \tau)$ is closed under finite intersections, it is easy to see that the collection $M \cup \{C\}$ has the FIP, and as M is maximal, we have $C \in M$. We show that C is irreducible. Suppose that $C = A \cup B$ and $A, B \in \delta$. If $M \cup \{A\}$ and $M \cup \{B\}$ do not have the FIP, then there exist $A_1, \ldots, A_n \in M$ with $A_1 \cap \cdots \cap A_n \cap A = \emptyset$ and $B_1, \ldots, B_m \in M$

with $B_1 \cap \cdots \cap B_m \cap B = \emptyset$. This implies that $A_1 \cap \cdots \cap A_n \cap B_1 \cap \cdots \cap B_m \cap C = \emptyset$, which is a contradiction. Therefore, either $M \cup \{A\}$ or $M \cup \{B\}$ has the FIP. Since M is maximal, either $A \in M$ or $B \in M$. Because of the choice of C, this implies that either $C \subseteq A$ or $C \subseteq B$, and so $C = A$ or $C = B$. Thus, C is irreducible. As (X, τ) is sober, $C = \mathrm{Cl}(x)$ for some $x \in X$. It is clear that x belongs to all $F \in M \cap \delta$ since $C \subseteq F$ for all such F. Moreover, for each $U \in M \cap \mathcal{E}(X, \tau)$, we have $U \cap \mathrm{Cl}(x) = U \cap C \neq \emptyset$. Since U is open in (X, τ), this implies that $x \in U$. Therefore, $x \in \bigcap M$, so $x \in \bigcap K$, as $K \subseteq M$, and so $\bigcap K \neq \emptyset$. Consequently, (X, τ_1, τ_2) is pairwise compact.

We show that $\beta_1 = \mathcal{E}(X, \tau)$ and $\beta_2 = \Delta(X, \tau)$, which establishes that (X, τ_1, τ_2) is pairwise zero-dimensional. By the definition of τ_2 we have $\mathcal{E}(X, \tau) \subseteq \delta_2$, and so $\mathcal{E}(X, \tau) \subseteq \beta_1$. Conversely, since (X, τ_1, τ_2) is pairwise compact, by Proposition 2.9, we have $\beta_1 = \tau_1 \cap \delta_2 \subseteq \tau_1 \cap \sigma_1 = \mathcal{E}(X, \tau)$. Therefore, $\beta_1 = \mathcal{E}(X, \tau)$. Moreover, $U \in \Delta(X, \tau) \iff U^c \in \mathcal{E}(X, \tau) = \tau_1 \cap \delta_2 \iff U \in \delta_1 \cap \tau_2 = \beta_2$. Thus, $\beta_2 = \Delta(X, \tau)$.

Lastly, we have for granted that (X, τ_1) is T_0. Therefore, by Lemma 2.5, (X, τ_1, τ_2) is pairwise T_2, so a pairwise Stone space, which concludes the proof. ⊣

Proposition 4.6. *Let (X, τ) and (X', τ') be two spectral spaces. If $f : (X, \tau) \to (X', \tau')$ is a spectral map, then $f : (X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ is bi-continuous.*

Proof. Since f is spectral, $f : (X, \tau_1) \to (X', \tau_1')$ is continuous. Moreover, for $U \in \beta_2'$ we have $U^c \in \beta_1'$. Therefore, $f^{-1}(U) = f^{-1}((U^c)^c) = f^{-1}(U^c)^c \in \beta_2$ since $f^{-1}(U^c) \in \beta_1$, as f is spectral. Consequently, $f : (X, \tau_2) \to (X', \tau_2')$ is continuous, and so $f : (X, \tau_1, \tau_2) \to (X', \tau_1', \tau_2')$ is bi-continuous. ⊣

Now we define the functor $\mathsf{G} : \mathbf{Spec} \to \mathbf{PStone}$ as follows. For a spectral space (X, τ), we put $\mathsf{G}(X, \tau) = (X, \tau_1, \tau_2)$, and for $f : (X, \tau) \to (X', \tau')$ a spectral map, we put $\mathsf{G}(f) = f$. It follows from Propositions 4.5 and 4.6 that G is well-defined.

Theorem 4.7. *The functors F and G establish isomorphism of the categories* **PStone** *and* **Spec**.

Proof. We already verified that F and G are well-defined. That they are natural is easy to see. Moreover, for each pairwise Stone space (X, τ_1, τ_2) we have $\mathsf{GF}(X, \tau_1, \tau_2) = \mathsf{G}(X, \tau_1) = (X, \tau_1, \tau_2)$, by Proposition 4.2. Also, for each spectral space (X, τ) we have $\mathsf{FG}(X, \tau) = \mathsf{F}(X, \tau_1, \tau_2) = (X, \tau_1) = (X, \tau)$. Thus, F and G establish isomorphism of **PStone** and **Spec**. ⊣

5. Distributive lattices and pairwise Stone spaces

Since **PStone** is isomorphic to **Spec** and **Spec** is dually equivalent to **DLat**, it follows that **PStone** is also dually equivalent to **DLat**. We give an explicit proof of this result. It will show that of the dual equivalences of **DLat** with **Spec**, **Pries**, and **PStone**, the dual equivalence of **DLat** with **PStone** is the easiest to establish. Indeed, as we will see below, the proof of compactness of the bitopoligical dual of a bounded distributive lattice L does not require the use of Alexander's Lemma, hence is simpler than in the Priestley case; moreover, the complicated proof of sobriety of the dual spectral space of L is completely avoided in the bitopological setting.

Let L be a bounded distributive lattice and let $X = \mathsf{pf}(L)$ be the set of prime filters of L. We define $\phi_+, \phi_- : L \to \wp(X)$ by

$$\phi_+(a) = \{x \in X \mid a \in x\} \text{ and } \phi_-(a) = \{x \in X \mid a \notin x\}.$$

If we think of L as a Lindenbaum algebra and of $a \in L$ as (an equivalence class of) a formula, then we can think of $\phi_+(a)$ as the set of points a is true at, and of $\phi_-(a)$ as the set of points a is false at. It is easy to check that $\phi_+(a) = \phi_-(a)^c$, and that the following identities hold:

$$\begin{array}{llll} 1_+ : & \phi_+(0) = \emptyset, & 1_- : & \phi_-(0) = X, \\ 2_+ : & \phi_+(1) = X, & 2_- : & \phi_-(1) = \emptyset, \\ 3_+ : & \phi_+(a \wedge b) = \phi_+(a) \cap \phi_+(b), & 3_- : & \phi_-(a \wedge b) = \phi_-(a) \cup \phi_-(b), \\ 4_+ : & \phi_+(a \vee b) = \phi_+(a) \cup \phi_+(b), & 4_- : & \phi_-(a \vee b) = \phi_-(a) \cap \phi_-(b). \end{array}$$

Let $\beta_+ = \phi_+[L] = \{\phi_+(a) \mid a \in L\}$, $\beta_- = \phi_-[L] = \{\phi_-(a) \mid a \in L\}$, τ_+ be the topology generated by β_+, and τ_- be the topology generated by β_-.

Proposition 5.1. *(X, τ_+, τ_-) is a pairwise Stone space.*

Proof. We start by showing that (X, τ_+, τ_-) is pairwise Hausdorff. Suppose that $x \neq y$. Without loss of generality we may assume that $x \not\subseteq y$. Therefore, there exists $a \in L$ with $a \in x$ and $a \notin y$. Thus, $x \in \phi_+(a) \in \tau_+$ and $y \in \phi_-(a) \in \tau_-$. Since $\phi_-(a) = \phi_+(a)^c$, $\phi_+(a)$ and $\phi_-(a)$ are disjoint. Consequently, (X, τ_+, τ_-) is pairwise Hausdorff.

Next we show that (X, τ_+, τ_-) is pairwise compact. For this it is sufficient to show that for each cover of X by elements of $\beta_+ \cup \beta_-$, there is a finite subcover. Suppose that $X = \bigcup\{\phi_+(a_i) \mid i \in I\} \cup \bigcup\{\phi_-(b_j) \mid j \in J\}$ for some $a_i, b_j \in L$. Let Δ be the ideal generated by $\{a_i \mid i \in I\}$ and ∇ be the filter generated by $\{b_j \mid j \in J\}$. If $\Delta \cap \nabla = \emptyset$, then by the prime filter lemma, there is a prime filter x of L such that $\nabla \subseteq x$ and $x \cap \Delta = \emptyset$. Therefore, $x \in \phi_+(b_j)$ and $x \in \phi_-(a_i)$ for each $j \in J$ and $i \in I$. Thus, $x \notin \phi_-(b_j)$ and $x \notin \phi_+(a_i)$ for each $j \in J$ and $i \in I$. Consequently, $\{\phi_+(a_i) \mid i \in I\} \cup \{\phi_-(b_j) \mid j \in J\}$ is not a cover of X, a contradiction. This shows that $\nabla \cap \Delta \neq \emptyset$, and so there exist $b_{j_1}, \ldots, b_{j_n}$ and $a_{i_1}, \ldots, a_{i_m}$ such that $b_{j_1} \wedge \cdots \wedge b_{j_n} \leq a_{i_1} \vee \cdots \vee a_{i_m}$. Therefore, $\phi_+(b_{j_1}) \cap \cdots \cap \phi_+(b_{j_n}) \subseteq \phi_+(a_{i_1}) \cup \cdots \cup \phi_+(a_{i_m})$, implying that $\phi_-(b_{j_1}) \cup \ldots \phi_-(b_{j_n}) \cup \phi_+(a_{i_1}) \cup \cdots \cup \phi_+(a_{i_m}) = X$. Thus, $\{\phi_+(a_{i_1}), \ldots, \phi_+(a_{i_m}), \phi_-(b_{j_1}), \ldots, \phi_-(b_{j_n})\}$ is a finite subcover of $\{\phi_+(a_i) \mid i \in I\} \cup \{\phi_-(b_j) \mid j \in J\}$, and so (X, τ_+, τ_-) is pairwise compact.

Let δ_+ denote the set of closed subsets and σ_+ denote the set of compact subsets of (X, τ_+); δ_- and σ_- are defined similarly. We show that $\beta_+ = \tau_+ \cap \delta_-$. If $U \in \beta_+$, then it is clear that $U \in \tau_+$. Moreover, since $U = \phi_+(a)$ for some $a \in L$, we have $U^c = \phi_-(a)$, and so $U^c \in \beta_-$. Thus, $U \in \delta_-$, so $U \in \tau_+ \cap \delta_-$, and so $\beta_+ \subseteq \tau_+ \cap \delta_-$. Conversely, let $U \in \tau_+ \cap \delta_-$. Since (X, τ_+, τ_-) is pairwise compact, by Proposition 2.9, $U \in \tau_+ \cap \sigma_+$. As β_+ is a basis for τ_+, we have that U is a union of elements of β_+. Because U is compact, it is a finite such union, thus an element of β_+ as β_+ is closed under finite unions. Consequently, $\tau_+ \cap \delta_- \subseteq \beta_+$, and so $\beta_+ = \tau_+ \cap \delta_-$. A similar argument shows that $\beta_- = \tau_- \cap \delta_+$. It follows that (X, τ_+, τ_-) is pairwise zero-dimensional, and so (X, τ_+, τ_-) is a pairwise Stone space. ⊣

For a bounded lattice homomorphism $h : L \to L'$, let $f_h : \mathsf{pf}(L') \to \mathsf{pf}(L)$ be given by $f_h(x) = h^{-1}(x)$. It is easy to check that f_h is well-defined.

Proposition 5.2. *The map f_h is bi-continuous.*

Proof. Let $a \in L$. Then it is easy to verify that $f_h^{-1}(\phi_+(a)) = {\phi_+}'(ha)$ and $f_h^{-1}(\phi_-(a)) = {\phi_-}'(ha)$. Therefore, the inverse image of each element of β_+ is in ${\beta_+}'$ and the inverse image of each element of β_- is in ${\beta_-}'$. Thus, f_h is bi-continuous. ⊣

This allows us to define the contravariant functor $(-)_* : \mathbf{DLat} \to \mathbf{PStone}$ as follows. For a bounded distributive lattice L, we let $L_* = (X, \tau_+, \tau_-)$, where $X = \mathsf{pf}(L)$, τ_+ is the topology generated by the basis $\beta_+ = \phi_+[L]$, and τ_- is the topology generated by the basis

$\beta_- = \phi_-[L]$. For $h \in \hom(L, L')$, we let $h_* = h^{-1}$. It follows from Propositions 5.1 and 5.2 that the functor $(-)_*$ is well-defined.

For a pairwise Stone space (X, τ_1, τ_2) it is easy to see that $(\beta_1, \cap, \cup, \emptyset, X)$ is a bounded distributive lattice. (Note that $(\beta_2, \cap, \cup, \emptyset, X)$ is also a bounded distributive lattice dually isomorphic to $(\beta_1, \cap, \cup, \emptyset, X)$.) If $f : X \to X'$ is a bi-continuous map, then for each $U \in \beta_1'$, we have $U \in \tau_1' \cap \delta_2'$. Since f is bi-continuous, $f^{-1}(U) \in \tau_1 \cap \delta_2$. Therefore, $f^{-1}(U) \in \beta_1$. Moreover, it is clear that $f^{-1} : \beta_1' \to \beta_1$ is a bounded lattice homomorphism. We define the contravariant functor $(-)^* : \mathbf{PStone} \to \mathbf{DLat}$ as follows. For a pairwise Stone space (X, τ_1, τ_2), we let $(X, \tau_1, \tau_2)^* = (\beta_1, \cap, \cup, \emptyset, X)$, and for $f \in \hom(X, X')$, we let $f^* = f^{-1}$. Then the functor $(-)^*$ is well-defined.

Theorem 5.3. *The functors $(-)_*$ and $(-)^*$ set dual equivalence of* **DLat** *and* **PStone**.

