Meaningless Sets in Infinitary Combinatory Logic

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—— Abstract ——

In this paper we study meaningless sets in infinitary combinatory logic. So far only a handful of meaningless sets were known. We show that there are uncountably many meaningless sets. As an application to the semantics of finite combinatory logics, we show that there exist uncountably many combinatory algebras that are not a lambda algebra. We also study ways of weakening the axioms of meaningless sets to get, not only sufficient, but also necessary conditions for having confluence and normalisation.

1   Introduction

In this paper, we study *meaningless sets* for *infinitary combinatory logic* [8, 5]. This is of interest because for infinitary combinatory logic, only a handful of meaningless sets are known so far, in stark contrast to the current situation for infinitary lambda calculus [14, 16]. Meaningless sets play an important role in the construction of *syntactic models* of finite lambda calculus and combinatory logic. The *interpretation* of a term is its infinitary normal form and it is well-defined if the corresponding infinitary lambda calculus or infinitary rewriting system is *confluent* and *normalising*. The extension with infinite terms and infinite reductions ruins the confluence property. We can recover confluence by extending the syntax with a fresh symbol $\perp$ and by extending the reduction rules with a $\perp$-rule that can reduce terms from a well chosen set $\mathcal{U}$ of meaningless terms to $\perp$. The papers [8, 5] looked simultaneously at orthogonal infinitary term rewriting systems and infinitary lambda calculus and gave a sufficient set of conditions for $\mathcal{U}$ so that the corresponding infinitary term rewriting systems or infinitary lambda calculi are confluent and normalising. Later for infinitary lambda calculus it was found in [16] that there exists a set of *necessary and sufficient conditions* for $\mathcal{U}$ such that the corresponding infinitary lambda calculus with β and $\perp$-reduction is confluent and normalising. And in [14, 15] it has been shown that there are uncountably many of such sets and hence *uncountably* many syntactic models of the lambda calculus. These sufficient and necessary conditions in the case of lambda calculus do not immediately carry over to orthogonal term rewriting systems. This hinges on the fact that in infinitary lambda calculus the set of rootactive terms coincides with the set of hypercollapsing terms. This is in general not the case for arbitrary orthogonal infinitary rewriting systems: infinitary

combinatory logic is an example of an infinitary rewriting system with rootactive terms that are not hypercollapsing. The reduction for combinatory logic is called w-reduction [1, 9]. In this paper we look at the following questions: what are the (necessary and) sufficient conditions for the infinitary combinatory logic with $w\perp$-reduction to be (1) confluent?, (2) normalising? and (3) the combination, confluent and normalising?.

Regarding confluence we will show that infinitary combinatory logic with $w\perp$-reduction is confluent for the traditional set of axioms of meaningless sets without assuming the axiom of rootactiveness. We give *uncountably* many examples of meaningless sets satisfying rootactiveness. Each one gives rise to a model of finite combinatory logic, called *combinatory algebra* [1]. We also show that these models are not λ-algebras, i.e. there exist two terms whose corresponding translations into lambda calculus are β-convertible but they have different interpretation in the model.

By studying the overlap cases between w and $\perp$-reduction, we realise that we can weaken the axiom of overlap. Combining weak overlap with rootactiveness and the axioms of closure under reduction, expansion and indiscernibility gives a *sufficient condition* so that the infinitary combinatory logic with $w\perp$-reduction is confluent and normalising. We also show that the axioms of hypercollapseness, overlap and closure under reduction are *necessary conditions* for confluence.

The paper is organised as follows. Section 2 gives a brief overview of infinitary rewriting in the setting of combinatory logic. Section 3 works out what the traditional theory of meaningless sets means for infinitary combinatory logic. Section 4 studies the axioms of closure under expansion and substitution. Section 5 gives concrete examples of meaningless sets and describes some of the structure of the lattice of meaningless sets. Section 6 shows an application to combinatory algebras. Section 7 explores sufficient and necessary conditions for confluence. Section 8 discusses some open problems and shows an example of a normalising infinitary combinatory logic that does not satisfy rootactiveness.

2 Infinitary Combinatory Logic

We will define infinitary combinatory logic assuming familiarity with basic notions and notations from combinatory logic, lambda calculus [1, 9] and some familiarity of infinite term rewriting [5]. The set $\mathcal{CL}$ of finite CL-terms is defined by induction from the following grammar:

$$M ::= x \mid \mathbf{K} \mid \mathbf{S} \mid MM.$$

The set $\mathcal{CL}^\infty$ of finite and infinite CL-terms over a given set of variables is defined by coinduction from the same grammar.
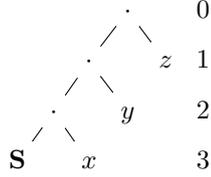
▶ **Definition 2.1** (w-reduction). The w-rules are extended to $\mathcal{CL}^\infty$.

$$\mathbf{K}MN \to_w M$$
$$\mathbf{S}MNP \to_w MP(NP)$$

The w in $\to_w$ stands for weak reduction [1, 3]. The first rule is called $\mathbf{K}$-*rule* and the second $\mathbf{S}$-*rule*. A term is a $\mathbf{K}$-redex (or $\mathbf{S}$-redex) if it is of the form $\mathbf{K}MN$ (or $\mathbf{S}MNP$).

Above we have presented $\mathcal{CL}$ in the traditional applicative format: this means that the infix application symbol $\cdot$ is suppressed, outermost brackets dropped and the usual bracket convention of association to the left is followed: e.g., xyz actually stands for $((x \cdot y) \cdot z)$. Alternatively we could have presented the terms of $\mathcal{CL}$ and $\mathcal{CL}^\infty$ and the two rules in the

format of first order term rewriting by adding an explicit binary application symbol $\mathbf{Ap}$ to the syntax $M ::= x \mid \mathbf{K} \mid \mathbf{S} \mid \mathbf{Ap}(M, N)$ with the proviso that we will read MN for $\mathbf{Ap}(M, N)$ [9]. Terminology and notation for infinite term rewriting in the latter format translate readily to CL presented in its applicative format. E.g. we say that $\mathbf{S}$ and x have depth 3 in the term $\mathbf{S}xyz$, whereas the depth of y and z is respectively 1 and 2.

$$
\begin{array}{cc}
\cdot & 0 \\
/ \ \backslash & \\
\cdot \quad z & 1 \\
/ \ \backslash & \\
\cdot \quad y & 2 \\
/ \ \backslash & \\
\mathbf{S} \quad x & 3
\end{array}
$$

The distance $d(M, N)$ between two terms is defined as 2^{-n} where n is the length of the shortest common position p of M and N such that M and N differ at p. With this notion of distance the set $\mathcal{CL}$ of finite terms becomes a metric space and $\mathcal{CL}^\infty$ its metric completion.

Metavariables in (infinitary) combinatory logic will be denoted by $M, N, P, \dots$ and contexts by $C, D, \dots$. Simultaneous substitution will be denoted by M^σ, where σ is a substitution of variables by terms.

One step reduction $\to_w$ on $\mathcal{CL}$ and $\mathcal{CL}^\infty$ is the smallest binary relation on $\mathcal{CL}$, respectively $\mathcal{CL}^\infty$ containing the w-rules and closed under contexts. Let α be an ordinal.

▶ **Definition 2.2** (Strongly converging reduction sequence). A *strongly converging reduction sequence* of length α is a sequence of reduction steps $(M_\beta \to_w M_{\beta+1})_{\beta < \alpha}$ such that for every limit ordinal $\lambda \leq \alpha$:

1. the sequence of terms $(M_\beta)_{\beta \leq \lambda}$ is a transfinite Cauchy sequence, that is

$$
\lim_{\beta \to \lambda} M_\beta = M_\lambda
$$

2. $\lim_{\beta \to \lambda} d_\beta = \infty$ where d_β is the depth of the redex contracted at $M_\beta \to_w M_{\beta+1}$.

