Advanced Topics in Automata
Exercise 8

Submission: June 24, 2003

Exercise

1. Given that language L has *semi-linear range*, prove that L^* also has semi-linear range.

2. Prove that the the universality of nondeterministic pushdown automata is undecidable.

Discussion.
We have covered for finite automata the questions of emptiness, validity, and containment. It is simple to see that the question of equivalence of finite automata is a simple generalization of the question of containment. We have seen that the question of containment is the most general one, as both other problems can be solved by it. The solution to the containment problem involved complementation of the 'containing' automaton.

When we try to study these problems in the context of pushdown automata, things are not so simple. The first thing that we note, is that context-free languages are not closed under complementation. Hence, our algorithm for deciding containment no longer works. Context-free languages are not closed also under intersection, so even for deterministic pushdown automata (where complementation is easy) the question of containment is not simple.

From the context of finite automata we know that the emptiness problem is the easiest to solve. Indeed, this question is decidable for pushdown automata (and CFL's).

The universality problem (as you have to prove) is undecidable. This immediately implies the undecidability of equivalence and containment. The proof uses strongly the fact that the automaton is nondeterministic. Indeed, very early in the study of pushdown automata the question was raised: is the equivalence problem for deterministic pushdown automata decidable [GG66]?

After being open for 30 years, the question was solved in 1997 by Géraud Sénizergues. His proof appeared first in the proceedings of *International Conference on Automata, Languages, and Programming (ICALP) 1997* and a full version appeard in *Theoretical Computer Science 251* in 2001. This proof also gained him the Gödel prize in 2002.

Géraud Sénizergues list of publications includes many versions of this proof (http://dept-info.labri.u-bordeaux.fr/ ges/publis_automates.html). His proof is extremely complex (and long). There is also the presentation that he gave on the occasion of receiving the Gödel prize that includes many related open problems.

Computations of Turing Machines.
A Turing machine is $M = \langle S, \Sigma, \delta, q_0, B, F_{acc}, F_{reg} \rangle$ where S is the set of states, Σ is the alphabet, the transition function $\delta : \Sigma \times Q \to \Sigma \times Q \times \{\leftarrow, \rightarrow, \downarrow\}$ associates with every letter and every state a triplet of letter, new state, and the direction in which the reading head should move, q_0 is the initial state, B is the blank symbol and F_{acc} and F_{rej} are disjoint sets of the accepting and rejecting states respectively.

Every configuration of a Turing machine can be represented by a finite word over the alphabet $\Sigma \cup (\Sigma \times Q)$ where in every configuration only one letter in $\Sigma \times Q$ appears, and this letter represents the location of the reading head and the state of the machine. The infinite suffix of B is (obviously) not included in this representation of the configuration. A computation of a Turing machine can be represented by a finite word over the alphabet $\{\sharp\} \cup \Sigma \cup (\Sigma \times Q)$ where every two configurations are separated by the $\sharp$ symbol. The computation is accepting if it is valid and the last configuration is an accpeting configuration (i.e. labeld by an accepting state).

The proof that universality of nondeterministic pushdown automata is undecidable should resemble the proof that universality for nondeterminitic finite automata is PSPACE-complete (at least the proof that I have in mind, other proofs are welcome).

References

[GG66] S. Ginsburg and S. Greibach. Deterministic Context Free Langugages. *Information and Control*, 1966.