Proof. For a bounded distributive lattice L, we have $L_*{}^* = \phi_+[L]$, and so ϕ_+ is a lattice isomorphism from L to $L_*{}^*$. For a pairwise Stone space (X, τ_1, τ_2), let $\psi : X \to X^*{}_*$ be given by $\psi(x) = \{U \in X^* \mid x \in U\}$. It is easy to see that ψ is well-defined. Since X is pairwise Hausdorff, ψ is 1-1. To see that ψ is onto, let P be a prime filter of β_1. We let $Q = \{V \in \beta_2 \mid Q^c \notin P\}$. It is easy to see that Q is a prime filter of β_2, and that $P \cup Q$ has the FIP. Since X is pairwise compact and pairwise Hausdorff, there is $x \in X$ such that $\bigcap(P \cup Q) = \{x\}$. Therefore, $\psi(x) = P$, and so ψ is onto. Moreover, for $U \in \beta_1$ we have $\psi^{-1}(\phi_+(U)) = U \in \beta_1$ and $\psi^{-1}(\phi_-(U)) = U^c \in \beta_2$. Therefore, f is bi-continuous. Furthermore, for $U \in \beta_1$, because ψ is a bijection, $\psi^{-1}(\phi_+(U)) = U$ implies $\psi(U) = \phi_+(U)$, and $\psi^{-1}(\phi_-(U)) = U^c$ implies $\psi(U^c) = \phi_-(U)$. Thus, f is bi-open, and so f is a bi-homeomorphism from X to $X^*{}_*$. That the functors $(-)_*$ and $(-)^*$ are natural is standard to prove. Consequently, $(-)_*$ and $(-)^*$ set dual equivalence of **DLat** and **PStone**. $\dashv$

Remark 5.4. It is worth pointing out that as in the case of the spectral and Priestley dualities, the dual equivalence between **DLat** and **PStone** is also induced by the *schizophrenic object* $\mathbf{2} = \{0, 1\}$. It has many lives: In **DLat** it is the two-element lattice; in **Spec** it is the *Sierpinski space* with the spectral topology $\tau_1 = \{\emptyset, \{1\}, \{0, 1\}\}$; in **Pries** it is the two-element ordered topological space with the discrete topology and the order $\leq$ given by $x \leq y$ iff $x = y$ or $x = 0$ and $y = 1$; finally in **PStone** it is the two element bitopological space with two Sierpinski topologies τ_1 and $\tau_2 = \{\emptyset, \{0\}, \{0, 1\}\}$.

6. Duality

In this section we use the isomorphism of **Pries**, **PStone**, and **Spec**, and their dual equivalence to **DLat** to obtain the dual description of the algebraic concepts important for the study of distributive lattices. In particular, we give the dual descriptions of filters, ideals, homomorphic images, sublattices, canonical completions, and MacNeille completions of bounded distributive lattices. We also give the dual description of complete bounded distributive lattices. The dual description of these concepts by means of Priestley spaces is known. Some of these concepts have also been described by means of spectral spaces. We complete the picture by giving the spectral description of the remaining concepts as well as describe them all by means of pairwise Stone spaces. At the end of the section we give a table, which serves as a dictionary of duality theory for distributive lattices, complementing the dictionary given in [22].

6.1. **Filters and ideals.** We start by the dual description of filters, prime filters, and maximal filters, as well as ideals, prime ideals, and maximal ideals of bounded distributive lattices. Let L be a bounded distributive lattice and let $(X, \tau, \leq)$ be the Priestley space of L. We recall that the poset $(\mathsf{Fi}(L), \supseteq)$ of filters of L is isomorphic to the poset $(\mathsf{ClUp}(X), \subseteq)$ of closed upsets of X, that the poset $(\mathsf{Id}(L), \subseteq)$ of ideals of L is isomorphic to the poset $(\mathsf{OpUp}(X), \subseteq)$ of open upsets of X, and that the isomorphisms are obtained as follows. With each filter F of L we associate the closed upset $C_F = \bigcap\{\varphi(a) \mid a \in L\}$ of X, and with each closed upset C of X we associate the filter $F_C = \{a \in L \mid C \subseteq \varphi(a)\}$ of L. Then $F \subseteq G$ iff $C_F \supseteq C_G$, $F_{C_F} = F$, and $C_{F_C} = C$. Therefore, $(\mathsf{Fi}(L), \supseteq)$ is isomorphic to $(\mathsf{ClUp}(X), \subseteq)$. Also, with each ideal I of L we associate the open upset $U_I = \bigcup\{\varphi(a) \mid a \in I\}$ of X, and with each open upset U of X we associate the ideal $I_U = \{a \in L \mid \varphi(a) \subseteq U\}$ of L. Then $I \subseteq J$ iff $U_I \subseteq U_J$, $I_{U_I} = I$, and $U_{I_U} = U$. Thus, $(\mathsf{Id}(L), \subseteq)$ is isomorphic to $(\mathsf{OpUp}(X), \subseteq)$.

Let (X, τ_1, τ_2) be the pairwise Stone space corresponding to $(X, \tau, \leq)$. By Proposition 3.6, $\beta_1 = \mathsf{CpUp}(X)$ and $\beta_2 = \mathsf{CpDo}(X)$. Therefore, $\tau_1 = \mathsf{OpUp}(X)$ and $\tau_2 = \mathsf{OpDo}(X)$, and so $\delta_1 = \mathsf{ClDo}(X)$ and $\delta_2 = \mathsf{ClUp}(X)$. Thus, $(\mathsf{Fi}(L), \supseteq)$ is isomorphic to $(\delta_2, \subseteq)$ and $(\mathsf{Id}(L), \subseteq)$ is isomorphic to $(\tau_1, \subseteq)$. Let (X, τ_1) be the spectral space corresponding to (X, τ_1, τ_2). Then clearly $(\mathsf{Id}(L), \subseteq)$ is isomorphic to the poset of τ_1-open sets. In order to characterize $(\mathsf{Fi}(L), \supseteq)$ in terms of (X, τ_1), we recall [10, Def. O-5.3] that a subset A of a topological space is *saturated* if it is an intersection of open subsets of the space; alternatively, A is saturated if it is an upset in the specialization order. We define A to be *co-saturated* if A is a union of closed subsets; alternatively, A is co-saturated if it is a downset in the specialization order.

Let $(X, \tau, \leq)$ be a Priestley space, (X, τ_1, τ_2) be the corresponding pairwise Stone space, and (X, τ_1) be the corresponding spectral space. Then it is clear that for $A \subseteq X$, we have that the following four conditions are equivalent: (i) A is an upset of $(X, \tau, \leq)$, (ii) A is a τ_1-saturated subset of (X, τ_1, τ_2), (iii) A is a τ_2-co-saturated subset of (X, τ_1, τ_2), and (iv) A is a saturated subset of (X, τ_1). Similarly, for $B \subseteq X$, we have that the following four conditions are equivalent: (i) B is a downset of $(X, \tau, \leq)$, (ii) B is a τ_1-co-saturated subset of (X, τ_1, τ_2), (iii) B is a τ_2-saturated subset of (X, τ_1, τ_2), and (iv) B is a co-saturated subset of (X, τ_1).

For a pairwise Stone space (X, τ_1, τ_2) and for $i = 1, 2$, let $\mathsf{S}_i(X)$ denote the set of τ_i-saturated sets and $\mathsf{CS}_i(X)$ denote the set of τ_i-co-saturated sets. Then $\mathrm{Up}(X) = \mathsf{S}_1(X) = \mathsf{CS}_2(X)$ and $\mathrm{Do}(X) = \mathsf{CS}_1(X) = \mathsf{S}_2(X)$. This gives us the following characterization of closed upsets and closed downsets of $(X, \tau, \leq)$.

Theorem 6.1. *Let $(X, \tau, \leq)$ be a Priestley space, (X, τ_1, τ_2) be the corresponding pairwise Stone space, and (X, τ_1) be the corresponding spectral space. For $C \subseteq X$, the following conditions are equivalent:*

(1) *C is a closed upset of $(X, \tau, \leq)$.*
(2) *C is a τ_2-closed set of (X, τ_1, τ_2).*
(3) *C is a compact and saturated set of (X, τ_1).*

Proof. As we already observed, (1)$\Leftrightarrow$(2) follows from Proposition 3.6. Next we show that (1)$\Rightarrow$(3). Since C is an upset of X, C is saturated in (X, τ_1). As C is closed in (X, τ) and (X, τ) is Hausdorff, C is a compact subset of (X, τ). Therefore, C is also compact in (X, τ_1). Thus, C is compact and saturated in (X, τ_1). Finally, we show that (3)$\Rightarrow$(1). Since C is saturated in (X, τ_1), C is an upset of X. We show that C is closed in (X, τ). Let $x \notin C$. Then for each $c \in C$ we have $c \not\leq x$. Therefore, there is a clopen upset U_c of X such that

$c \in U_c$ and $x \notin U_c$. Thus, $C \subseteq \bigcup\{U_c \mid c \in C\}$. By Propositions 3.6 and 4.2, each U_c belongs to $\mathcal{E}(X, \tau_1)$. Since C is compact, there are $c_1, \dots c_n \in C$ such that $C \subseteq U_{c_1} \cup \dots \cup U_{c_n}$. But then $V = U_{c_1}^c \cap \dots \cap U_{c_n}^c$ is a clopen downset of X containing x and having the empty intersection with C. Thus, C is closed. ⊣

A similar argument gives us:

Theorem 6.2. *Let $(X, \tau, \leq)$ be a Priestley space, (X, τ_1, τ_2) be the corresponding pairwise Stone space, and (X, τ_1) be the corresponding spectral space. For $D \subseteq X$, the following conditions are equivalent:*

(1) *D is a closed downset of $(X, \tau, \leq)$.*
(2) *D is a τ_1-closed set of (X, τ_1, τ_2).*
(3) *D is a compact and saturated set of (X, τ_2).*

For a pairwise Stone space (X, τ_1, τ_2) and $i = 1, 2$, let $\mathsf{KS}_i(X)$ denote the set of compact saturated subsets of X. Then the following characterization of filters and ideals of a bounded distributive lattice is an immediate consequence of the results obtained above.

Corollary 6.3. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be its Priestley space, (X, τ_1, τ_2) be its pairwise Stone space, and (X, τ_1) be its spectral space. Then:*

(1) $(\mathsf{Fi}(L), \supseteq) \simeq (\mathsf{ClUp}(X), \subseteq) = (\delta_2, \subseteq) = (\mathsf{KS}_1(X), \subseteq)$.
(2) $(\mathsf{Id}(L), \subseteq) \simeq (\mathsf{OpUp}(X), \subseteq) = (\tau_1, \subseteq)$.

Now we turn to the dual description of prime filters and prime ideals of L. Let $(X, \tau, \leq)$ be the Priestley space of L. It is well known that a filter F of L is prime iff $C_F = {\uparrow}x$ for some $x \in X$, and that an ideal I of L is prime iff $U_I = ({\downarrow}x)^c$ for some $x \in X$. Now we give the dual description of prime filters and prime ideals of L by means of pairwise Stone and spectral spaces of L.

Lemma 6.4. *Let $(X, \tau, \leq)$ be a Priestley space, (X, τ_1, τ_2) be the corresponding pairwise Stone space, and (X, τ_1) be the corresponding spectral space. Then for each $A \subseteq X$ we have:*

(1) $\mathrm{Cl}_1(A) = {\downarrow}\mathrm{Cl}(A)$.
(2) $\mathrm{Cl}_2(A) = {\uparrow}\mathrm{Cl}(A)$.

Proof. (1) We have $\mathrm{Cl}_1(A) = \bigcap\{B \in \delta_1 \mid A \subseteq B\} = \bigcap\{B \in \mathsf{ClUp}(X) \mid A \subseteq B\}$. By Lemma 3.2.2, ${\downarrow}\mathrm{Cl}(A)$ is a closed downset, and clearly $A \subseteq {\downarrow}\mathrm{Cl}(A)$. Therefore, $\mathrm{Cl}_1(A) \subseteq {\downarrow}\mathrm{Cl}(A)$. Conversely, suppose that $x \notin \mathrm{Cl}_1(A)$. Then there is $U \in \beta_1$ such that $x \in U$ and $U \cap A = \emptyset$. Since $\beta_1 = \mathsf{CpUp}(X)$, U is a clopen upset of X. As U is open in (X, τ), $U \cap A = \emptyset$ implies $U \cap \mathrm{Cl}(A) = \emptyset$. Because U is an upset, $U \cap \mathrm{Cl}(A) = \emptyset$ implies $U \cap {\downarrow}\mathrm{Cl}(A) = \emptyset$. Thus, $x \notin {\downarrow}\mathrm{Cl}(A)$, and so $\mathrm{Cl}_1(A) = {\downarrow}\mathrm{Cl}(A)$.

(2) is proved similarly. ⊣

Let (X, τ_1, τ_2) be a bitopological space. Following [10, Def. O-5.3], for $A \subseteq X$ and $i = 1, 2$, we define the *τ_i-saturation* of A as $\mathrm{Sat}_i(A) = \bigcap\{U \in \tau_i \mid A \subseteq U\}$. Obviously $\mathrm{Sat}_1(A) = {\uparrow}_1 A$ and $\mathrm{Sat}_2(A) = {\downarrow}_2 A$. This immediately gives us the following corollary to Lemma 6.4.

Corollary 6.5. *Let $(X, \tau, \leq)$ be a Priestley space, (X, τ_1, τ_2) be the corresponding pairwise Stone space, and (X, τ_1) be the corresponding spectral space. Then for each closed set A of (X, τ) we have:*

(1) ${\downarrow}A = \mathrm{Cl}_1(A) = \mathrm{Sat}_2(A)$.
(2) ${\uparrow}A = \mathrm{Cl}_2(A) = \mathrm{Sat}_1(A)$.

In particular, for each $x \in X$ we have:

(1) $\downarrow x = \mathrm{Cl}_1(x) = \mathrm{Sat}_2(x)$.
(2) $\uparrow x = \mathrm{Cl}_2(x) = \mathrm{Sat}_1(x)$.

Putting these results together, we obtain the following dual description of prime filters and prime ideals of L.