We will denote this by $M_0 \twoheadrightarrow_w^\alpha M_\alpha$ (or just $M_0 \twoheadrightarrow_w M_\alpha$). We will use the notation $M \twoheadrightarrow_w N$ for a finite reduction from M to N.

Because $\mathcal{CL}$ is a left-linear system the Compression Lemma holds, which says that whenever $M_0 \twoheadrightarrow_w^\alpha M_\alpha$ then $M_0 \twoheadrightarrow_w^{\leq \omega} M_\alpha$, i.e. there is an strongly converging reduction from M_0 to M_α of length at most ω.

We will use the following abbreviations for terms:

$$
\begin{array}{ll}
\mathbf{I} = \mathbf{SKK}, & \mathbf{\Omega} = \mathbf{SII}(\mathbf{SII}), \\
M^\omega = M(M(M(\dots))) & C^\omega = C[C[C[\dots]]] \text{ if } C[\ ] \text{ is a context.}
\end{array}
$$

Although the names $\mathbf{I}$ and $\mathbf{\Omega}$ may feel familiar from lambda calculus, one should beware that their reduction behaviour is slightly different: $\mathbf{I}x \twoheadrightarrow_w x$ takes two steps, while $\mathbf{\Omega} \twoheadrightarrow_w \mathbf{\Omega}$ takes five. For the complete reduction graph of $\mathbf{\Omega}$ in $\mathcal{CL}$ see, see [9]. We give some examples of interesting infinite terms.

▶ **Example 2.3. 1.** It is easy to verify that the infinite term $\mathbf{Y} = (\mathbf{SI})^\omega$ is an infinite fixed point combinator

$$
(\mathbf{SI})^\omega M = \mathbf{SI}(\mathbf{SI})^\omega M \to_w \mathbf{I}M((\mathbf{SI})^\omega M) \to_w M((\mathbf{SI})^\omega M).
$$

To verify that a finite term is a fixed point combinators usually takes more steps to prove. See for instance the shortest fixed point combinator [17].

$$
\mathbf{Y}_{Tromp} = \mathbf{SSK}(\mathbf{S}(\mathbf{K}(\mathbf{SS}(\mathbf{S}(\mathbf{SSK})))) \mathbf{K}).
$$

The infinite term $(\mathbf{SI})^\omega$ can be obtained as limit of a strongly convergent reduction starting from the finite term $\mathbf{Y}_{Tromp}(\mathbf{SI})$.

2. Since $\mathbf{Y}(\mathbf{SII}) \twoheadrightarrow_w (\mathbf{Y}(\mathbf{SII}))(\mathbf{Y}(\mathbf{SII}))$ we find that if we continue this process, $\mathbf{Y}(\mathbf{SII})$ strongly converges to an infinite normal form, the binary tree of applications:

$$\mathbf{X} = (((\ldots)(\ldots))((\ldots)(\ldots)))(((\ldots)(\ldots))((\ldots)(\ldots))).$$

3. The term $\mathbf{S}^\omega \mathbf{XX}$ is self-looping: $\mathbf{S}^\omega \mathbf{XX} = \mathbf{S}(\mathbf{S}^\omega)\mathbf{XX} \to_w (\mathbf{S}^\omega)\mathbf{X}(\mathbf{XX}) = \mathbf{S}^\omega \mathbf{XX}$.

4. Let $C_M[\ ]$ abbreviate the context $\mathbf{K}[\ ]M$. Then C_M^ω is self-looping for any M. Redexes of the form $C_M[x] \to_w x$ are called *collapsing*.

Let Λ^∞ is defined by coinduction from the following grammar.

$$M ::= \bot \mid x \mid (\lambda x M) \mid (MM)$$

Combinator terms translate directly into lambda terms.

▶ **Definition 2.4** (Translation from $\mathcal{CL}^\infty$ to Λ^∞). For $M \in \mathcal{CL}^\infty$ we define $M_\lambda \in \Lambda^\infty$ by coinduction as follows: $x_\lambda = x$, $\mathbf{K}_\lambda = \lambda xy.x$, $\mathbf{S}_\lambda = \lambda xyz.xz(yz)$, $(PQ)_\lambda = P_\lambda Q_\lambda$.

We define $\to_w^h$ as the restriction of $\to_w$ to w-redexes in the head position where M is in the head position of N if $N = MP_1 \ldots P_n$.

▶ **Definition 2.5** (Skeleton). We define the skeleton of a term by corecursion as follows.

$\mathsf{skel}(M) =$	N	if $M \twoheadrightarrow_w^h N$, and N is either x, $\mathbf{K}$ or $\mathbf{S}$
$\mathsf{skel}(M) =$	$\mathsf{skel}(N)\mathsf{skel}(P)$	if $M \twoheadrightarrow_w^h NP$, while $NP \not\twoheadrightarrow_w^h Q$ for any w-redex Q
$\mathsf{skel}(M) =$	M	otherwise

The skeleton of a term can be computed by a depth-first leftmost strategy that contracts all w-redexes not contained in a rootactive term (see Definition 3.4). We have that $M \twoheadrightarrow_w \mathsf{skel}(M)$.

3 Axioms of Meaningless Sets in Infinitary Combinatory Logic

Finite combinatory logic $\mathcal{CL}$ is confluent for finite w-reduction. In contrast infinitary combinatory logic $\mathcal{CL}^\infty$ is not confluent for strongly converging w-reduction. The reason behind this negative result is that the $\mathbf{K}$-rule is a collapsing rule with a multivariable left-hand side. The infinite tower of collapsing redexes $C_x[C_y[C_x[C_y[\ldots]]]]$, that is the term $\mathbf{K}(\mathbf{K}(\mathbf{K}(\mathbf{K}(\ldots)y)x)y)x$, can strongly converge to both C_x^ω and C_y^ω, which are self-looping by Example 2.3. Hence C_x^ω and C_y^ω can not reduce further to a common reduct [6].

▶ **Definition 3.1** (Hypercollapsing Term). We say that a term $M \in \mathcal{CL}_\bot^\infty$ is hypercollapsing, if any reduct of M can further reduce to a collapsing redex. We denote the set of hypercollapsing terms by $\mathcal{H}$.

Confluence can be restored if we identify hypercollapsing terms with $\bot$ [8]. For this, we extend $\mathcal{CL}^\infty$ with a fresh symbol $\bot$. We define $\mathcal{CL}_\bot^\infty$ by coinduction from the grammar: $M ::= x \mid \bot \mid \mathbf{K} \mid \mathbf{S} \mid MN$. We also add a reduction rule $\to_{\bot_\mathcal{U}}$ that allows us reduce terms from a chosen subset $\mathcal{U} \subseteq \mathcal{CL}^\infty$ to $\bot$. This is reminiscent to $B\Omega$-reduction in [1] where the unsolvables play the role of undefined terms. We need some notation to define this $\bot$-rule.

▶ **Definition 3.2** (⊥-instance). Let $\mathcal{U} \subseteq \mathcal{C}L^\infty$ and $M, N \in \mathcal{C}L_\perp^\infty$. We say that M is a ⊥-instance of N, notation $M \preceq_\mathcal{U} N$, if M is obtained from N by replacing some (possibly infinitely many) subterms of N in $\mathcal{U}$ by ⊥. We define $\mathcal{U}_\perp \subseteq \mathcal{C}L_\perp^\infty$ as $\mathcal{U}_\perp = \{M \mid \exists N \in \mathcal{U}. M \preceq_\mathcal{U} N\}$.

▶ **Definition 3.3** (⊥$_\mathcal{U}$-reduction). Let $\mathcal{U} \subseteq \mathcal{C}L^\infty$. We define the ⊥$_\mathcal{U}$-rule on $\mathcal{C}L_\perp^\infty$: $M \to \perp$ if $M \in \mathcal{U}_\perp$ and $M \neq \perp$. The one step reduction $\to_{\perp_\mathcal{U}}$ is the smallest binary relation containing ⊥$_\mathcal{U}$ and closed under contexts and substitutions.