Corollary 6.6. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be its Priestley space, (X, τ_1, τ_2) be its pairwise Stone space, and (X, τ_1) be its spectral space. For a filter F of L, the following conditions are equivalent:*

(1) *F is a prime filter of L.*
(2) *$C_F = \uparrow x$ for some $x \in X$.*
(3) *$C_F = \mathrm{Cl}_2(x)$ for some $x \in X$.*
(4) *$C_F = \mathrm{Sat}_1(x)$ for some $x \in X$.*

Also, for an ideal I of L, the following conditions are equivalent:

(1) *I is a prime ideal of L.*
(2) *$U_I = (\downarrow x)^c$ for some $x \in X$.*
(3) *$U_I = [\mathrm{Cl}_1(x)]^c$ for some $x \in X$.*
(4) *$U_I = [\mathrm{Sat}_2(x)]^c$ for some $x \in X$.*

Another consequence of our results is the dual description of maximal filters and maximal ideals of L. Let $(X, \tau, \leq)$ be the Priestley space of L. We let $\max X$ and $\min X$ denote the sets of maximal and minimal points of X, respectively. From the dual description of prime filters and prime ideals of L it immediately follows that a filter F of L is maximal iff $C_F = \{x\}(= \uparrow x)$ for some $x \in \max X$, and that an ideal I of L is maximal iff $U_I = \{x\}^c(= (\downarrow x)^c)$ for some $x \in \min X$. This together with the above corollary immediately give us:

Corollary 6.7. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be its Priestley space, (X, τ_1, τ_2) be its pairwise Stone space, and (X, τ_1) be its spectral space. For a filter F of L, the following conditions are equivalent:*

(1) *F is a maximal filter of L.*
(2) *$C_F = \{x\}$ for some $x \in X$ with $\uparrow x = \{x\}$.*
(3) *$C_F = \{x\}$ for some $x \in X$ with $\mathrm{Cl}_2(x) = \{x\}$.*
(4) *$C_F = \{x\}$ for some $x \in X$ with $\mathrm{Sat}_1(x) = \{x\}$.*

Also, for an ideal I of L, the following conditions are equivalent:

(1) *I is a prime ideal of L.*
(2) *$U_I = \{x\}^c$ for some $x \in X$ with $\downarrow x = \{x\}$.*
(3) *$U_I = \{x\}^c$ for some $x \in X$ with $\mathrm{Cl}_1(x) = \{x\}$.*
(4) *$U_I = \{x\}^c$ for some $x \in X$ with $\mathrm{Sat}_2(x) = \{x\}$.*

6.2. **Homomorphic images.** It is well-known (see, e.g., [22, Cor. 2.5]) that homomorphic images of a bounded distributive lattice L are in a 1-1 correspondence with closed subsets of the Priestley space $(X, \tau, \leq)$ of L. Now we give the dual description of homomorphic images of L in terms of the pairwise Stone space and spectral space of L.

Lemma 6.8. *Let $(X, \tau, \leq)$ be a Priestley space and let (X, τ_1, τ_2) be its corresponding pairwise Stone space. For $C \subseteq X$, the following conditions are equivalent.*

(1) *C is closed in $(X, \tau, \leq)$.*

(2) *C is compact in $(X, \tau, \leq)$.*
(3) *C is pairwise compact in (X, τ_1, τ_2).*

Proof. That (1)$\Leftrightarrow$(2) is obvious since (X, τ) is compact and Hausdorff. That (2)$\Rightarrow$(3) is straightforward. To see that (3)$\Rightarrow$(2), it follows from (3) that each cover $\{U_i \mid i \in I\}$ of C, with $U_i \in \tau_1 \cup \tau_2$, has a finite subcover. Now use Alexander's Lemma. $\dashv$

For a topological space (X, τ) and a subset Y of X, let τ^Y denote the subspace topology on Y; that is, $\tau^Y = \{U \cap Y \mid U \in \tau\}$.

Definition 6.9. *Let (X, τ) be a spectral space. We call a subset Y of X a* spectral subset *of X if (Y, τ^Y) is a spectral space and $U \in \mathcal{E}(X, \tau)$ implies $U \cap Y \in \mathcal{E}(Y, \tau^Y)$.*

Theorem 6.10. *Let (X, τ_1, τ_2) be a pairwise Stone space and let (X, τ_1) be its corresponding spectral space. For $Y \subseteq X$, the following conditions are equivalent.*

(1) *Y is pairwise compact in (X, τ_1, τ_2).*
(2) *Y is a spectral subset of (X, τ_1).*

Proof. (1)$\Rightarrow$(2): Since Y is pairwise compact, by Theorem 6.8, Y is closed in the corresponding Priestley space $(X, \tau, \leq)$. Let $\leq^Y$ denote the restriction of $\leq$ to Y. Then $(Y, \tau^Y, \leq^Y)$ is a Priestley space. By Propositions 3.6 and 4.2, (Y, τ_1^Y) is a spectral space. Let $U \in \mathcal{E}(X)$. Again using Propositions 3.6 and 4.2 we obtain $U \in \mathsf{CpUp}(X, \tau, \leq)$. Therefore, $U \cap Y \in \mathsf{CpUp}(Y, \tau^Y, \leq^Y)$. Thus, $U \cap Y \in \mathcal{E}(Y, \tau_1^Y)$, and so Y is a spectral subset of (X, τ_1).

(2)$\Rightarrow$(1): Let Y be a spectral subset of (X, τ_1) and let $\Delta(Y, \tau_1^Y) = \{Y - U \mid U \in \mathcal{E}(Y, \tau^Y)\}$. We show that τ_2^Y is the topology generated by $\Delta(Y, \tau_1^Y)$. For this we show that $\mathcal{E}(Y, \tau_1^Y) = \{U \cap Y \mid U \in \mathcal{E}(X, \tau_1)\}$. Since Y is a spectral subset, we have $\{U \cap Y \mid U \in \mathcal{E}(X, \tau_1)\} \subseteq \mathcal{E}(Y, \tau_1^Y)$. Conversely, suppose that $U \in \mathcal{E}(Y, \tau_1^Y)$. Then there is $V \in \tau_1$ such that $U = V \cap Y$. From $V \in \tau_1$ it follows that $V = \bigcup\{V_i \mid i \in I\}$ for some family $\{V_i \mid i \in I\} \subseteq \mathcal{E}(X, \tau_1)$. Then $U = \bigcup\{V_i \mid i \in I\} \cap Y = \bigcup\{V_i \cap Y \mid i \in I\}$. Since U is compact and $V_i \cap Y$ are open in (Y, τ_1^Y), there exist $i_1, \ldots, i_n \in I$ such that $U = (V_{i_1} \cap Y) \cup \cdots \cup (V_{i_n} \cap Y) = (V_{i_1} \cup \cdots \cup V_{i_n}) \cap Y$. Let $W = V_{i_1} \cup \cdots \cup V_{i_n}$. Since $\mathcal{E}(X, \tau_1)$ is closed under finite unions, $W \in \mathcal{E}(X, \tau_1)$. Therefore, $U = W \cap Y$ for some $W \in \mathcal{E}(X, \tau_1)$. Thus, $\mathcal{E}(Y, \tau_1^Y) \subseteq \{U \cap Y \mid U \in \mathcal{E}(X, \tau_1)\}$, and so $\mathcal{E}(Y, \tau_1^Y) = \{U \cap Y \mid U \in \mathcal{E}(X, \tau_1)\}$. Consequently, $\Delta(Y, \tau_1^Y) = \{Y - U \mid U \in \mathcal{E}(Y, \tau_1^Y)\} = \{Y - (V \cap Y) \mid V \in \mathcal{E}(X, \tau_1)\} = \{Y - V \mid V \in \mathcal{E}(X, \tau_1)\}$, and so τ_2^Y is the topology generated by $\Delta(Y, \tau_1^Y)$. Now, since (Y, τ_1^Y) is a spectral space, by Proposition 4.5, (Y, τ_1^Y, τ_2^Y) is pairwise compact. It follows that Y is pairwise compact in (X, τ_1, τ_2). $\dashv$

Now putting the above results together, we obtain the following dual description of homomorphic images of L by means of all three dual spaces of L.

Corollary 6.11. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be its Priestley space, (X, τ_1, τ_2) be its pairwise Stone space, and (X, τ_1) be its spectral space. Then there is a 1-1 correspondence between (i) homomorphic images of L, (ii) closed subsets of $(X, \tau, \leq)$, (iii) pairwise compact subsets of (X, τ_1, τ_2), and (iv) spectral subsets of (X, τ_1).*

Proof. As follows from [22, Cor. 2.5], homomorphic images of L are in a 1-1 correspondence with closed subsets of $(X, \tau, \leq)$. Lemma 6.8 and Theorem 6.10 imply that closed subsets of $(X, \tau, \leq)$ are in a 1-1 correspondence with pairwise compact subsets of (X, τ_1, τ_2), which are in a 1-1 correspondence with spectral subsets of (X, τ_1). The result follows. $\dashv$

We conclude this subsection by giving an example of a subset Y of a spectral space (X, τ) such that (Y, τ^Y) is a spectral space, but there exists $U \in \mathcal{E}(X, \tau)$ such that $U \cap Y \notin \mathcal{E}(Y, \tau^Y)$.

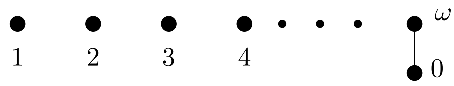

Figure 1

Therefore, the condition "$U \in \mathcal{E}(X, \tau)$ implies $U \cap Y \in \mathcal{E}(Y, \tau^Y)$" can not be omitted from Definition 6.9.

Example 6.12. Let (X, τ) be the ordinal $\omega + 1 = \omega \cup \{\omega\}$ with the interval topology. Then each $n \in \omega$ is an isolated point of X and ω is the only limit point of X. For $x, y \in X$ we set $x \leq y$ iff $x = y$ or $x = 0$ and $y = \omega$ (see Figure 1). It is easy to verify that $(X, \tau, \leq)$ is a Priestley space. Let (X, τ_1, τ_2) be the corresponding pairwise Stone space and (X, τ_1) be the corresponding spectral space. We let $Y = (\omega - \{0\}) \cup \{\omega\}$. Then Y is a closed subset of $(X, \tau, \leq)$, so $(Y, \tau^Y, \leq^Y)$ is a Priestley space, and so (Y, τ_1^Y) is a spectral space. On the other hand, $\omega \subseteq X$ is compact open in (X, τ_1). However, $\omega \cap Y = \omega - \{0\}$ is not compact in (Y, τ^Y). Therefore, Y is not a spectral subset of (X, τ_1).

6.3. **Sublattices.** The dual description of bounded sublattices of a bounded distributive lattice by means of its Priestley space can be found in [1, 3, 26]. We will rephrase it in our terminology. We recall that a *quasi-order* Q on a set X is a reflexive and transitive relation on X. We call the pair (X, Q) a *quasi-ordered set*. For a quasi-ordered set (X, Q), we call $A \subseteq X$ a *Q-upset* of X if $x \in A$ and xQy imply $y \in A$.

Definition 6.13. *Let X be a topological space and Q be a quasi-order on X. We call Q a* Priestley quasi-order *on X if for each $x, y \in X$ with $x \not\!Q y$ there exists a clopen Q-upset A of X such that $x \in A$ and $y \notin A$.*

Theorem 6.14. [26, Thm. 3.7] *Let L be a bounded distributive lattice and $(X, \tau, \leq)$ be the Priestley space of L. Then there is a dual isomorphism between the poset $(\mathsf{S}_L, \subseteq)$ of bounded sublattices of L and the poset $(\mathsf{Q}_X, \subseteq)$ of Priestley quasi-orders on X extending $\leq$.*

Proof. (Sketch) For $S \in \mathsf{S}_L$, we define Q_S on X by xQ_Sy iff $x \cap S \subseteq y \cap S$. Then $Q_S \in \mathsf{Q}_X$, and $S \subseteq K$ implies $Q_K \subseteq Q_S$ for each $S, K \in \mathsf{S}_L$. Therefore, $S \mapsto Q_S$ is an order-reversing map from S_L to Q_X. For a Priestley quasi-order Q on X, we let $S_Q = \{a \in L \mid \phi(a)$ is a Q-upset of $X\}$. Then S_Q is a bounded sublattice of L, and $Q \subseteq R$ implies $S_R \subseteq S_Q$ for each $Q, R \in \mathsf{Q}_X$. Thus, $Q \mapsto S_Q$ is an order-reversing map from Q_X to S_L. Moreover, $S_{Q_S} = S$ and $Q_{S_Q} = Q$ for each $S \in \mathsf{S}_L$ and $Q \in \mathsf{Q}_X$. It follows that the order-reversing maps $S \mapsto Q_S$ and $Q \mapsto S_Q$ are inverses of each other. Consequently, $(\mathsf{S}_L, \subseteq)$ is dually isomorphic to $(\mathsf{Q}_X, \subseteq)$. ⊣

Now we characterize Priestley quasi-orders extending $\leq$ by means of pairwise Stone spaces and spectral spaces.

Definition 6.15. *Let (τ_1, τ_2) and (τ_1', τ_2') be two bitopologies on X. We say that (τ_1, τ_2) is finer than (τ_1', τ_2') and that (τ_1', τ_2') is coarser than (τ_1, τ_2) if $\tau_1' \subseteq \tau_1$ and $\tau_2' \subseteq \tau_2$.*

Lemma 6.16. *Let $(X, \tau, \leq)$ be a Priestley space and (X, τ_1, τ_2) be the corresponding pairwise Stone space. Then the poset $(\mathsf{Q}_X, \subseteq)$ of Priestley quasi-orders on X is dually isomorphic to the poset $(\mathsf{Z}_X, \subseteq)$ of pairwise zero-dimensional bi-topologies on X coarser than (τ_1, τ_2).*

Proof. For a Priestley quasi-order Q on X, let τ_1^Q be the set of open Q-upsets and τ_2^Q be the set of open Q-downsets of X. Clearly (τ_1^Q, τ_2^Q) is a bi-topology on X coarser than (τ_1, τ_2). Moreover, $\beta_1^Q = \tau_1^Q \cap \delta_2^Q$ is exactly the set of clopen Q-upsets of X and $\beta_2^Q = \tau_2^Q \cap \delta_1^Q$ is exactly the set of clopen Q-downsets of X. Since Q is a Priestley quasi-order, clopen Q-upsets are a basis for open Q-upsets and clopen Q-downsets are a basis for open Q-downsets. Therefore, (τ_1^Q, τ_2^Q) is pairwise zero-dimensional. For two Priestley quasi-orders Q and R on X, we show $Q \subseteq R$ implies $\tau_1^R \subseteq \tau_1^Q$ and $\tau_2^R \subseteq \tau_2^Q$. Let $U \in \tau_1^R$. Then U is an open R-upset of X. Since $Q \subseteq R$, U is also a Q-upset of X. Thus, $U \in \tau_1^Q$. That $\tau_2^R \subseteq \tau_2^Q$ is proved similarly. It follows that $Q \mapsto (\tau_1^Q, \tau_2^Q)$ is an order-reversing map from Q_X to Z_X.