Occasionally, we may denote ⊥$_\mathcal{U}$ just by ⊥. We denote finite ⊥-reductions by $\twoheadrightarrow_\perp$ and strongly convergent reductions by $\twoheadrightarrow\!\!\!\twoheadrightarrow_\perp$.

▶ **Definition 3.4** (Rootactive Term). We say that a term $M \in \mathcal{C}L_\perp^\infty$ is rootactive, if any reduct of M can further reduce to a redex, i.e. either a **K** or an **S**-redex. We denote the set of rootactive terms by $\mathcal{R}$.

   All hypercollapsing terms are rootactive. The converse is not true. Examples of terms that are rootactive but not hypercollapsing are the term $\mathbf{\Omega}$, any term of the form $\mathbf{S}^\omega MN$ (in particular, the term $\mathbf{S}^\omega \mathbf{XX}$) and terms of the form $\mathbf{D}^\omega M$ (where $\mathbf{D}^\omega$ is the infinite tower of contexts of the form $\mathbf{S}(\mathbf{K}[\ ])\mathbf{I})$) for any M.
   Note that unlike in lambda calculus [7, 8] we have that in combinatory logic the set of hypercollapsing terms does not coincide with the rootactive terms..

▶ **Definition 3.5** (Hypercollapseness). We say that a set $\mathcal{U} \subseteq \mathcal{C}L^\infty$ satisfies the Axiom of hypercollapseness if $\mathcal{H} \subseteq \mathcal{U}$.

▶ **Definition 3.6** (Rootactiveness). We say that a set $\mathcal{U} \subseteq \mathcal{C}L^\infty$ satisfies the Axiom of rootactiveness if $\mathcal{R} \subseteq \mathcal{U}$.

▶ **Definition 3.7** ($\xleftrightarrow{\mathcal{U}}$ and $\overset{\mathcal{U}}{=}$). Let $M, N \in \mathcal{C}L_\perp^\infty$. We write $M \xleftrightarrow{\mathcal{U}} N$, if N can be obtained from M by replacing some (possibly infinitely many) subterms of M in $\mathcal{U}$ by other terms in $\mathcal{U}$. We write $t \overset{\mathcal{U}}{=} s$ for the transitive closure of $\xleftrightarrow{\mathcal{U}}$.

▶ **Definition 3.8** (Indiscernibility). We say that a set $\mathcal{U} \subseteq \mathcal{C}L^\infty$ satisfies the Axiom of indiscernibility if for all $M, N \in \mathcal{C}L^\infty$ such that $M \xleftrightarrow{\mathcal{U}} N$, we have that $M \in \mathcal{U}$ iff $N \in \mathcal{U}$.

▶ **Definition 3.9** (Closure under w-reduction). We say that a set $\mathcal{U} \subseteq \mathcal{C}L^\infty$ satisfies the Axiom of closure under w-reduction if $M \in \mathcal{U}$ and $M \twoheadrightarrow_w N$ implies $N \in \mathcal{U}$ for all $M, N \in \mathcal{C}L^\infty$.

   The general formulation of the next axiom of overlap says that if a redex M overlaps a subterm, and this subterm is in $\mathcal{U}$ then $M \in \mathcal{U}$. For combinatory logic this means concretely:

▶ **Definition 3.10** (Overlap). We say that a set $\mathcal{U} \subseteq \mathcal{C}L^\infty$ satisfies the axiom of overlap if the following conditions holds for all $M, N, P \in \mathcal{C}L^\infty$:
1. If $\mathbf{K} \in \mathcal{U}$ or $\mathbf{K}M \in \mathcal{U}$ then $\mathbf{K}MN \in \mathcal{U}$.
2. If $\mathbf{S} \in \mathcal{U}$, $\mathbf{S}M \in \mathcal{U}$ or $\mathbf{S}MN \in \mathcal{U}$ then $\mathbf{S}MNP \in \mathcal{U}$.

▶ **Definition 3.11** (Meaningless Set). A set $\mathcal{U} \subseteq \mathcal{C}L^\infty$ is called a *set of meaningless terms* (meaningless set for short), if it satisfies the axioms of hypercollapseness, closure under w-reduction, indiscernibility and overlap. These four axioms are called *the axioms of meaninglessness.*

▶ **Theorem 3.12** (Meaninglessness implies Confluence modulo $\mathcal{U}$ [8, 5]). *If $\mathcal{U}$ is a set of meaningless terms, then $(\mathcal{C}L_\perp^\infty, \twoheadrightarrow_w)$ is confluent modulo $\mathcal{U}$, i.e. if $M \twoheadleftarrow_w \overset{\mathcal{U}}{=} \twoheadrightarrow_w N$ implies $P \twoheadrightarrow_w \overset{\mathcal{U}}{=} \twoheadleftarrow_w Q$.*

▶ **Theorem 3.13** (Indiscernibility implies Confluence of $\perp_{\mathcal{U}}$-reduction [8, 5]). *Let $\mathcal{U} \subseteq \mathcal{CL}^\infty$. If $\mathcal{U}$ satisfies indiscernibility then $(\mathcal{CL}_\perp^\infty, \twoheadrightarrow_{\perp_{\mathcal{U}}})$ is confluent.*

▶ **Theorem 3.14** (Postponement [8, 5]). *Let $\mathcal{U} \subseteq \mathcal{CL}^\infty$. If $M \twoheadrightarrow_{w\perp_{\mathcal{U}}} N$ then there exists P such that $M \twoheadrightarrow_w P \twoheadrightarrow_{\perp_{\mathcal{U}}} N$. (No properties of $\mathcal{U}$ need to be assumed.)*

▶ **Theorem 3.15** (Rootactiveness implies Normalization [8, 5]). *If $\mathcal{U} \subseteq \mathcal{CL}^\infty$ satisfies rootactiveness, then $(\mathcal{CL}_\perp^\infty, \twoheadrightarrow_{w\perp_{\mathcal{U}}})$ is normalising.*
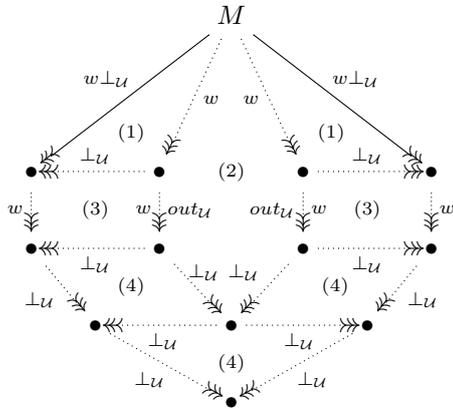
▶ **Theorem 3.16** (Rootactiveness and Meaninglessness implies Confluence [8, 5]). *If $\mathcal{U} \subseteq \mathcal{CL}^\infty$ is a meaningless set that satisfies rootactiveness, then $(\mathcal{CL}_\perp^\infty, \twoheadrightarrow_{w\perp_{\mathcal{U}}})$ is confluent.*

▶ **Notation 3.17** (Normal Form). If $(\mathcal{CL}_\perp^\infty, \twoheadrightarrow_{w\perp_{\mathcal{U}}})$ is confluent and normalising, then every term M in $\mathcal{CL}_\perp^\infty$ has a unique normal form, that we denote by $\mathsf{nf}_{\mathcal{U}}(M)$. We also write $\mathsf{nf}_{\mathcal{U}}(X) = \{\mathsf{nf}_{\mathcal{U}}(M) \mid M \in X\}$ for $X \subseteq \mathcal{CL}_\perp^\infty$.

Next we prove that $(\mathcal{CL}_\perp^\infty, \twoheadrightarrow_{w\perp_{\mathcal{U}}})$ is confluent for meaningless sets $\mathcal{U}$ that do not necessarily satisfy rootactiveness with tools from from [8, 5]. The earlier papers bypassed this result, because their interest lay in extensions $(\mathcal{CL}_\perp^\infty, \twoheadrightarrow_{w\perp_{\mathcal{U}}})$ that are both confluence and normalising.

▶ **Theorem 3.18** (Meaninglessness implies Confluence). *If $\mathcal{U}$ is a meaningless set, then $(\mathcal{CL}_\perp^\infty, \twoheadrightarrow_{w\perp_{\mathcal{U}}})$ is confluent.*

Proof. Let $M \xrightarrow{out_{\mathcal{U}}}_w N$ denote one w-step where the contracted redex in $M \rightarrow_w N$ is not contained in any subterm P of M which is in $\mathcal{U}$.

Suppose we are given two $w\perp_{\mathcal{U}}$-reductions starting from M. Using Postponement Theorem 3.14 we factor both reductions in a w-reduction followed by a $\perp_{\mathcal{U}}$ reduction. This gives us the two triangles (1). Next, from Theorem 3.12, we find two w-reductions ending in terms that are identical up to subterms in $\mathcal{U}$. If we factor both reductions into an outside $\mathcal{U}$ w-reduction (Lemma 12.9.20 in [5], which needs overlap and closure under reduction), followed by an inside $\mathcal{U}$ w-reduction we obtain (2). And (3) follows from commutation of outside $\mathcal{U}$ w-reduction with $\twoheadrightarrow_{\perp_{\mathcal{U}}}$ (Lemma 12.9.19 in [5], which needs overlap), while (4) is implied by Theorem 3.13 (confluence of $\twoheadrightarrow_\perp$). ◀

Note that the previous theorem and proof holds for arbitrary orthogonal term rewriting systems as well.

▶ Remark. In [16], we could prove that confluence implies normalisation for infinitary lambda calculus with $\beta\perp_{\mathcal{U}}$-reduction defined with any set of finite and infinite terms $U \subseteq \Lambda^\infty$. In the case of combinatory logic, this does not hold. For example, take $\mathcal{U} = \mathcal{H}$. By Theorem 3.18 we have that $(\mathcal{CL}^\infty_\perp, \twoheadrightarrow_{w\perp_\mathcal{H}})$ is confluent. But it is not normalising, because the term $\mathbf{\Omega}$, which is rootactive but not hypercollapsing, is not normalising.

4   Axioms of Closure under Expansion and Substitution

We now introduce and study some axioms involving closure under expansion and substitution.

▶ **Definition 4.1** (Closure under Expansion and Substitution)**.** Let $\mathcal{U} \subseteq \mathcal{CL}^\infty$.

1. $\mathcal{U}$ satisfies the **axiom of closure under w-expansion**, if for all $M \in \mathcal{U}$ $N \in \mathcal{U}$ whenever $M \twoheadrightarrow_w N$.
2. $\mathcal{U}$ satisfies the **axiom of closure under substitution**, if $M \in \mathcal{U}$ implies $M^\sigma \in \mathcal{U}$ for all $M \in \mathcal{CL}^\infty$ and substitutions σ from variables to terms in $\mathcal{CL}^\infty$.
3. We say that $\mathcal{U}$ satisfies the **axiom of closure under $w\perp$-expansion from $\perp$**, if for all $M \in \mathcal{CL}^\infty$, if $M \twoheadrightarrow_{w\perp_\mathcal{U}} \perp$ then $M \in \mathcal{U}$.

▶ **Lemma 4.2.** *Let $\mathcal{U}$ be a subset of $\mathcal{CL}^\infty$. Then $\mathcal{U}$ satisfies both the axiom of indiscernibility and the axiom of closure under w-expansion if and only if satisfies the axiom of closure under $w\perp$-expansion from $\perp$.*

Proof. Assume $M \twoheadrightarrow_{w\perp} \perp$ for some $M \in \mathcal{CL}^\infty$. Then $M \twoheadrightarrow_w N \twoheadrightarrow_{\perp_\mathcal{U}} \perp$ for some $N \in \mathcal{CL}^\infty_\perp$ by postponement. We prove that $N \in \mathcal{U}$ by induction on the length β of $N \twoheadrightarrow_{\perp_\mathcal{U}} \perp$. Note that the last step should be a successor ordinal $\beta = \alpha + 1$. Hence, $N \twoheadrightarrow_\perp N' \to_\perp \perp$ for some N'. Let $N|_p$ denote the subterm of N at position p. For every position p of N' such that $N'|_p = \perp$, we have that $N|_p \twoheadrightarrow_\perp \perp$ is shorted than β. By induction hypothesis, $N|p \in \mathcal{U}$. By indiscernibility, $N \in \mathcal{U}$. Closure under w-expansion then gives us the desired $M \in \mathcal{U}$. For the converse, assume first that $M, N \in \mathcal{CL}^\infty$ such that $M \xleftrightarrow{\mathcal{U}} N$. There exists P such that $P \preceq_\mathcal{U} M$ and $P \preceq_\mathcal{U} N$. If $M \in \mathcal{U}$ then $P \to_\perp \perp$. Hence, $N \twoheadrightarrow_\perp P \to_\perp \perp$. Since $\mathcal{U}$ satisfies closure under $w\perp$-expansion from $\perp$, we get $N \in \mathcal{U}$. Hence indiscernibility holds. Second, assuming that $M \twoheadrightarrow_w N$ and $N \in \mathcal{U}$, we get $M \twoheadrightarrow_w N \to_{\perp_\mathcal{U}} \perp$. By closure under $w\perp$-expansion from $\perp$ we can conclude $M \in \mathcal{U}$. Hence closure under w-reduction holds. ◀

▶ **Lemma 4.3.** *If a meaningless set $\mathcal{U} \subseteq \mathcal{CL}^\infty$ satisfies the axiom of closure under $w\perp$-expansion from $\perp$, then it also satisfies closure under substitution.*

Proof. Suppose $M \in \mathcal{U}$. Then $M \to_\perp \perp$. Because the reduction $\to_\perp$ is closed under substitution by definition we get $M^\sigma \to_\perp \perp$. But then $M^\sigma \in \mathcal{U}$ follows from closure under $w\perp$-expansion from $\perp$. ◀

▶ **Definition 4.4** (Set of $w\perp$-expansions from $\perp$)**.** Let $\mathcal{U} \subseteq \mathcal{CL}^\infty$. We define $\overline{\mathcal{U}} = \{M \in \mathcal{CL}^\infty \mid M \twoheadrightarrow_{w\perp_\mathcal{U}} \perp\}$.

It is a immediate that $\overline{\mathcal{U}}$ satisfies closure under $w\perp$-expansion from $\perp$.

▶ **Lemma 4.5** ($\mathcal{U}$ and $\overline{\mathcal{U}}$ define the same reduction)**.** *Let $\mathcal{U}$ be a meaningless set. We have that $M \twoheadrightarrow_{w\perp_{\overline{\mathcal{U}}}} N$ iff $M \twoheadrightarrow_{w\perp_\mathcal{U}} N$.*

Proof. By induction on the length of the reduction sequence. We only show the case when the length is one. Let $M = C[P] \to_{\perp_{\overline{\mathcal{U}}}} C[\perp] = N$. Then, $P \preceq_{\overline{\mathcal{U}}} Q$ and $Q \in \overline{\mathcal{U}}$. By definition of $\overline{\mathcal{U}}$, we have that $Q \twoheadrightarrow_{w\perp_{\mathcal{U}}} \perp$. Since $P \preceq_{\overline{\mathcal{U}}} Q$, we have that P is obtained from Q by replacing some of its subterms in $\overline{\mathcal{U}}$ by $\perp$. Each of these subterms $w\perp_{\mathcal{U}}$-reduces to $\perp$ so that we can construct a reduction sequence $Q \twoheadrightarrow_{w\perp_{\mathcal{U}}} P$. By Confluence (Theorem 3.13), we have that $P \twoheadrightarrow_{w\perp_{\mathcal{U}}} \perp$. Hence, $C[P] \twoheadrightarrow_{w\perp_{\mathcal{U}}} C[\perp]$. The converse is trivial. ◄

As consequence of this lemma we have $\overline{\overline{\mathcal{U}}} = \overline{\mathcal{U}}$ and we obtain the corollary:

▶ **Corollary 4.6.** *If $\mathcal{U}$ is meaningless then $(\mathcal{CL}_\perp^\infty, \twoheadrightarrow_{w\perp_{\overline{\mathcal{U}}}})$ is confluent. Moreover, if $\mathcal{U}$ satisfies rootactiveness, then $(\mathcal{CL}_\perp^\infty, \twoheadrightarrow_{w\perp_{\overline{\mathcal{U}}}})$ is normalising.*

5 Meaningless Sets

In this section we will show that there are uncountably many meaningless sets in infinitary combinatory logic. This contrasts with the handful of meaningless sets (among which the rootactive terms) known for orthogonal infinitary term rewriting systems [8].