Let (τ_1', τ_2') be a pairwise zero-dimensional bi-topology on X coarser than (τ_1, τ_2). We define $Q_{(\tau_1', \tau_2')}$ to be the specialization order of τ_1'. Since (τ_1', τ_2') is pairwise zero-dimensional, $Q_{(\tau_1', \tau_2')}$ is the dual of the specialization order of τ_2'. Because $Q_{(\tau_1', \tau_2')}$ is a specialization order, it is clear that $Q_{(\tau_1', \tau_2')}$ is a quasi-order. From $\tau_1' \subseteq \tau_1$ it follows that $Q_{(\tau_1', \tau_2')}$ extends the specialization order of τ_1. Consequently, $Q_{(\tau_1', \tau_2')}$ extends $\leq$. We show that $Q_{(\tau_1', \tau_2')}$ is a Priestley quasi-order. If $x \not{Q}_{(\tau_1', \tau_2')} y$, then there exists $U \in \tau_1'$ such that $x \in U$ and $y \notin U$. Since (τ_1', τ_2') is pairwise zero-dimensional, we may assume that $U \in \beta_1'$. Therefore, U is clopen in τ. Clearly each $U \in \tau_1'$ is a $Q_{(\tau_1', \tau_2')}$-upset. Thus, there exists a clopen $Q_{(\tau_1', \tau_2')}$-upset U of X such that $x \in U$ and $y \notin U$. For $(\tau_1', \tau_2'), (\tau_1'', \tau_2'') \in \mathsf{Z}_X$, we show $(\tau_1', \tau_2') \subseteq (\tau_1'', \tau_2'')$ implies $Q_{(\tau_1'', \tau_2'')} \subseteq Q_{(\tau_1', \tau_2')}$. Let $x Q_{(\tau_1'', \tau_2'')} y$. Then $x \in U$ implies $y \in U$ for each $U \in \tau_1''$. Therefore, $x \in U$ implies $y \in U$ for each $U \in \tau_1'$. Thus, $x Q_{(\tau_1', \tau_2')} y$. It follows that $(\tau_1', \tau_2') \mapsto Q_{(\tau_1', \tau_2')}$ is an order-reversing map from Z_X to Q_X.

We show that $Q_{(\tau_1^Q, \tau_2^Q)} = Q$ and $(\tau_1^{Q_{(\tau_1', \tau_2')}}, \tau_2^{Q_{(\tau_1', \tau_2')}}) = (\tau_1', \tau_2')$ for each $Q \in \mathsf{Q}_X$ and $(\tau_1', \tau_2') \in \mathsf{Z}_X$. Indeed, $x Q_{(\tau_1^Q, \tau_2^Q)} y$ iff $(\forall U \in \tau_1^Q)(x \in U \Rightarrow y \in U)$, which is equivalent to xQy since Q is a Priestley quasi-order. Thus, $Q_{(\tau_1^Q, \tau_2^Q)} = Q$. Moreover, $U \in \tau_1^{Q_{(\tau_1', \tau_2')}}$ iff U is an open $Q_{(\tau_1', \tau_2')}$-upset of X. Clearly $U \in \tau_1'$ implies U is an open $Q_{(\tau_1', \tau_2')}$-upset of X. Conversely, let U be an open $Q_{(\tau_1', \tau_2')}$-upset of X. We show that $U = \bigcup\{V \in \tau_1' \mid V \subseteq U\}$. Clearly $\bigcup\{V \in \tau_1' \mid V \subseteq U\} \subseteq U$. Let $x \in U$. Since U is a $Q_{(\tau_1', \tau_2')}$-upset, for each $y \in U^c$ we have $x \not{Q}_{(\tau_1', \tau_2')} y$. Therefore, there exists $V_y \in \tau_1'$ such that $x \in V_y$ and $y \notin V_y$. Since β_1' is a basis for τ_1', we may assume that $V_y \in \beta_1'$. Thus, $\bigcap\{V_y \mid y \in U^c\} \cap U^c = \emptyset$. Since U^c and each V_y is closed in τ and τ is compact, there exist $V_1, \dots, V_n \in \beta_1'$ such that $V_1 \cap \cdots \cap V_n \cap U^c = \emptyset$. So $x \in V_1 \cap \cdots \cap V_n \subseteq U^c$, and so $U \subseteq \bigcup\{V \in \tau_1' \mid V \subseteq U\}$. Consequently, $U \in \tau_1'$. This implies that $\tau_1^{Q_{(\tau_1', \tau_2')}} = \tau_1'$. A similar argument shows that $\tau_2^{Q_{(\tau_1', \tau_2')}} = \tau_2'$. Thus, $(\tau_1^{Q_{(\tau_1', \tau_2')}}, \tau_2^{Q_{(\tau_1', \tau_2')}}) = (\tau_1', \tau_2')$. It follows that the order-reversing maps $Q \mapsto (\tau_1^Q, \tau_2^Q)$ and $(\tau_1', \tau_2') \mapsto Q_{(\tau_1', \tau_2')}$ are inverses of each other. Thus, $(\mathsf{Q}_X, \subseteq)$ is dually isomorphic to $(\mathsf{Z}_X, \subseteq)$. ⊣

Definition 6.17. *Let τ be a spectral topology on X and let τ' be a coherent topology on X coarser than τ. We call τ'* strongly coherent *if the set $\mathcal{E}(X, \tau')$ of compact open subsets of (X, τ') is equal to the set $\tau' \cap \sigma$ of open subsets of (X, τ') that are compact in (X, τ).*

Lemma 6.18. *Let (X, τ_1, τ_2) be a pairwise Stone space and (X, τ_1) be the corresponding spectral space. Then the poset $(\mathsf{Z}_X, \subseteq)$ of pairwise zero-dimensional bi-topologies (τ_1', τ_2') on X coarser than (τ_1, τ_2) is isomorphic to the poset $(\mathsf{SC}_X, \subseteq)$ of strongly coherent topologies τ_1' on X coarser than τ_1.*

Proof. Let (τ_1', τ_2') be a pairwise zero-dimensional bi-topology on X coarser than (τ_1, τ_2). Then τ_1' is a topology on X coarser than τ_1. Let $\beta_1' = \tau_1' \cap \delta_2'$. We show that $\mathcal{E}(X, \tau_1') = \beta_1' = \tau_1' \cap \sigma_1$. Let $U \in \mathcal{E}(X, \tau_1')$. Since β_1' is a basis for τ_1', U is the union of elements of β_1' contained in U. As U is compact in (X, τ_1'), U is a finite union of elements of β_1', so U is an element of β_1', and so $\mathcal{E}(X, \tau_1') \subseteq \beta_1'$. Now let $U \in \beta_1'$. Because (X, τ_1, τ_2) is pairwise compact, $\delta_2 \subseteq \sigma_1$. Therefore, $\delta_2' \subseteq \delta_2 \subseteq \sigma_1$, and so $\beta_1' \subseteq \tau_1' \cap \delta_2' \subseteq \tau_1' \cap \sigma_1$. Finally, let $U \in \tau_1' \cap \sigma_1$. Since $U \in \tau_1'$ and $\mathcal{E}(X, \tau_1')$ is a basis for τ_1', U is the union of elements of $\mathcal{E}(X, \tau_1')$ contained in U. Because $U \in \sigma_1$ and $\tau_1' \subseteq \tau_1$, U is a finite union of elements of $\mathcal{E}(X, \tau_1')$. Therefore, $U \in \mathcal{E}(X, \tau_1')$, and so $\tau_1' \cap \sigma_1 \subseteq \mathcal{E}(X, \tau_1')$. Thus, $\mathcal{E}(X, \tau_1') = \beta_1' = \tau_1' \cap \sigma_1$, implying that τ_1' is a strongly coherent topology. For $(\tau_1', \tau_2'), (\tau_1'', \tau_2'') \in \mathsf{Z}_X$, if $(\tau_1', \tau_2') \subseteq (\tau_1'', \tau_2'')$, then it is obvious that $\tau_1' \subseteq \tau_1''$. It follows that $(\tau_1', \tau_2') \mapsto \tau_1'$ is an order-preserving map from Z_X to SC_X.

For a strongly coherent topology τ_1' on X coarser than τ_1, we let τ_2' be the topology generated by the basis $\Delta(X, \tau_1') = \{U^c \mid U \in \mathcal{E}(X, \tau_1')\}$. Let δ_1' denote the set of closed subsets of (X, τ_1') and δ_2' denote the set of closed subsets of (X, τ_2'). We set $\beta_1' = \tau_1' \cap \delta_2'$ and $\beta_2' = \tau_2' \cap \delta_1'$. We show that $\beta_1' = \mathcal{E}(X, \tau_1')$ and $\beta_2' = \Delta(X, \tau_1')$. It follows from the definition that $\mathcal{E}(X, \tau_1') \subseteq \beta_1'$. Conversely, $\beta_1' = \tau_1' \cap \delta_2' \subseteq \tau_1' \cap \delta_2 \subseteq \tau_1' \cap \sigma_1 = \mathcal{E}(X, \tau_1')$. Therefore, $\beta_1' = \mathcal{E}(X, \tau_1')$. Also, $U \in \Delta(X, \tau_1')$ iff $U^c \in \mathcal{E}(X, \tau_1')$ iff $U^c \in \beta_1'$ iff $U^c \in \tau_1' \cap \delta_2'$ iff $U \in \delta_1' \cap \tau_2'$ iff $U \in \beta_2'$. Thus, $\beta_2' = \Delta(X, \tau_1')$. Consequently, β_1' is a basis for τ_1' and β_2' is a basis for τ_2', and so (τ_1', τ_2') is pairwise zero-dimensional. For $\tau_1', \tau_1'' \in \mathsf{SC}_X$, we show $\tau_1' \subseteq \tau_1''$ implies $(\tau_1', \tau_2') \subseteq (\tau_1'', \tau_2'')$. Let $U \in \Delta(X, \tau_1')$. Then $U^c \in \mathcal{E}(X, \tau_1')$. Therefore, $U^c \in \tau_1' \cap \sigma_1 \subseteq \tau_1'' \cap \sigma_1$, and so $U^c \in \mathcal{E}(X, \tau_1'')$. Thus, $U \in \Delta(X, \tau_1'')$, so $\Delta(X, \tau_1') \subseteq \Delta(X, \tau_1'')$, and so $\tau_2' \subseteq \tau_2''$. It follows that $\tau_1' \mapsto (\tau_1', \tau_2')$ is an order-preserving map from SC_X to Z_X.

Finally, if $(\tau_1', \tau_2') \in \mathsf{Z}_X$, then $\mathcal{E}(X, \tau_1') = \beta_1'$, so $\Delta(X, \tau_1') = \beta_2'$, and so the composition $\mathsf{Z}_X \to \mathsf{SC}_X \to \mathsf{Z}_X$ is an identity. Moreover, it is clear that the composition $\mathsf{SC}_X \to \mathsf{Z}_X \to \mathsf{SC}_X$ is also an identity. Thus, $(\mathsf{Z}_X, \subseteq)$ is isomorphic to $(\mathsf{SC}_X, \subseteq)$. $\dashv$

Putting Theorem 6.14 and Lemmas 6.16 and 6.18 together, we obtain the following dual description of bounded sublattices of L by means of all three dual spaces of L.

Corollary 6.19. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of L, (X, τ_1, τ_2) be the pairwise Stone space of L, and (X, τ_1) be the spectral space of L. Then the poset $(\mathsf{S}_L, \subseteq)$ of bounded sublattices of L is dually isomorphic to the poset $(\mathsf{Q}_X, \subseteq)$ of Priestley quasi-orders on X extending $\leq$, and is isomorphic to the poset $(\mathsf{Z}_X, \subseteq)$ of pairwise zero-dimensional bi-topologies on X coarser than (τ_1, τ_2), and to the poset $(\mathsf{SC}_X, \subseteq)$ of strongly coherent topologies on X coarser than τ_1.*

6.4. **Canonical completions, MacNeille completions, and complete lattices.** In the theory of completions of lattices, or more generally of posets, the MacNeille and canonical completions play a prominent role. Let L be a lattice. We recall that a subset S of L is *join-dense* in L if for each $a \in L$ we have $a = \bigvee({\downarrow} a \cap S)$, and that S is *meet-dense* in L if for each $a \in L$ we have $a = \bigwedge({\uparrow} a \cap S)$. We further recall that the *MacNeille completion* of L is a unique up to isomorphism complete lattice $\overline{L}$ together with a lattice embedding $i : L \to \overline{L}$ such that $i[L]$ is both join-dense and meet-dense in L. Furthermore, we recall that the *canonical completion* of L is a unique up to isomorphism complete lattice L^σ together with a lattice embedding $j : L \to L^\sigma$ such that (i) for each filter F and ideal I of L, from $F \cap I = \emptyset$ it follows that $\bigwedge j[F] \not\leq \bigvee j[I]$, (ii) the set $K_L = \{\bigwedge j[S] \mid S \subseteq L\}$ of closed elements of L^σ is join-dense in L^σ, and (iii) the set $O_L = \{\bigvee j[S] \mid S \subseteq L\}$ of open elements of L^σ is meet-dense in L^σ.