We will identify sets of meaningless terms that define the same reduction. Hence, by Lemma 4.5, we will consider those sets of meaningless terms that satisfy $\mathcal{U} = \overline{\mathcal{U}}$. Or, equivalently, using Lemma 4.2, we will restrict ourselves to sets of meaningless terms that satisfy the extra axiom of closure under w-expansion.

We will show that the class $\mathcal{L}_{CL}$ of sets of meaningless terms that satisfy w-expansion is a bounded lattice ordered by set inclusion. The meet (denoted by $\sqcap$) coincides with the intersection. The join $\mathcal{U} \sqcup \mathcal{V}$ is the least meaningless set containing $\mathcal{U} \cup \mathcal{V}$. By construction the set $\mathcal{CL}^\infty$ is the largest element of $\mathcal{L}_{CL}$ and $\mathcal{H}$ the smallest one.

The lattice $\mathcal{L}_{CL}$ has a somewhat richer structure than the corresponding lattice of meaningless sets of lambda calculus [16]. Below $\mathcal{CL}^\infty$ we find the unsolvables as the next largest set, as the notion of solvability for finite combinatory logic extends to $\mathcal{CL}^\infty$ and $\mathcal{CL}_\perp^\infty$ [1, 3].

▶ **Definition 5.1** (Solvable Term). Let $M \in \mathcal{CL}_\perp^\infty$.
1. A closed term M is *solvable* if there exist $P_1 \ldots P_k$ such that $MP_1 \ldots P_k \twoheadrightarrow_w \mathbf{I}$.
2. A term M is *solvable* if M^σ is solvable for some substitution σ replacing all variables in M by closed terms from $\mathcal{CL}^\infty$. Terms that are not solvable are called *unsolvable*.
3. $\mathcal{NS} = \{M \in \mathcal{CL}^\infty \mid M$ is not solvable$\}$.

We will first show that $\mathcal{NS}$ is indeed an element of $\mathcal{L}_{CL}$, before showing that $\mathcal{NS}$ is the largest element in $\mathcal{L}_{CL}$ below $\mathcal{CL}^\infty$. We need some useful lemmas to do so. The first lemma follows by induction on the length of the reduction sequence and it holds for any set $\mathcal{U}$.

▶ **Lemma 5.2.** *If $P \twoheadrightarrow_w P'$ and $P \preceq_{\mathcal{U}} M$ then $M \twoheadrightarrow_w M'$ and $P' \preceq_{\mathcal{U}} M'$ for some M'*

▶ **Lemma 5.3.** *If $Q \preceq_{\mathcal{NS}} N$ and $N \twoheadrightarrow_w \mathbf{I}$ then $Q \twoheadrightarrow_w \mathbf{I}$.*

Proof. By induction on the length n of the reduction sequence. If $n = 0$ then $Q \preceq_{\mathcal{NS}} N = \mathbf{I}$ and $Q = \mathbf{I}$. If $n > 0$ then we can only have that either $N = \mathbf{K}N_1 \ldots N_k$ or $N = \mathbf{S}N_1 \ldots N_{k+1}$ with $k \geq 2$ because the normal form of N is $\mathbf{I}$. We do the case $N = \mathbf{S}N_1 \ldots N_k$ with $k \geq 3$ where the reduction sequence is $N = \mathbf{S}N_1 \ldots N_k \to_w N_1 N_3 (N_2 N_3) N_4 \ldots N_k \twoheadrightarrow_w \mathbf{I}$. Hence N and all its prefixes $\mathbf{S}N_1 \ldots N_i$ for $i \leq k$ are solvable. If $Q \preceq_{\mathcal{NS}} N$ then $Q = \mathbf{S}Q_1 \ldots Q_k$ with $Q_i \preceq_{\mathcal{NS}} N_i$ for $1 \leq i \leq k$. By IH, $Q_1 Q_3 (Q_2 Q_3) Q_4 \ldots Q_k \twoheadrightarrow_w \mathbf{I}$. Hence $Q \twoheadrightarrow_w \mathbf{I}$. ◄

▶ **Lemma 5.4.** *Let $P \in \mathcal{CL}_\perp^\infty$ such that $P \preceq_{\mathcal{NS}} M$. Then, P is solvable iff M is solvable.*

Proof. Suppose P is solvable. Then there exist a substitution σ and terms $Q_1, \ldots, Q_n$ such that $P^\sigma Q_1 \ldots Q_n \twoheadrightarrow_w \mathbf{I}$. By Lemma 5.2, $M^\sigma Q_1 \ldots Q_n \twoheadrightarrow_w \mathbf{I}$. Hence, M is solvable.

Suppose M is solvable. Then, there exist a substitution σ and terms $Q_1, \ldots Q_n$ such that $M^\sigma Q_1 \ldots Q_n \twoheadrightarrow_w \mathbf{I}$. Assume that P is unsolvable. Then whenever $P^\sigma Q_1 \ldots Q_n \twoheadrightarrow_w Q$ it follows that $Q \neq \mathbf{I}$. Since $P^\sigma Q_1 \ldots Q_n \preceq_{\mathcal{NS}} M^\sigma Q_1 \ldots Q_n$ we get by Lemma 5.2 that $M^\sigma Q_1 \ldots Q_n \twoheadrightarrow_w N$ for some $Q \preceq_{\mathcal{NS}} N$. By confluence of $w\perp_\mathcal{R}$-reduction, we have that $N \twoheadrightarrow_{w\perp_\mathcal{R}} \mathbf{I}$. Hence by postponement we can factor this reduction into a w-reduction followed by a $\perp_\mathcal{U}$-reduction to $\mathbf{I}$. Because $\mathbf{I}$ does not contain any $\perp$, the latter must be necessarily must be of zero length. Hence $N \twoheadrightarrow_w \mathbf{I}$. Applying Lemma 5.3, we find that then also $Q \twoheadrightarrow_w \mathbf{I}$. Hence P is solvable as well.   ◀

▶ **Lemma 5.5.** *If $M \in \mathcal{U}$ and $MP_1 \ldots P_n \twoheadrightarrow_w Q$ then $Q = NP_{i+1} \ldots P_n$ where $N \in \mathcal{U}$.*

Proof. This is proved by induction on the length of the reduction. It is enough to consider one step $MP_1 \ldots P_n \to_w Q$. If the w-redex is inside M or one of the P_i's, the claim is immediate. If the w-redex is of the form $MP_1 \ldots P_i$ for $i > 0$ then by overlap, we have that $MP_1 \ldots P_i \in \mathcal{U}$. By closure under reduction, the contracted redex is in $\mathcal{U}$.   ◀

▶ **Theorem 5.6** (Meaningless Set $\mathcal{NS}$). *The set $\mathcal{NS}$ is a meaningless set. Moreover, it is the largest meaningless set satisfying the axiom of closure under w-expansion which is a proper subset of $\mathcal{CL}^\infty$. In particular, $\mathcal{NS} = \overline{\mathcal{NS}} \subset \mathcal{CL}^\infty$.*

Proof. Let R be a rootactive term and σ a substitution. Then R^σ is also rootactive and $R^\sigma P_1 \ldots P_n$ can not reduce to $\mathbf{I}$. Therefore all rootactive terms are unsolvable and $\mathcal{NS}$ satisfies rootactiveness.

To prove closure under w-reduction, suppose M is unsolvable and $M \twoheadrightarrow_w N$. Assume N is solvable. Then $N^\sigma P_1 \ldots P_n \twoheadrightarrow_w \mathbf{I}$ for some $\sigma, P_1, \ldots, P_n$. Now M is solvable too, because $M^\sigma P_1 \ldots P_n \twoheadrightarrow_w N^\sigma P_1 \ldots P_n \twoheadrightarrow_w \mathbf{I}$. Contradiction. Hence N is unsolvable.

Overlap is also easy to prove. Suppose $\mathbf{S}MNP$ is solvable. Then there exist $\sigma, Q_1, \ldots, Q_n$ such that $(\mathbf{S}MNP)^\sigma Q_1 \ldots Q_n \twoheadrightarrow_w \mathbf{I}$. Hence, $\mathbf{S}$, $\mathbf{S}M$ and $\mathbf{S}MN$ are also solvable. Overlap with $\mathbf{K}$ goes similar.
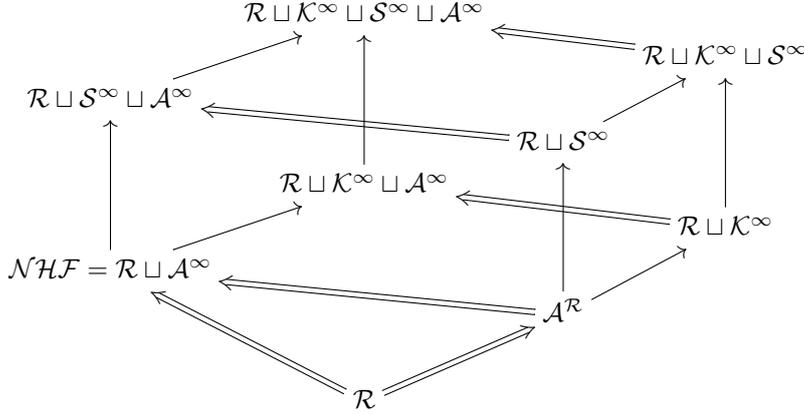
We now prove indiscernibility. If $M \overset{\mathcal{NS}}{\longleftrightarrow} N$, then both $P \preceq_{\mathcal{NS}} M$ and $P \preceq_{\mathcal{NS}} N$ for some $P \in \mathcal{CL}_\perp^\infty$. It follows from Lemma 5.4 that $M \in \mathcal{NS}$ iff $N \in \mathcal{NS}$.

Next we prove that $\mathcal{NS}$ is closed under under w-expansion. Suppose $M \twoheadrightarrow_w N$. Assume M is solvable. If we show that N is solvable, then closure under w-reduction follows by contraposition. Solvability of M implies that $MP_1 \ldots P_n \twoheadrightarrow_w \mathbf{I}$ for some P_i. Applying Theorem 3.12 on confluence modulo hypercollapsing terms, we get $N \twoheadrightarrow_w \mathbf{I}$, as $\mathbf{I}$ does not contain any hypercollapsing subterms. Hence N is solvable.

Finally we prove that $\mathcal{NS}$ is the largest meaningless set closed under under w-expansion, which is a proper subset of $\mathcal{CL}^\infty$. Now, suppose that $M \in \mathcal{U}$ and $M \notin \mathcal{NS}$. Hence, there exists a substitution σ and terms $P_1, \ldots P_n$ such that $M^\sigma P_1 \ldots P_n \twoheadrightarrow_w \mathbf{I} = \mathbf{SKK}$. By Lemma 5.5, we have that either $\mathbf{S}$, $\mathbf{SK}$ or $\mathbf{SKK}$ are in $\mathcal{U}$. For any term N, we have that $\mathbf{SKK}N \twoheadrightarrow_w N$. By the axioms of overlap and closure under reduction, $N \in \mathcal{U}$.   ◀

Note that a term in $\mathcal{CL}^\infty$ has either one of the following forms: (1) $\mathbf{S}$, $\mathbf{S}P$, $\mathbf{S}PQ$, $\mathbf{K}$, $\mathbf{K}P$ or $\mathbf{K}PQ$; (2) $xP_1 \ldots P_n$ for $n \geq 0$; (3) $((\ldots)P_2)P_1$; (4) $\mathbf{S}P_1 \ldots P_n$ for $n \geq 3$; (5) $\mathbf{K}P_1 \ldots P_n$ for $n \geq 2$. Having this in mind we give the following definition, reminiscent of the analogous Definition 4.1 in [16] for infinitary lambda calculus.

▶ **Definition 5.7** (Head Normal Form). *Let $M \in \mathcal{CL}_\perp^\infty$.*

Figure 1 A fragment of $\mathcal{L}_{CL}$: single (double) arrows indicate that the corresponding open intervals are singletons (uncountable). A similar fragment can be built by replacing $\mathcal{R}$ by $\mathcal{H}$ in the above diagram.

1. M is in *head normal form (hnf or w-hnf)* if it is one of the forms $\mathbf{K}$, $\mathbf{K}P$, $\mathbf{S}$, $\mathbf{S}P$, $\mathbf{S}PQ$ or $xP_1 \ldots P_n$.
2. $\mathcal{NHF} = \{M \in \mathcal{CL}^\infty \mid$ there is no N such that $M \twoheadrightarrow_w N$ and N is in hnf$\}$.

The set $\mathcal{NHF}$ is a set of meaningless terms that satisfies closure under w-expansion. Terms in $\mathcal{NHF}$ are exactly the opaque terms, i.e. their reducts cannot overlap any redex [8]. Unlike lambda calculus, terms with head normal forms do not correspond to solvable terms in combinatory logic. For example, the head normal form $\mathbf{K\Omega}$ is unsolvable in combinatory logic. The reason is that head normal forms in combinatory logic are related to weak head normal forms in lambda calculus: M is a hnf in combinatory logic iff M_λ is a weak head normal form in lambda calculus [4]. To define many more elements of $\mathcal{L}_{CL}$ we define the following forms and sets:

▶ **Definition 5.8.** 1. $\mathcal{A}_X^Y = \{M \in \mathcal{CL}^\infty \mid M \twoheadrightarrow_w N$ and N is a X, Y-ha$\}$ where M is a *head active form relative to X and Y* (X, Y-ha) if $M = RP_1 \ldots P_k$, $R \in Y$ and $P_i \in X$ for $1 \le i \le k$.
2. $\mathcal{A}_X^\infty = \{M \in \mathcal{CL}^\infty \mid M \twoheadrightarrow_w N$ and N is a X-il $\}$ where M is an *infinite left spine form relative to X* (X-il) if $M = (\ldots P_2)P_1$ and $P_i \in X$ for all i.
3. $\mathcal{K}^\infty = \{M \in \mathcal{CL}^\infty \mid M \twoheadrightarrow_w \mathbf{K}^\omega\}$ where $\mathbf{K}^\omega = \mathbf{K}(\mathbf{K}(\mathbf{K}(\ldots)))$.
4. $\mathcal{S}^\infty = \{M \in \mathcal{CL}^\infty \mid M \twoheadrightarrow_w \mathbf{S}^\omega\}$ where $\mathbf{S}^\omega = \mathbf{S}(\mathbf{S}(\mathbf{S}(\ldots)))$.
   When $X = \mathcal{CL}^\infty$ in $\mathcal{A}_X^Y$ or $\mathcal{A}_X^\infty$ we drop the phrase *relative to X* and the subscript X.