For a Priestley space $(X, \tau, \leq)$, following [11, Sec. 3], we define two maps $\mathbf{D} : \mathsf{OpUp}(X) \to \mathsf{ClUp}(X)$ and $\mathbf{J} : \mathsf{ClUp}(X) \to \mathsf{OpUp}(X)$ by $\mathbf{D}(U) = {\uparrow}\mathrm{Cl}(U)$ and $\mathbf{J}(K) = ({\downarrow}(\mathrm{Int}K)^c)^c$ for $U \in \mathsf{OpUp}(X)$ and $K \in \mathsf{ClUp}(X)$. Then it follows from [11, Lemma 3.4] that $\mathbf{D}$ and $\mathbf{J}$ form a Galois connection between $(\mathsf{OpUp}(X), \subseteq)$ and $(\mathsf{ClUp}(X), \supseteq)$. Let $\mathsf{RgOpUp}(X)$ denote the set of fixpoints of $\mathbf{J} \circ \mathbf{D}$; that is, $\mathsf{RgOpUp}(X) = \{U \in \mathsf{OpUp}(X) \mid \mathbf{JD}U = U\}$. The next theorem is well-known. The first half of it can be found in [11, Thm. 3.5], and the second half in [9, Sec. 2].

Theorem 6.20. *Let L be a bounded distributive lattice and $(X, \tau, \leq)$ be the Priestley space of L. Then $\overline{L}$ is isomorphic to $\mathsf{RgOpUp}(X)$ and L^σ is isomorphic to $\mathsf{Up}(X)$.*

Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of L, (X, τ_1, τ_2) be the pairwise Stone space of L, and (X, τ_1) be the spectral space of L. Since $\mathsf{Up}(X) = \mathsf{S}_1(X) = \mathsf{CS}_2(X)$, we immediately obtain the following dual description of the canonical completion of L.

Theorem 6.21. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of L, (X, τ_1, τ_2) be the pairwise Stone space of L, and (X, τ_1) be the spectral space of L. Then L^σ is isomorphic to $\mathsf{Up}(X) = \mathsf{S}_1(X) = \mathsf{CS}_2(X)$.*

Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of L, and (X, τ_1, τ_2) be the pairwise Stone space of L. Since $\mathsf{OpUp}(X) = \tau_1$, $\mathsf{ClUp}(X) = \delta_2$, $\mathbf{D}(U) = \mathrm{Cl}_2(U)$, and $\mathbf{J}(U) = \mathrm{Int}_1(U)$ for $U \subseteq X$, we obtain that $\mathrm{Cl}_2 : \tau_1 \to \delta_2$ and $\mathrm{Int}_1 : \delta_2 \to \tau_1$ form a Galois connection between $(\tau_1, \subseteq)$ and $(\delta_2, \supseteq)$, and so the MacNeille completion $\overline{L}$ of L is isomorphic to the fixpoints of $\mathrm{Int}_1 \circ \mathrm{Cl}_2$, we denote by $\mathsf{RgOp}_{12}(X)$.

Let (X, τ_1) be the spectral space corresponding to the pairwise Stone space (X, τ_1, τ_2). Then $\delta_2 = \mathsf{KS}_1(X)$ and $\mathrm{Cl}_2(U) = \mathrm{Sat}_1\mathrm{Cl}(U)$ for $U \subseteq X$. Let $\mathrm{S}_1 = \mathrm{Sat}_1 \circ \mathrm{Cl}$. Then $\mathrm{S}_1 : \tau_1 \to \mathsf{KS}_1(X)$ and $\mathrm{Int}_1 : \mathsf{KS}_1(X) \to \tau_1$ form a Galois connection between $(\tau_1, \subseteq)$ and $(\mathsf{KS}_1(X), \supseteq)$, and so the MacNeille completion $\overline{L}$ of L is isomorphic to the fixpoints of $\mathrm{Int}_1 \circ \mathrm{S}_1$, we denote by $\mathsf{SatOp}_1(X)$. Consequently, we obtain the following dual description of the MacNeille completion of L.

Theorem 6.22. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of L, (X, τ_1, τ_2) be the pairwise Stone space of L, and (X, τ_1) be the spectral space of L. Then $\overline{L}$ is isomorphic to $\mathsf{RgOpUp}(X) = \mathsf{RgOp}_{12}(X) = \mathsf{SatOp}_1(X)$.*

The bitopological description of $\overline{L}$ provides a nice generalization of the characterization of the MacNeille completion of a Boolean algebra B by means of the regular open subsets of the Stone space (X, τ) of B. We recall that the regular open subsets of (X, τ) are exactly the fixpoints of the maps $\mathrm{Cl} : \tau \to \delta$ and $\mathrm{Int} : \delta \to \tau$. When working with a pairwise Stone space (X, τ_1, τ_2), we consider the fixpoints of the maps Cl_2 and Int_1 between τ_1 and δ_2, respectively. Therefore, whenever $\tau_1 = \tau_2$, the pairwise Stone space (X, τ_1, τ_2) becomes the Stone space (X, τ), where $\tau = \tau_1 = \tau_2$. So $\tau_1 = \tau$, $\delta_2 = \delta$, $\mathrm{Cl}_2 = \mathrm{Cl}$, $\mathrm{Int}_1 = \mathrm{Int}$, and the fixpoints of $\mathrm{Int}_1 \circ \mathrm{Cl}_2$ are exactly the regular open subsets of (X, τ). As a corollary, we obtain the well-known dual description of the MacNeille completion of a Boolean algebra:

Corollary 6.23. *Let B be a Boolean algebra and (X, τ) be the Stone space of B. Then the MacNeille completion $\overline{B}$ of B is isomorphic to the regular open subsets $\mathsf{RgOp}(X, \tau)$ of (X, τ).*

Since L is a complete lattice iff L is isomorphic to $\overline{L}$, it follows from the construction of $\overline{L}$ that L is complete iff in the dual Priestley space $(X, \tau, \leq)$ of L we have $\mathsf{RgOpUp}(X) =$

DLat	**Pries**	**PStone**	**Spec**
filter	closed upset	τ_2-closed set	compact saturated set
ideal	open upset	τ_1-open set	open set
prime filter	$\uparrow x$	$\mathrm{Cl}_2(x)$	$\mathrm{Sat}(x)$
prime ideal	$(\downarrow x)^c$	$[\mathrm{Cl}_1(x)]^c$	$[\mathrm{Cl}(x)]^c$
maximal filter	$\uparrow x = \{x\}$	$\mathrm{Cl}_2(x) = \{x\}$	$\mathrm{Sat}(x) = \{x\}$
maximal ideal	$(\downarrow x)^c = \{x\}^c$	$[\mathrm{Cl}_1(x)]^c = \{x\}^c$	$[\mathrm{Cl}(x)]^c = \{x\}^c$
homomorphic image	closed subset	pairwise compact subset	spectral subset
subalgebra	$Q \in \mathsf{Q}_X$	$(\tau_1', \tau_2') \in \mathsf{Z}_X$	$\tau' \in \mathsf{SC}_X$
canonical completion	$\mathsf{Up}(X)$	$\mathsf{S}_1(X) = \mathsf{CS}_2(X)$	$\mathsf{S}(X)$
MacNeille complition	$\mathsf{RgOpUp}(X)$	$\mathsf{RgOp}_{12}(X)$	$\mathsf{SatOp}(X)$
complete lattice	$\mathsf{RgOpUp}(X) = \mathsf{CpUp}(X)$	$\beta_1 = \mathsf{RgOp}_{12}(X)$	$\mathcal{E}(X) = \mathsf{SatOp}(X)$

TABLE 1. Dictionary for **DLat**, **Pries**, **PStone**, and **Spec**.

$\mathsf{ClUp}(X)$ (see [21, Prop. 16] and [11, p. 948]). Such Priestley spaces were called *extremally order disconnected* in [21, p. 521]. This together with Theorem 6.22 immediately give us the following dual description of complete distributive lattices.

Theorem 6.24. *Let L be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of L, (X, τ_1, τ_2) be the pairwise Stone space of L, and (X, τ_1) be the spectral space of L. Then the following conditions are equivalent:*

1. *L is complete.*
2. *$\mathsf{RgOpUp}(X) = \mathsf{ClUp}(X)$.*
3. *$\mathsf{RgOp}_{12}(X) = \beta_1$.*
4. *$\mathsf{SatOp}_1(X) = \mathcal{E}(X, \tau_1)$.*

In Table 1 we gather together the dual descriptions of different algebraic concepts for bounded distributive lattices by means of their Priestley spaces, pairwise Stone spaces, and spectral spaces obtained in this section. This can be thought of as a dictionary of duality theory for bounded distributive lattices, complementing the dictionary given in [22].

7. Duality for Heyting algebras, co-Heyting algebras, and bi-Heyting algebras

A rather natural subclass of distributive lattices is the class of Heyting algebras (resp. co-Heyting algebras/bi-Heyting algebras), which plays an important role in the study of superintuitionistic logics. The first duality for Heyting algebras (resp. co-Heyting algebras/bi-Heyting algebras) was developed by Esakia [5] (resp. [6]). It is a restricted version of Priestley's duality. In this section we develop duality for Heyting algebras (resp. co-Heyting algebras/bi-Heyting algebras) by means of pairwise Stone spaces and spectral spaces, thus providing the bitopological and spectral alternatives of the Esakia duality.

We recall that a *Heyting algebra* is a bounded distributive lattice $(A, \wedge, \vee, 0, 1)$ with a binary operation $\to: A^2 \to A$ such that for all $a, b, c \in A$ we have:

$$c \leq a \to b \text{ iff } a \wedge c \leq b.$$

Similarly a *co-Heyting algebra* is a bounded distributive lattice A with a binary operation $\leftarrow: A^2 \to A$ such that for all $a, b, c \in A$ we have:

$$c \geq a \leftarrow b \text{ iff } b \vee c \geq a.$$

We call $(A, \to, \leftarrow)$ a *bi-Heyting algebra* if $(A, \to)$ is a Heyting algebra and $(A, \leftarrow)$ is a co-Heyting algebra.

Let A and A' be two Heyting algebras. We recall that a map $h : A \to A'$ is a *Heyting algebra homomorphism* if h is a bounded lattice homomorphism and $h(a \to b) = h(a) \to' h(b)$ for each $a, b \in A$. Similarly, if A and A' are two co-Heyting algebras, then $h : A \to A'$ is a *co-Heyting algebra homomorphism* if h is a bounded lattice homomorphism and $h(a \leftarrow b) = h(a) \leftarrow' h(b)$ for each $a, b \in A$. If A and A' are two bi-Heyting algebras, then h is a *bi-Heyting algebra homomorphism* if it is both a Heyting algebra homomorphism and a co-Heyting algebra homomorphism. Let **Heyt** denote the category of Heyting algebras and Heyting algebra homomorphisms, **coHeyt** denote the category of co-Heyting algebras and co-Heyting algebra homomorphisms, and **biHeyt** denote the category of bi-Heyting algebras and bi-Heyting algebra homomorphisms. Clearly **biHeyt** = **Heyt** $\cap$ **coHeyt**.

For a topological space (X, τ), let $\mathsf{Cp}(X)$ denote the set of clopen subsets of X.

Definition 7.1. *Let $(X, \tau, \leq)$ be a Priestley space.*

(1) *We call $(X, \tau, \leq)$ an* Esakia space *if $A \in \mathsf{Cp}(X)$ implies $\downarrow A \in \mathsf{Cp}(X)$.*
(2) *We call $(X, \tau, \leq)$ a* co-Esakia space *if $A \in \mathsf{Cp}(X)$ implies $\uparrow A \in \mathsf{Cp}(X)$.*
(3) *We call $(X, \tau, \leq)$ a* bi-Esakia space *if it is both an Esakia space and a co-Esakia space.*

Let $(X, \leq)$ and $(X', \leq')$ be two posets. We recall that a map $f : X \to X'$ is a *p-morphism* if it is order-preserving and for each $x \in X$ and $x' \in X'$, from $f(x) \leq x'$ it follows that there is $y \in X$ such that $x \leq y$ and $f(y) = x'$. We call $f : X \to X'$ a *co-p-morphism* if it is order-preserving and for each $x \in X$ and $x' \in X'$, from $x' \leq f(x)$ it follows that there is $y \in X$ such that $y \leq x$ and $f(y) = x'$. For two Esakia spaces (resp. co-Esakia spaces) $(X, \tau, \leq)$ and $(X', \tau', \leq')$, we call a map $f : X \to X'$ an *Esakia morphism* (resp. a *co-Esakia morphism*) if it is a continuous p-morphism (resp. a continuous co-p-morphism). We call f a *bi-Esakia morphism* if it is both an Esakia morphism and a co-Esakia morphism. Let **Esa** denote the category of Esakia spaces and Esakia morphisms, **coEsa** denote the category of co-Esakia spaces and co-Esakia morphisms, and **biEsa** denote the category of bi-Esakia spaces and bi-Esakia morphisms. Then we have the following theorem established in [5] and [6]:

Theorem 7.2. **Heyt** *is dually equivalent to* **Esa***,* **coHeyt** *is dually equivalent to* **coEsa***, and* **biHeyt** *is dually equivalent to* **biEsa***.*

In fact, the same functors establishing the dual equivalence of **DLat** and **Pries** restricted to **Heyt** (resp. **coHeyt**/**biHeyt**) establish the required dual equivalences. In order to describe the pairwise Stone spaces and spectral spaces dual to Heyting algebras (resp. coHeyting algebras/bi-Heyting algebras), it is sufficient to characterize those pairwise Stone spaces and spectral spaces that correspond to Esakia spaces (resp. coEsakia spaces/biEsakia spaces). As an immediate consequence of Lemma 6.8 and Theorem 6.10, we obtain:

Lemma 7.3. *Let $(X, \tau, \leq)$ be a Priestley space, (X, τ_1, τ_2) be the corresponding pairwise Stone space, and (X, τ_1) be the corresponding spectral space. For $Y \subseteq X$, the following conditions are equivalent:*

(1) *Y is clopen in $(X, \tau, \leq)$.*
(2) *Y and Y^c are pairwise compact in (X, τ_1, τ_2).*
(3) *Y and Y^c are spectral subsets of (X, τ_1).*

Let (X, τ_1, τ_2) be a pairwise Stone space. We call $Y \subseteq X$ *doubly pairwise compact* if both Y and Y^c are pairwise compact in (X, τ_1, τ_2). Let $\mathsf{DPC}(X)$ denote the set of doubly pairwise compact subsets of (X, τ_1, τ_2).

Definition 7.4. *Let (X, τ_1, τ_2) be a pairwise Stone space.*

(1) *We call (X, τ_1, τ_2) a* Heyting bitopological space *if $A \in \mathsf{DPC}(X)$ implies $\mathrm{Cl}_1(A) \in \mathsf{DPC}(X)$.*
(2) *We call (X, τ_1, τ_2) a* co-Heyting bitopological space *if $A \in \mathsf{DPC}(X)$ implies $\mathrm{Cl}_2(A) \in \mathsf{DPC}(X)$.*
(3) *We call (X, τ_1, τ_2) a* bi-Heyting bitopological space *if it is both a Heyting bitopological space and a co-Heyting bitopological space.*

Theorem 7.5. *Let (X, τ_1, τ_2) be a pairwise Stone space.*

(1) *(X, τ_1, τ_2) is a Heyting bitopological space iff for each $A \in \beta_1$ and $B \in \beta_2$ we have $\mathrm{Cl}_1(A \cap B) \in \beta_2$.*
(2) *(X, τ_1, τ_2) is a co-Heyting bitopological space iff for each $A \in \beta_1$ and $B \in \beta_2$ we have $\mathrm{Cl}_2(A \cap B) \in \beta_1$.*
(3) *(X, τ_1, τ_2) is a bi-Heyting bitopological space iff for each $A \in \beta_1$ and $B \in \beta_2$ we have $\mathrm{Cl}_1(A \cap B) \in \beta_2$ and $\mathrm{Cl}_2(A \cap B) \in \beta_1$.*

Proof. (1) Let $(X, \tau, \leq)$ be the Priestley space corresponding to (X, τ_1, τ_2). Suppose that (X, τ_1, τ_2) is a Heyting bitopological space, $A \in \beta_1$, and $B \in \beta_2$. Then $A \in \delta_2$ and $A^c \in \delta_1$. Therefore, both A and A^c are closed in $(X, \tau, \leq)$. A similar argument shows that both B and B^c are closed in $(X, \tau, \leq)$. Thus, both $A \cap B$ and $(A \cap B)^c = A^c \cup B^c$ are closed in $(X, \tau, \leq)$. By Lemma 6.8, both $A \cap B$ and $(A \cap B)^c$ are pairwise compact in $(X, \tau, \leq)$, implying that $A \cap B \in \mathsf{DPC}(X)$. Since (X, τ_1, τ_2) is a Heyting bitopological space, we have $\mathrm{Cl}_1(A \cap B) \in \mathsf{DPC}(X)$. By Lemma 7.3, $\mathrm{Cl}_1(A \cap B)$ is clopen in $(X, \tau, \leq)$. Moreover, since $\leq$ is the specialization order of (X, τ_1), we have that $\mathrm{Cl}_1(A \cap B)$ is a downset of $(X, \tau, \leq)$. Therefore, $\mathrm{Cl}_1(A \cap B) \in \mathsf{CpDo}(X)$. By Proposition 3.4, $\mathsf{CpDo}(X) = \beta_2$. Thus, $\mathrm{Cl}_1(A \cap B) \in \beta_2$.

Conversely, suppose that (X, τ_1, τ_2) is a pairwise Stone space and for each $A \in \beta_1$ and $B \in \beta_2$ we have $\mathrm{Cl}_1(A \cap B) \in \beta_2$. Let $A \in \mathsf{DPC}(X)$. By Lemma 7.3, A is clopen in $(X, \tau, \leq)$. Since $\mathsf{CpUp}(X) \cup \mathsf{CpDo}(X)$ is a subbasis for τ and A is compact in (X, τ), we have $A = (U_1 \cap V_1) \cup \cdots \cup (U_n \cap V_n)$ for some $U_1, \ldots, U_n \in \mathsf{CpUp}(X)$ and $V_1, \ldots, V_n \in \mathsf{CpDo}(X)$. By Proposition 3.4, $\mathsf{CpUp}(X) = \beta_1$ and $\mathsf{CpDo}(X) = \beta_2$. Therefore, for each $i = 1, \ldots, n$ we have $\mathrm{Cl}_1(U_i \cap V_i) \in \beta_2$. Thus, $\mathrm{Cl}_1(A) = \mathrm{Cl}_1[(U_1 \cap V_1) \cup \cdots \cup (U_n \cap V_n)] = \mathrm{Cl}_1(U_1 \cap V_1) \cup \cdots \cup \mathrm{Cl}_1(U_n \cap V_n) \in \beta_2 = \mathsf{CpDo}(X)$. This implies that $\mathrm{Cl}_1(A)$ is clopen in $(X, \tau, \leq)$, so by Lemma 7.3, $\mathrm{Cl}_1(A) \in \mathsf{DPC}(X)$, and so (X, τ_1, τ_2) is a Heyting bitopological space.

(2) is proved similarly.

(3) is an immediate consequence of (1) and (2). ⊣

From now on we will call a pairwise Stone space a Heyting bitopological space (resp. co-Heyting bitopological space/bi-Heyting bitopological space) if it satisfies the condition of Theorem 7.5.1 (resp. Theorem 7.5.2/Theorem 7.5.3).

Theorem 7.6. *Let $(X, \tau, \leq)$ be a Priestley space and (X, τ_1, τ_2) be the corresponding pairwise Stone space. Then:*

(1) *$(X, \tau, \leq)$ is an Esakia space iff (X, τ_1, τ_2) is a Heyting bitopological space.*
(2) *$(X, \tau, \leq)$ is a co-Esakia space iff (X, τ_1, τ_2) is a co-Heyting bitopological space.*

(3) $(X, \tau, \leq)$ *is a bi-Esakia space iff* (X, τ_1, τ_2) *is a bi-Heyting bitopological space.*

Proof. Since $\mathsf{Cp}(X) = \mathsf{DPC}(X)$ and for $A \in \mathsf{DPC}(X)$ we have $\mathrm{Cl}_1(A) = {\downarrow}A$ and $\mathrm{Cl}_2(A) = {\uparrow}A$, the results follow. ⊣

In order to characterize morphisms between Esakia (resp. co-Esakia) bitopological spaces, we recall the following characterization of p-morphisms (resp. co-p-morphisms).

Lemma 7.7. [7, pp. 17-18] *For two posets* $(X, \leq)$ *and* $(X', \leq')$ *and a map* $f : X \to X'$, *the following conditions are equivalent:*

(1) f *is a* p*-morphism* (*resp.* f *is a co-*p*-morphism*).
(2) *For each* $x \in X$ *we have* $f({\uparrow}x) = {\uparrow}f(x)$ (*resp.* $f({\downarrow}x) = {\downarrow}f(x)$).
(3) *For each* $x' \in X'$ *we have* $f^{-1}({\downarrow}x') = {\downarrow}f^{-1}(x')$ (*resp.* $f^{-1}({\uparrow}x') = {\uparrow}f^{-1}(x')$).

Definition 7.8.

(1) *Let* (X, τ_1, τ_2) *and* (X', τ_1', τ_2') *be two Heyting bitopological spaces. We call a map* $f : X \to X'$ *a* Heyting morphism *if* f *is bi-continuous and* $f(\mathrm{Cl}_2(x)) = \mathrm{Cl}_2'(f(x))$ *for each* $x \in X$.
(2) *Let* (X, τ_1, τ_2) *and* (X', τ_1', τ_2') *be two co-Heyting bitopological spaces. We call a map* $f : X \to X'$ *a* co-Heyting morphism *if* f *is bi-continuous and* $f(\mathrm{Cl}_1(x)) = \mathrm{Cl}_1'(f(x))$ *for each* $x \in X$.
(3) *Let* (X, τ_1, τ_2) *and* (X', τ_1', τ_2') *be two bi-Heyting bitopological spaces. We call a map* $f : X \to X'$ *a* bi-Heyting morphism *if* f *is bi-continuous,* $f(\mathrm{Cl}_2(x)) = \mathrm{Cl}_2'(f(x))$, *and* $f(\mathrm{Cl}_1(x)) = \mathrm{Cl}_1'(f(x))$ *for each* $x \in X$.

Let $(X, \tau, \leq)$ and $(X', \tau', \leq')$ be two Esakia spaces, (X, τ_1, τ_2) and (X', τ_1', τ_2') be the corresponding Heyting bitopological spaces, and $f : X \to X'$ be bi-continuous. By Corollary 6.5, for each $x \in X$ we have ${\uparrow}x = \mathrm{Cl}_2(x)$ and ${\downarrow}x = \mathrm{Cl}_1(x)$. Therefore, by Lemma 7.7, f is an Esakia morphism iff f is a Heyting morphism iff $f^{-1}(\mathrm{Cl}_1(x')) = \mathrm{Cl}_1(f^{-1}(x'))$. Similarly, for two co-Esakia spaces $(X, \tau, \leq)$ and $(X', \tau', \leq')$ and their corresponding co-Heyting bitopological spaces (X, τ_1, τ_2) and (X', τ_1', τ_2'), a bi-continuous map $f : X \to X'$ is a co-Esakia morphism iff f is a co-Heyting morphism iff $f^{-1}(\mathrm{Cl}_2(x')) = \mathrm{Cl}_2(f^{-1}(x'))$. Putting these together, for two bi-Esakia spaces $(X, \tau, \leq)$ and $(X', \tau', \leq')$ and their corresponding bi-Heyting bitopological spaces (X, τ_1, τ_2) and (X', τ_1', τ_2'), a bi-continuous map $f : X \to X'$ is a bi-Esakia morphism iff f is a bi-Heyting morphism iff $f^{-1}(\mathrm{Cl}_1(x')) = \mathrm{Cl}_1(f^{-1}(x'))$ and $f^{-1}(\mathrm{Cl}_2(x')) = \mathrm{Cl}_2(f^{-1}(x'))$.

Let **HPStone** denote the category of Heyting bitopological spaces and Heyting morphisms, **coHPStone** denote the category of co-Heyting bitopological spaces and co-Heyting morphisms, and **biHPStone** denote the category of bi-Heyting bitopological spaces and bi-Heyting morphisms. Clearly each of **HPStone**, **coHPStone**, and **HPStone** is a proper subcategory of **PStone**. Moreover, **biHPStone** = **HPStone** ∩ **coHPStone**. Furthermore, putting the results obtained above together, we obtain:

Theorem 7.9.

(1) *The categories* **Esa** *and* **HPStone** *are isomorphic. Consequently,* **Heyt** *is dually equivalent to* **HPStone**.
(2) *The categories* **coEsa** *and* **coHPStone** *are isomorphic. Consequently,* **coHeyt** *is dually equivalent to* **coHPStone**.
(3) *The categories* **biEsa** *and* **biHPStone** *are isomorphic. Consequently,* **biHeyt** *is dually equivalent to* **biHPStone**.

Let (X,τ) be a spectral space. We call $Y \subseteq X$ a *doubly spectral subset* of (X,τ) if both Y and Y^c are spectral subsets of (X,τ). Let $\mathsf{DS}(X)$ denote the set of doubly spectral subsets of X.

Definition 7.10. *Let (X,τ) be a spectral space.*

(1) *We call (X,τ)* H-spectral *if $A \in \mathsf{DS}(X)$ implies $\mathrm{Cl}(A) \in \mathsf{DS}(X)$.*
(2) *We call (X,τ)* coH-spectral *if $A \in \mathsf{DS}(X)$ implies $\mathrm{Sat}(A) \in \mathsf{DS}(X)$.*
(3) *We call (X,τ)* biH-spectral *if it is both H-spectral and coH-spectral.*

Theorem 7.11. *Let (X,τ_1,τ_2) be a pairwise Stone space and (X,τ_1) be the corresponding spectral space. Then:*

(1) *(X,τ_1,τ_2) is a Heyting bitopological space iff (X,τ_1) is H-spectral.*
(2) *(X,τ_1,τ_2) is a co-Heyting bitopological space iff (X,τ_1) is coH-spectral.*
(3) *(X,τ_1,τ_2) is a bi-Heyting bitopological space iff (X,τ_1) is biH-spectral.*

Proof. By Lemma 7.3, $\mathsf{DPC}(X) = \mathsf{DS}(X)$. The results follow. $\dashv$

For two H-spectral spaces (X,τ) and (X',τ'), we call a map $f : X \to X'$ *H-spectral* if f is spectral and $f(\mathrm{Sat}(x)) = \mathrm{Sat}'(f(x))$. Moreover, for two coH-spectral spaces (X,τ) and (X',τ'), we call a map $f : X \to X'$ *coH-spectral* if f is spectral and $f(\mathrm{Cl}(x)) = \mathrm{Cl}'(f(x))$. Furthermore, for two biH-spectral spaces (X,τ) and (X',τ'), we call a map $f : X \to X'$ *biH-spectral* if f is spectral, $f(\mathrm{Sat}(x)) = \mathrm{Sat}'(f(x))$, and $f(\mathrm{Cl}(x)) = \mathrm{Cl}'(f(x))$.