Using the $\mathcal{H}$ or $\mathcal{R}$ as a base we can make two sublattices of $\mathcal{L}_{CL}$ as depicted in Figure 1. The top elements of these fragments, $\mathcal{R} \sqcup \mathcal{K}^\infty \sqcup \mathcal{S}^\infty \sqcup \mathcal{A}^\infty$ and $\mathcal{H} \sqcup \mathcal{K}^\infty \sqcup \mathcal{S}^\infty \sqcup \mathcal{A}^\infty$ are proper subsets of $\mathcal{NS}$, because neither of the two sets contain the unsolvable $\mathbf{S}(\mathbf{K}(\mathbf{S}(\mathbf{K}(\ldots))))$. Note that $\mathcal{NHF} = \mathcal{A}^\mathcal{R} \cup \mathcal{A}^\infty = \mathcal{R} \sqcup \mathcal{A}^\infty$. The intervals of Figure 1 that have uncountable cardinality, indicated by $\Rightarrow$, follow from Theorem 5.9 and the fact that $\mathsf{nf}_{\mathcal{NS}}(\mathcal{CL}^\infty) \cap \mathcal{CL}^\infty$ is uncountable.

▶ **Theorem 5.9** (Meaningless Sets). *Let $X \subseteq \mathsf{nf}_{\mathcal{NS}}(\mathcal{CL}^\infty) \cap \mathcal{CL}^\infty$ and $Y \in \{\mathcal{R}, \mathcal{H}\}$. The following sets are sets of meaningless terms that are closed under w-expansion: $\mathcal{A}_X^Y$, $\mathcal{A}_X^Y \cup \mathcal{A}_X^\infty$, $\mathcal{A}^Y \cup \mathcal{A}_X^\infty$, $\mathcal{A}^Y \cup \mathcal{K}^\infty \cup \mathcal{A}_X^\infty$, $\mathcal{A}^Y \cup \mathcal{S}^\infty \cup \mathcal{A}_X^\infty$ and $\mathcal{A}^Y \cup \mathcal{S}^\infty \cup \mathcal{K}^\infty \cup \mathcal{A}_X^\infty$.*

Proof. It is straightforward to show that these sets satisfy rootactiveness, overlap, closure under w-reduction and closure under w-expansion. We show that $\mathcal{U} = \mathcal{A}_X^{\mathcal{R}}$ satisfies indiscernibility for $X \subseteq \mathsf{nf}_{\mathcal{NS}}(\mathcal{C}L^\infty) \cap \mathcal{C}L^\infty$. If $M \xleftrightarrow{u} N$ then there exists P such that $P \preceq_{\mathcal{U}} M$ and $P \preceq_{\mathcal{U}} N$. By Lemma 5.2, we can assume that $P = \mathsf{skel}(P)$. We discuss cases according to the shape of P. The interesting case is when $P = \bot P_1 \ldots P_n$. Then $M = M_0 M_1 \ldots M_n$ and $N = N_0 N_1 \ldots N_n$. Since $X \subseteq \mathsf{nf}_{\mathcal{NS}}(\mathcal{C}L^\infty)$, we have that $P_i = M_i = N_i$ for all $1 \leq i \leq n$. It is clear that $M \in \mathcal{A}_X^{\mathcal{R}}$ iff $N \in \mathcal{A}_X^{\mathcal{R}}$. ◀

6   Application to Combinatory Algebras

Models for combinatory logic have a simple structure: they are *combinatory algebras*. Models for lambda calculus are λ-algebras which are combinatory algebras that satisfy some further properties. We show that there is an uncountable number of combinatory algebras that are not λ-algebras. We recall the definition of combinatory and λ-algebras from [1].

▶ **Definition 6.1** (Combinatory Algebra). A *combinatory algebra* is a structure $(X, \cdot, k, s)$ where $\cdot$ is a binary operation on X and $k, s \in X$ satisfy $kxy = x$ and $sxyz = xz(yz)$.

Given a valuation ρ mapping variables to terms in X we interpret terms of $\mathcal{C}L$ in a combinatory algebra $(X, \cdot, k, s)$ as follows: $[\![x]\!]_\rho = \rho(x)$, $[\![\mathbf{K}]\!]_\rho = k$, $[\![\mathbf{S}]\!]_\rho = s$ and $[\![MN]\!]_\rho = [\![M]\!]_\rho \cdot [\![N]\!]_\rho$.

▶ **Definition 6.2** (Lambda Algebra). A combinatory algebra is a λ-*algebra* if $M_\lambda =_\beta N_\lambda$ then $[\![M]\!]_\rho = [\![N]\!]_\rho$ for all $M, N \in \mathcal{C}L$.

▶ **Theorem 6.3** (Combinatory Algebra induced by Infinitary Combinatory Logic). *Let* $\mathcal{U} \subset \mathcal{C}L^\infty$ *be closed under* $w\bot$-*expansions from* $\bot$. *If* $(\mathcal{C}L_\bot^\infty, \twoheadrightarrow_{w\bot_{\mathcal{U}}})$ *is confluent and normalising, it induces a combinatory algebra that is not a λ-algebra which is given by* $(\mathsf{nf}_{\mathcal{U}}(\mathcal{C}L^\infty), \cdot, \mathbf{K}, \mathbf{S})$ *where* $M \cdot N = \mathsf{nf}_{\mathcal{U}}(MN)$.

Proof. Assume $\mathcal{U} \subset \mathcal{C}L^\infty$. A term is interpreted by its normal form. In a λ-algebra, the term $\mathbf{K}$ should be equal to $\mathbf{S}(\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{KK})\mathbf{K}))(\mathbf{K}(\mathbf{S}(\mathbf{KK})))$. Since these two terms are in w-normal form, they will have the same $w\bot_{\mathcal{U}}$-normal form only if they both reduce to $\bot$. This is not possible, as then $\mathbf{K}$ should belong to $\mathcal{U}$, which would imply by Lemma 7.2 that $\mathcal{U} = \mathcal{C}L^\infty$. Contradiction. ◀

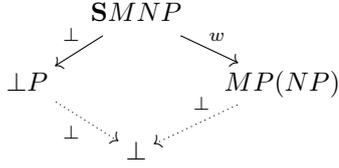▶ **Corollary 6.4** (Uncountable Combinatory Algebras). *There is an uncountable number of combinatory algebras that are not λ-algebras.*

Proof. By Theorem 5.9, there are uncountably many meaningless sets satisfying rootactiveness. Each of them gives rise to a different, confluent and normalising infinitary $\bot$-extension of $\mathcal{C}L$ by Theorems 5.9 and 3.16. By Theorem 6.3 none of the induced combinatory algebras is a λ-model. ◀

7   Weakening the Axioms of Meaningless Terms

In this section, we first show that confluence implies hypercollapseness, closure under w-reduction, indiscernibility and a weaker form of overlap. We can prove the converse only under the extra condition of rootactiveness.

To weaken the axiom of overlap, we inspect when overlap between $\perp$-reduction and w-reduction occurs. Overlap happens when the $\perp$-redex is of the form $\mathbf{K}M$, $\mathbf{S}M$ or $\mathbf{S}MN$. This gives a divergence that can be resolved with the axiom of overlap, e.g.

There is, however, another way of resolving this divergence. Suppose that $M = (\mathbf{K}W)$ for some $W \in \mathcal{U}$ and $N = \mathbf{I}$. Then, we have that $\mathbf{S}(\mathbf{K}W)\mathbf{I}P \twoheadrightarrow_w WP \to_\perp \perp$. This is not the only alternative to resolve the divergence. Suppose $M = \mathbf{S}(\mathbf{K}\mathbf{K})W$ for some $W \in \mathcal{U}$.. Then, we have that $\mathbf{S}(\mathbf{S}(\mathbf{K}\mathbf{K})W)NP \twoheadrightarrow_w WP \to_\perp \perp$. In general, there is an alternative resolution of the divergence whenever there exists a $W \in \mathcal{U}$ such that $MP(NP) \twoheadrightarrow_w WP$.