Let (X,τ_1,τ_2) and (X',τ_1',τ_2') be two Heyting bitopological spaces and (X,τ_1) and (X',τ_1') be the corresponding H-spectral spaces. By Corollary 6.5, for each $x \in X$ we have $\mathrm{Cl}_2(x) = \mathrm{Sat}_1(x)$ and $\mathrm{Cl}_1(x) = \mathrm{Sat}_2(x)$. Therefore, a bi-continuous map $f : X \to X'$ is a Heyting morphism iff f is H-spectral iff $f^{-1}(\mathrm{Cl}_1(x')) = \mathrm{Cl}_1(f^{-1}(x'))$. Similarly, for two co-Heyting bitopological spaces (X,τ_1,τ_2) and (X',τ_1',τ_2') and their corresponding coH-spectral spaces (X,τ_1) and (X',τ_1'), a bi-continuous map $f : X \to X'$ is a co-Heyting morphism iff f is coH-spectral iff $f^{-1}(\mathrm{Sat}_1(x')) = \mathrm{Sat}_1(f^{-1}(x'))$. Putting these together, for two bi-Heyting bitopological spaces (X,τ_1,τ_2) and (X',τ_1',τ_2') and their corresponding biH-spectral spaces (X,τ_1) and (X',τ_1'), a bi-continuous map $f : X \to X'$ is a bi-Heyting morphism iff f is biH-spectral iff $f^{-1}(\mathrm{Sat}_1(x')) = \mathrm{Sat}_1(f^{-1}(x'))$ and $f^{-1}(\mathrm{Cl}_1(x')) = \mathrm{Cl}_1(f^{-1}(x'))$.

Let **HSpec** denote the category of H-spectral spaces and H-spectral maps, **coHSpec** denote the category of coH-spectral spaces and coH-spectral maps, and **biHSpec** denote the category of biH-spectral spaces and biH-spectrals maps. Clearly each of **HSpec**, **coHSpec**, and **biHSpec** is a proper subcategory of **Spec**. Moreover, **biHSpec** = **HSpec** $\cap$ **coHSpec**. Furthermore, putting the results obtained above together, we obtain:

Theorem 7.12.

(1) *The categories* **Esa**, **HPStone**, *and* **HSpec** *are isomorphic. Consequently,* **Heyt** *is also dually equivalent to* **HSpec**.
(2) *The categories* **coEsa**, **coHPStone**, *and* **coHSpec** *are isomorphic. Consequently,* **coHeyt** *is also dually equivalent to* **coHSpec**.
(3) *The categories* **biEsa**, **biHPStone**, *and* **biHSpec** *are isomorphic. Consequently,* **biHeyt** *is also dually equivalent to* **biHSpec**.

The dual description of algebraic concepts important for the study of Heyting algebras (resp. co-Heyting algebras/bi-Heyting algebras) is similar to that of bounded distributive lattices. The dual description of filters, prime filters, and maximal filters as well as ideals, prime ideals, and maximal ideals is exactly the same. So is the dual description of the canonical completions. On the other hand, the dual description of the MacNeille completions gets simplified [11, Sec. 3]: In the case of Heyting algebras, we have $\mathbf{D} = \mathrm{Cl}$; and in the case of co-Heyting algebras, we have $\mathbf{J} = \mathrm{Int}$; consequently, in the case of bi-Heyting algebras we obtain that Cl and Int form a Galois connection between $\mathsf{OpUp}(X)$ and $\mathsf{ClOp}(X)$, and so the MacNeille completion $\overline{A}$ of a bi-Heyting algebra is dually characterized as the fixpoints of $\mathrm{Cl} \circ \mathrm{Int}$, which are exactly the regular open upsets of X. This provides a nice generalization of the Boolean case (see Corollary 6.23).

It is well-known that homomorphic images of a Heyting algebra A are characterized by its filters. Consequently, unlike the case of bounded distributive lattices, homomorphic images of a Heyting algebra A dually correspond to closed upsets of the Esakia space X of A. Similarly, homomorphic images of a co-Heyting algebra A are characterized by its ideals, and so homomorphic images of A dually correspond to open upsets of the co-Esakia space X of A. Therefore, homomorphic images of a bi-Heyting algebra A dually correspond to either closed upsets that are also downsets (denoted $\mathsf{ClUpDo}(X)$) or open upsets that are also downsets (denoted $\mathsf{OpUpDo}(X)$) of the bi-Esakia space X of A, thus generalizing the Boolean case, where homomorphic images of a Boolean algebra B dually correspond to either closed subsets or open subsets of the Stone space X of B. We give the dual description of subalgebras of a Heyting algebra (resp. co-Heyting algebra/bi-Heyting algebra). For a quasi-ordered set (X, Q), we define an equivalence relation E on X by xEy iff xQy and yQx.

Definition 7.13. *Let $(X, \tau, \leq)$ be a Priestley space and Q be a Priestley quasi-order on X extending $\leq$.*

(1) *We call Q an* Esakia quasi-order *if for each $x, y \in X$, from xQy it follows that there exists $z \in X$ such that $x \leq z$ and zEy.*
(2) *We call Q a* co-Esakia quasi-order *if for each $x, y \in X$, from xQy it follows that there exists $u \in X$ such that xEu and $u \leq y$.*
(3) *We call Q a* bi-Esakia quasi-order *if Q is both an Esakia quasi-order and a co-Esakia quasi-order.*

Remark 7.14. Let $(X, \tau, \leq)$ be a Priestley space and E be an equivalence relation on X. We call E an *Esakia (resp. co-Esakia) equivalence relation* if E viewed as a quasi-order is a Priestley quasi-order on X and ${\uparrow}E(x) \subseteq E({\uparrow}x)$ (resp. ${\downarrow}E(x) \subseteq E({\downarrow}x)$). We also call E a *bi-Esakia equivalence relation* if E is both an Esakia and a co-Esakia equivalence relation. It is easy to see that if Q is an Esakia (resp. co-Esakia/bi-Esakia) quasi-order, then E is an Esakia (resp. co-Esakia/bi-Esakia) equivalence relation. For an Esakia (resp. co-Esakia) equivalence relation E, we define Q on X by xQy iff there exists $z \in X$ such that $x \leq z$ and zEy (resp. there exists $u \in X$ such that xEu and $u \leq y$). Then for an Esakia (resp. co-Esakia) space X, it is easy to see that Q is an Esakia (resp. co-Esakia) quasi-order. Also if X is a bi-Esakia space and E is a bi-Esakia equivalence relation, then Q is a bi-Esakia quasi-order. Thus, for an Esakia (resp. co-Esakia/bi-Esakia) space X, there is an isomorphism between Esakia (resp. co-Esakia/bi-Esakia) quasi-orders on X ordered by inclusion and Esakia (resp. co-Esakia/bi-Esakia) equivalence relations on X ordered by inclusion.

Theorem 7.15.

(1) *Let A be a Heyting algebra and $(X, \tau, \leq)$ be the Esakia space of A. Then the poset $(\mathsf{HS}_A, \subseteq)$ of Heyting subalgebras of A is dually isomorphic to the poset $(\mathsf{EQ}_X, \subseteq)$ of Esakia quasi-orders on X.*

(2) *Let A be a co-Heyting algebra and $(X, \tau, \leq)$ be the co-Esakia space of A. Then the poset $(\mathsf{coHS}_A, \subseteq)$ of co-Heyting subalgebras of A is dually isomorphic to the poset $(\mathsf{coEQ}_X, \subseteq)$ of co-Esakia quasi-orders on X.*

(3) *Let A be a bi-Heyting algebra and $(X, \tau, \leq)$ be the bi-Esakia space of A. Then the poset $(\mathsf{biHS}_A, \subseteq)$ of bi-Heyting subalgebras of A is dually isomorphic to the poset $(\mathsf{biEQ}_X, \subseteq)$ of bi-Esakia quasi-orders on X.*

Proof. (1) In view of Theorem 6.14, it is sufficient to show that if $S \in \mathsf{HS}_A$, then $Q_S \in \mathsf{EQ}_X$, and that if $Q \in \mathsf{EQ}_X$, then $S_Q \in \mathsf{HS}_A$. Let $S \in \mathsf{HS}_A$. By Theorem 6.14, Q_S is a Priestley quasi-order on X extending $\leq$. Suppose that xQ_Sy. Then $x \cap S \subseteq y \cap S$. Let F be the filter of A generated by $x \cup (y \cap S)$. Then F is a proper filter of A with $x \subseteq F$ and $F \cap S = y \cap S$. By Zorn's lemma we can extend F to a maximal such filter z. The standard argument shows that z is prime. Therefore, there exists $z \in X$ such that $x \leq z$ and zE_Sy. Thus, $Q_S \in \mathsf{EQ}_X$. Now let $Q \in \mathsf{EQ}_X$. By Theorem 6.14, S_Q is a bounded distributive sublattice of A. For $a, b \in S_Q$ we have $\phi(a), \phi(b)$ are Q-upsets of X. We show that $\phi(a \to b) = \phi(a) \to \phi(b) = [\downarrow(\phi(a) - \phi(b))]^c = \{x \in X \mid \uparrow x \cap \phi(a) \subseteq \phi(b)\}$ is also a Q-upset of X. Let $x \in \phi(a \to b)$ and xQy. We show that $\uparrow y \cap \phi(a) \subseteq \phi(b)$. Let $u \in \uparrow y \cap \phi(a)$. Then $y \leq u$ and $u \in \phi(a)$. Therefore, xQu, and so there exists $z \in X$ such that $x \leq z$ and zEu. Since zEu, $u \in \phi(a)$, and $\phi(a)$ is a Q-upset, we have $z \in \phi(a)$. This implies that $z \in \uparrow x \cap \phi(a)$ an as $\uparrow x \cap \phi(a) \subseteq \phi(b)$, we obtain $z \in \phi(b)$. Now zEu and $\phi(b)$ being a Q-upset imply that $u \in \phi(b)$. Consequently, $\uparrow y \cap \phi(a) \subseteq \phi(b)$, so $y \in \phi(a \to b)$, and so $\phi(a \to b)$ is a Q-upset. It follows that $a, b \in S_Q$ implies $a \to b \in S_Q$, and so $S_Q \in \mathsf{HS}_A$.

(2) is proved similar to (1).

(3) is an immediate consequence of (1) and (2). $\dashv$

As a consequence of Remark 7.14 and Theorem 7.15, we obtain the following well-known dual description of subalgebras of Heyting (resp. co-Heyting/bi-Heyting) algebras [5, Thm. 4]: The poset of Heyting subalgebras of a Heyting algebra A is dually isomorphic to the poset of Esakia equivalence relations on the Esakia space X of A; the poset of co-Heyting subalgebras of a co-Heyting algebra A is dually isomorphic to the poset of co-Esakia equivalence relations on the co-Esakia space X of A; and the poset of bi-Heyting subalgebras of a bi-Heyting algebra A is dually isomorphic to the poset of bi-Esakia equivalence relations on the bi-Esakia space X of A.

Now we give the dual description of subalgebras of Heyting algebras (resp. co-Heyting algebras/bi-Heyting algebras) by means of Heyting bitopological spaces (resp. co-Heyting bitopological spaces/bi-Heyting bitopological spaces) and H-spectral spaces (resp. coH-spectral spaces/biH-spectral spaces). Let (X, τ_1, τ_2) be a Heyting bitopological space (resp. a co-Heyting bitopological space). We call a bi-topology (τ_1', τ_2') a *Heyting bi-topology (resp. a co-Heyting bi-topology)* on X if (τ_1', τ_2') is pairwise zero-dimensional and $A \in \beta_1', B \in \beta_2'$ imply $\mathrm{Cl}_1(A \cap B) \in \beta_2'$ (resp. $A \in \beta_1', B \in \beta_2'$ imply $\mathrm{Cl}_2(A \cap B) \in \beta_1'$). We also call (τ_1', τ_2') a *bi-Heyting bi-topology* on X if it is both a Heyting and a co-Heyting bi-topology on X. Let $(\mathsf{HB}_X, \subseteq)$ (resp. $(\mathsf{coHB}_X, \subseteq)/(\mathsf{biHB}_X, \subseteq)$) denote the poset of Heyting bi-topologies (resp. co-Heyting bi-topologies/bi-Heyting bi-topologies) on X coarser than (τ_1, τ_2).

Lemma 7.16.

(1) *Let $(X, \tau, \leq)$ be an Esakia space and (X, τ_1, τ_2) be the corresponding Heyting bitopological space. Then $(\mathsf{EQ}_X, \subseteq)$ is dually isomorphic to $(\mathsf{HB}_X, \subseteq)$.*

(2) *Let* $(X, \tau, \leq)$ *be a co-Esakia space and* (X, τ_1, τ_2) *be the corresponding co-Heyting bitopological space. Then* $(\mathsf{coEQ}_X, \subseteq)$ *is dually isomorphic to* $(\mathsf{coHB}_X, \subseteq)$*.*

(3) *Let* $(X, \tau, \leq)$ *be a bi-Esakia space and* (X, τ_1, τ_2) *be the corresponding bi-Heyting bitopological space. Then* $(\mathsf{biEQ}_X, \subseteq)$ *is dually isomorphic to* $(\mathsf{biHB}_X, \subseteq)$*.*

Proof. (1) In view of Lemma 6.16, we only need to show that if $Q \in \mathsf{EQ}_X$, then $(\tau_1^Q, \tau_2^Q) \in \mathsf{HB}_X$, and that if $(\tau_1', \tau_2') \in \mathsf{HB}_X$, then $Q_{(\tau_1', \tau_2')} \in \mathsf{EQ}_X$. Let $Q \in \mathsf{EQ}_X$. By Lemma 6.16, (τ_1^Q, τ_2^Q) is a zero-dimensional bi-topology coarser than (τ_1, τ_2). Moreover, β_1^Q coincides with the set of clopen Q-upsets and β_2^Q coincides with the set of clopen Q-downsets of $(X, \tau, \leq)$. Therefore, for $A \in \beta_1^Q$ and $B \in \beta_2^Q$ we have that A is a clopen Q-upset and B is a clopen Q-downset of $(X, \tau, \leq)$. Since Q is an Esakia quasi-order, by Theorem 7.15, the lattice of clopen Q-upsets of $(X, \tau, \leq)$ is a Heyting subalgebra of the Heyting algebra of all clopen upsets of $(X, \tau, \leq)$. Thus, $\downarrow(A \cap B)$ is a clopen Q-downset of $(X, \tau, \leq)$, and so $\downarrow(A \cap B) \in \beta_2^Q$. By Corollary 6.5, $\mathrm{Cl}_1(A \cap B) = \downarrow(A \cap B)$. Consequently, $\mathrm{Cl}_1(A \cap B) \in \beta_2^Q$, and so $(\tau_1^Q, \tau_2^Q) \in \mathsf{HB}_X$. Now suppose that $(\tau_1', \tau_2') \in \mathsf{HB}_X$. By Lemma 6.16, $Q_{(\tau_1', \tau_2')}$ is a Priestley quasi-order on X extending $\leq$. We show that the lattice of clopen $Q_{(\tau_1', \tau_2')}$-upsets of $(X, \tau, \leq)$ is closed under $\to$. Let A and B be clopen $Q_{(\tau_1', \tau_2')}$-upsets of $(X, \tau, \leq)$. Then $A \in \beta_1'$ and $B^c \in \beta_2'$. Therefore, $\mathsf{Cl}_1(A \cap B^c) \in \beta_2'$, and so $\mathsf{Cl}_1(A \cap B^c)$ is a clopen $Q_{(\tau_1', \tau_2')}$-downset of $(X, \tau, \leq)$. By Corollary 6.5, $\mathrm{Cl}_1(A \cap B^c) = \downarrow(A \cap B^c)$. Consequently, $\downarrow(A \cap B^c)$ is a clopen $Q_{(\tau_1', \tau_2')}$-downset of $(X, \tau, \leq)$, so $A \to B = [\downarrow(A \cap B^c)]^c$ is a clopen $Q_{(\tau_1', \tau_2')}$-upset of $(X, \tau, \leq)$, and so the lattice of clopen $Q_{(\tau_1', \tau_2')}$-upsets of $(X, \tau, \leq)$ is closed under $\to$. This implies that the lattice of clopen $Q_{(\tau_1', \tau_2')}$-upsets of $(X, \tau, \leq)$ is a Heyting subalgebra of the Heyting algebra of all clopen upsets of $(X, \tau, \leq)$, which, by Theorem 7.15, gives us that $Q_{(\tau_1', \tau_2')} \in \mathsf{EQ}_X$.

(2) is proved similar to (1), and (3) follows from (1) and (2). $\dashv$

Let (X, τ) be a H-spectral space (resp. a coH-spectral space). We call a topology τ' on X a *H-spectral topology (resp. a coH-spectral topology)* if τ' is strongly coherent and $A \in \mathcal{E}(X, \tau'), B \in \Delta(X, \tau')$ imply $\mathrm{Cl}(A \cap B) \in \Delta(X, \tau')$ (resp. $A \in \mathcal{E}(X, \tau'), B \in \Delta(X, \tau')$ imply $\mathrm{Sat}(A \cap B) \in \mathcal{E}(X, \tau')$). For a biH-spectral space (X, τ), we call τ' a *biH-spectral topology* if it is both a H-spectral topology and a coH-spectral topology. For a H-spectral (resp. coH-spectral/biH-spectral) space (X, τ), let $(\mathsf{HS}_X, \subseteq)$ (resp. $(\mathsf{coHS}_X, \subseteq)/(\mathsf{biHS}_X, \subseteq)$) denote the poset of H-spectral (resp. coH-spectral/biH-spectral) topologies on X coarser than τ.

Lemma 7.17.

(1) *Let* (X, τ_1, τ_2) *be a Heyting bitopological space and* (X, τ_1) *be the corresponding H-spectral space. Then* $(\mathsf{HB}_X, \subseteq)$ *is isomorphic to* $(\mathsf{HS}_X, \subseteq)$*.*

(2) *Let* (X, τ_1, τ_2) *be a co-Heyting bitopological space and* (X, τ_1) *be the corresponding coH-spectral space. Then* $(\mathsf{coHB}_X, \subseteq)$ *is isomorphic to* $(\mathsf{coHS}_X, \subseteq)$*.*

(3) *Let* (X, τ_1, τ_2) *be a bi-Heyting bitopological space and* (X, τ_1) *be the corresponding biH-spectral space. Then* $(\mathsf{biHB}_X, \subseteq)$ *is isomorphic to* $(\mathsf{biHS}_X, \subseteq)$*.*

Proof. (1) In view of Lemma 6.18, we only need to show that if $(\tau_1', \tau_2') \in \mathsf{HB}_X$, then $\tau_1' \in \mathsf{HS}_X$, and that if $\tau_1' \in \mathsf{HS}_X$, then $(\tau_1', \tau_2') \in \mathsf{HB}_X$. Let $(\tau_1', \tau_2') \in \mathsf{HB}_X$. By Lemma 6.18, τ_1' is a strongly coherent topology coarser than τ_1. Moreover, since $\beta_1' = \mathcal{E}(X, \tau_1')$ and $\beta_2' = \Delta(X, \tau_1')$, for $A \in \mathcal{E}(X, \tau_1')$ and $B \in \Delta(X, \tau_1')$, we have $A \in \beta_1'$ and $B \in \beta_2'$, so $\mathrm{Cl}_1(A \cap B) \in \beta_2'$, and so $\mathrm{Cl}_1(A \cap B) \in \Delta(X, \tau_1')$. Therefore, $\tau_1' \in \mathsf{HS}_X$. Now let $\tau_1' \in \mathsf{HS}_X$. By Lemma 6.18, (τ_1', τ_2') is a zero-dimensional bi-topology coarser than (τ_1, τ_2). Moreover,

Heyt	**Esa**	**HPStone**	**HSpec**
filter	closed upset	τ_2-closed set	compact saturated set
prime filter	$\uparrow x$	$\mathrm{Cl}_2(x)$	$\mathrm{Sat}(x)$
maximal filter	$\uparrow x = \{x\}$	$\mathrm{Cl}_2(x) = \{x\}$	$\mathrm{Sat}(x) = \{x\}$
ideal	open upset	τ_1-open set	open set
prime ideal	$(\downarrow x)^c$	$[\mathrm{Cl}_1(x)]^c$	$[\mathrm{Cl}(x)]^c$
maximal ideal	$(\downarrow x)^c = \{x\}^c$	$[\mathrm{Cl}_1(x)]^c = \{x\}^c$	$[\mathrm{Cl}(x)]^c = \{x\}^c$
homomorphic image	closed upset	τ_2-closed set	compact saturated set
subalgebra	$Q \in \mathsf{EQ}_X$	$(\tau_1', \tau_2') \in \mathsf{HB}_X$	$\tau' \in \mathsf{HS}_X$
canonical completion	$\mathsf{Up}(X)$	$\mathsf{S}_1(X) = \mathsf{CS}_2(X)$	$\mathsf{S}(X)$
MacNeille completion	$\mathsf{RgOpUp}(X)$	$\mathsf{RgOp}_{12}(X)$	$\mathsf{SatOp}(X)$
complete lattice	$\mathsf{RgOpUp}(X) = \mathsf{CpUp}(X)$	$\beta_1 = \mathsf{RgOp}_{12}(X)$	$\mathcal{E}(X) = \mathsf{SatOp}(X)$

TABLE 2. Dictionary for **Heyt**, **Esa**, **HPStone**, and **HSpec**.

since $\mathcal{E}(X, \tau_1') = \beta_1'$ and $\Delta(X, \tau_1') = \beta_2'$, for $A \in \beta_1'$ and $B \in \beta_2'$, we have $A \in \mathcal{E}(X, \tau_1')$ and $B \in \Delta(X, \tau_1')$, so $\mathrm{Cl}_1(A \cap B) \in \Delta(X, \tau_1')$, and so $\mathrm{Cl}_1(A \cap B) \in \beta_2'$. Thus, $(\tau_1', \tau_2') \in \mathsf{HB}_X$.

(2) is proved similar to (1), and (3) follows from (1) and (2). $\dashv$

Putting Lemmas 7.16 and 7.17 together, we obtain the following dual description of Heyting (resp. co-Heyting/bi-Heyting) subalgebras of a Heyting algebra (resp. co-Heyting algebra/bi-Heyting algebra).

Corollary 7.18.

(1) *Let A be a Heyting algebra, $(X, \tau, \leq)$ be the Esakia space of A, (X, τ_1, τ_2) be the Heyting bitopological space of A, and (X, τ_1) be the H-spectral space of A. Then $(\mathsf{HS}_A, \subseteq)$ is dually isomorphic to $(\mathsf{EQ}_X, \subseteq)$, and is isomorphic to $(\mathsf{HB}_X, \subseteq)$ and $(\mathsf{HS}_X, \subseteq)$.*

(2) *Let A be a co-Heyting algebra, $(X, \tau, \leq)$ be the co-Esakia space of A, (X, τ_1, τ_2) be the co-Heyting bitopological space of A, and (X, τ_1) be the coH-spectral space of A. Then $(\mathsf{coHS}_A, \subseteq)$ is dually isomorphic to $(\mathsf{coEQ}_X, \subseteq)$, and is isomorphic to $(\mathsf{coHB}_X, \subseteq)$ and $(\mathsf{coHS}_X, \subseteq)$.*

(3) *Let A be a bi-Heyting algebra, $(X, \tau, \leq)$ be the bi-Esakia space of A, (X, τ_1, τ_2) be the bi-Heyting bitopological space of A, and (X, τ_1) be the biH-spectral space of A. Then $(\mathsf{biHS}_A, \subseteq)$ is dually isomorphic to $(\mathsf{biEQ}_X, \subseteq)$, and is isomorphic to $(\mathsf{biHB}_X, \subseteq)$ and $(\mathsf{biHS}_X, \subseteq)$.*

We conclude the paper with Tables 2,3, and 4, in which we gather together the dual descriptions of different algebraic concepts for Heyting algebras (resp. co-Heyting algebras/bi-Heyting algebras) by means of their Esakia spaces (resp. co-Esakia spaces/bi-Esakia spaces), Heyting bitopological spaces (resp. co-Heyting bitopological spaces/bi-Heyting bitopological spaces), and H-spectral spaces (resp. coH-spectral spaces/biH-spectral spaces) obtained in this section. This can be thought of as a dictionary of duality theory for Heyting algebras (resp. co-Heyting algebras/bi-Heyting algebras).

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coHeyt	**coEsa**	**coHPStone**	**coHSpec**
filter	closed upset	τ_2-closed set	compact saturated set
prime filter	$\uparrow x$	$\mathrm{Cl}_2(x)$	$\mathrm{Sat}(x)$
maximal filter	$\uparrow x = \{x\}$	$\mathrm{Cl}_2(x) = \{x\}$	$\mathrm{Sat}(x) = \{x\}$
ideal	open upset	τ_1-open set	open set
prime ideal	$(\downarrow x)^c$	$[\mathrm{Cl}_1(x)]^c$	$[\mathrm{Cl}(x)]^c$
maximal ideal	$(\downarrow x)^c = \{x\}^c$	$[\mathrm{Cl}_1(x)]^c = \{x\}^c$	$[\mathrm{Cl}(x)]^c = \{x\}^c$
homomorphic image	open upset	τ_1-open set	open set
subalgebra	$Q \in \mathsf{coEQ}_X$	$(\tau_1', \tau_2') \in \mathsf{coHB}_X$	$\tau' \in \mathsf{coHS}_X$
canonical completion	$\mathsf{Up}(X)$	$\mathsf{S}_1(X) = \mathsf{CS}_2(X)$	$\mathsf{S}(X)$
MacNeille completion	$\mathsf{RgOpUp}(X)$	$\mathsf{RgOp}_{12}(X)$	$\mathsf{SatOp}(X)$
complete lattice	$\mathsf{RgOpUp}(X) = \mathsf{CpUp}(X)$	$\beta_1 = \mathsf{RgOp}_{12}(X)$	$\mathcal{E}(X) = \mathsf{SatOp}(X)$

TABLE 3. Dictionary for **coHeyt**, **coEsa**, **coHPStone**, and **coHSpec**.

biHeyt	**biEsa**	**biHPStone**	**biHSpec**
filter	closed upset	τ_2-closed set	compact saturated set
prime filter	$\uparrow x$	$\mathrm{Cl}_2(x)$	$\mathrm{Sat}(x)$
maximal filter	$\uparrow x = \{x\}$	$\mathrm{Cl}_2(x) = \{x\}$	$\mathrm{Sat}(x) = \{x\}$
ideal	open upset	τ_1-open set	open set
prime ideal	$(\downarrow x)^c$	$[\mathrm{Cl}_1(x)]^c$	$[\mathrm{Cl}(x)]^c$
maximal ideal	$(\downarrow x)^c = \{x\}^c$	$[\mathrm{Cl}_1(x)]^c = \{x\}^c$	$[\mathrm{Cl}(x)]^c = \{x\}^c$
homomorphic image	$\mathsf{ClUpDo}(X) \simeq \mathsf{OpUpDo}(X)$	$\delta_1 \cap \delta_2 \simeq \tau_1 \cap \tau_2$	$\delta \cap \mathsf{KS}(X)$
subalgebra	$Q \in \mathsf{biEQ}_X$	$(\tau_1', \tau_2') \in \mathsf{biHB}_X$	$\tau' \in \mathsf{biHS}_X$
canonical completion	$\mathsf{Up}(X)$	$\mathsf{S}_1(X) = \mathsf{CS}_2(X)$	$\mathsf{S}(X)$
MacNeille completion	$\mathsf{RgOpUp}(X)$	$\mathsf{RgOp}_{12}(X)$	$\mathsf{SatOp}(X)$
complete lattice	$\mathsf{RgOpUp}(X) = \mathsf{CpUp}(X)$	$\beta_1 = \mathsf{RgOp}_{12}(X)$	$\mathcal{E}(X) = \mathsf{SatOp}(X)$

TABLE 4. Dictionary for **biHeyt**, **biEsa**, **biHPStone**, and **biHSpec**.

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Guram Bezhanishvili
Department of Mathematical Sciences
New Mexico State University
Las Cruces NM 88003-8001, USA
Email: gbezhani@nmsu.edu

Nick Bezhanishvili
Department of Computer Science
University of Leicester
University Road, Leicester LE1 7RH, UK
Email: nick@mcs.le.ac.uk

David Gabelaia
Department of Mathematical Logic
Razmadze Mathematical Institute
M. Aleksidze Str. 1, Tbilisi 0193, Georgia
Email: gabelaia@gmail.com

Alexander Kurz
Department of Computer Science
University of Leicester
University Road, Leicester LE1 7RH, UK
Email: kurz@mcs.le.ac.uk