▶ **Definition 7.1** (Weak Overlap). Let $\mathcal{U} \subseteq \mathcal{CL}^\infty$. We say that $\mathcal{U}$ satisfies the axiom of weak overlap if we have that for all $M, N, P \in \mathcal{CL}^\infty$:
1. If $\mathbf{K}$ or $\mathbf{K}M \in \mathcal{U}$ then $\mathbf{K}MN \in \mathcal{U}$.
2. If $\mathbf{S}$ or $\mathbf{S}M \in \mathcal{U}$ then $\mathbf{S}MN \in \mathcal{U}$ or $\mathbf{S}MNP \in \mathcal{U}$.
3. If $\mathbf{S}MN \in \mathcal{U}$ then $\mathbf{S}MNP \in \mathcal{U} \cup \mathcal{R}$ or $\mathbf{S}MNP \to_w MP(NP) \twoheadrightarrow_w WP$ and $W \in \mathcal{U} \cap (\mathcal{A}^\mathcal{R} \cup \mathcal{A}^\infty)$.

Clearly, overlap implies weak overlap.

▶ **Lemma 7.2** ($\mathbf{K}$ or $\mathbf{S}$ in $\mathcal{U}$ implies all Terms are in $\mathcal{U}$). *Let $\mathcal{U} \subseteq \mathcal{CL}^\infty$ satisfy closure under $w\perp$-expansion from $\perp$ and $(\mathcal{CL}^\infty_\perp, \twoheadrightarrow_{w\perp_\mathcal{U}})$ be confluent. If $\mathbf{K} \in \mathcal{U}$ or $\mathbf{S} \in \mathcal{U}$ then $\mathcal{U} = \mathcal{CL}^\infty$.*

Proof. Suppose that $\mathbf{K} \in \mathcal{U}$. $\mathbf{K}xy$ reduces to $\perp xy$ and to x. By confluence, these two terms should have a common reduct. The only possibility is that this reduct is $\perp$. By closure under $w\perp$-expansion from $\perp$, $x \in \mathcal{U}$. By Lemma 4.3, $M \in \mathcal{U}$ for all $M \in \mathcal{CL}^\infty$. Suppose now that $\mathbf{S} \in \mathcal{U}$. We have that $\mathbf{S}\mathbf{K}xy \twoheadrightarrow_w y$ and $\mathbf{S}\mathbf{K}xy \to_\perp \perp\mathbf{K}xy$. By confluence, y and $\perp\mathbf{K}xy$ have a common reduct Q. The only possibility is that $Q = \perp$. Hence, $y \in \mathcal{U}$. By Lemma 4.3, $\mathcal{U}$ satisfies closure under substitution and hence, $M \in \mathcal{U}$ for all $M \in \mathcal{CL}^\infty$. ◀

The next lemma is proved by induction on the length of the reduction sequence.

▶ **Lemma 7.3.** *If $M \twoheadrightarrow_w N$ and all occurrences of x in M are in a term of the form xy then so are all occurrences of x in N.*

▶ **Theorem 7.4** (Necessary Conditions for Confluence). *Let $\mathcal{U} \subseteq \mathcal{CL}^\infty$ be closed under $w\perp$-expansion from $\perp$. If $(\mathcal{CL}^\infty_\perp, \twoheadrightarrow_{w\perp_\mathcal{U}})$ is confluent, then $\mathcal{U}$ satisfies the axioms of hypercollapseness, closure under w-reduction, indiscernibility and weak overlap.*

Proof. If $\mathcal{U} = \mathcal{CL}^\infty$ then it satisfies all the axioms. From now on, suppose that $\mathcal{U} \subsetneq \mathcal{CL}^\infty$. We have that $\mathcal{U}$ satisfies indiscernibility by Lemma 4.2.

We now prove that $\mathcal{U}$ satisfies closure under w-reduction as follows. Suppose $M \twoheadrightarrow_w N$ and $M \in \mathcal{U}$. It follows from $M \to_\perp \perp$ and confluence that $N \twoheadrightarrow_{w\perp_\mathcal{U}} \perp$. By closure under $w\perp$-expansion from $\perp$, we have that $N \in \mathcal{U}$.

We prove that $\mathcal{U}$ satisfies hypercollapseness by showing that all hypercollapsing terms reduce to $\perp$. The infinite term $\mathbf{C}_x[\mathbf{C}_y[\mathbf{C}_x[\mathbf{C}_y[\ldots]]]]$ can reduce to both $\mathbf{C}_x^\omega$ and $\mathbf{C}_y^\omega$. By confluence, both $\mathbf{C}_x^\omega$ and $\mathbf{C}_y^\omega$ reduce to some common term Q. By postponement, $\mathbf{C}_x^\omega \twoheadrightarrow_\perp Q$

and $\mathbf{C}_y^\omega \twoheadrightarrow_\perp Q$ because $\mathbf{C}_x^\omega$ and $\mathbf{C}_y^\omega$ can only ω-reduce to themselves. Since $\mathbf{C}_x$ and $\mathbf{C}_y$ have no prefix in common, Q should be $\perp$.

Now suppose P is an arbitrary hypercollapsing term. Then, we have $P \twoheadrightarrow \mathbf{C}_{M_1}[\mathbf{C}_{M_2}[\ldots]]$ where $\mathbf{C}_{M_i}[\ ] = (\mathbf{K}[\ ]M_i)$ for some collapsing tower $\mathbf{C}_{M_1}[\mathbf{C}_{M_2}[\ldots]]$. The infinite collapsing tower $\mathbf{C}_{M_1}[\mathbf{C}_x[\mathbf{C}_{M_2}[\mathbf{C}_x[\ldots]]]]$ can reduce both to $\mathbf{C}_{M_1}[\mathbf{C}_{M_2}[\ldots]]$ and $\mathbf{C}_x[\mathbf{C}_x[\ldots]]$. Since $\mathbf{C}_x[\mathbf{C}_x[\ldots]]$ reduces to $\perp$, we find that by confluence, also $\mathbf{C}_{M_1}[\mathbf{C}_{M_2}[\ldots]]$ should reduce to $\perp$. Hence, all the hypercollapsing terms reduce to $\perp$. By closure under $\omega\perp$-expansion from $\perp$, we have that $\mathcal{H} \subseteq \mathcal{U}$.

We prove that $\mathcal{U}$ satisfies weak overlap by proving the clauses of Definition 7.1.

1. If $\mathbf{K} \in \mathcal{U}$ then by Lemma 7.2, we would have $\mathcal{U} = \mathcal{C}L^\infty$. Assume $\mathbf{K}M \in \mathcal{U}$. Then, $\mathbf{K}MN \to_w M$ and $\mathbf{K}MN \to_\perp \perp N$ for all N. Hence by confluence M and $\perp N$ must have a common reduct for any N. Hence in particular $\perp x$ and $\perp y$ must have a common reduct, which can only be $\perp$. Hence also M reduces to $\perp$ and therefore we see that $\mathbf{K}MN \in \mathcal{U}$ by closure under w-expansion.

2. If $\mathbf{S} \in \mathcal{U}$ then by Lemma 7.2 we would have $\mathcal{U} = \mathcal{C}L^\infty$. Suppose $\mathbf{S}M \in \mathcal{U}$ and x, y do not occur in M. Then, $\mathbf{S}Mxy$ reduces to $\perp xy$ and to $My(xy)$. They should both have a common reduct N. We have three cases. The first case is when $N = \perp xy$. Then, $My(xy) \twoheadrightarrow_w Wxy$ for some $W \in \mathcal{U}$. By Lemma 7.3, this is not possible. The second case is when $N = \perp y$. By closure under $w\perp$-expansion from $\perp$, $\mathbf{S}Mx \in \mathcal{U}$. By Lemma 4.3, $\mathbf{S}MN \in \mathcal{U}$ for all $N \in \mathcal{U}$. The third case is when $N = \perp$. By closure under $w\perp$-expansion from $\perp$, $\mathbf{S}Mxy \in \mathcal{U}$. By Lemma 4.3, $\mathbf{S}MNP \in \mathcal{U}$ for all $N, P \in \mathcal{C}L^\infty$.

3. By closure under substitutions (Lemma 4.3), it is enough to consider $P = x$. Assume $\mathbf{S}MN \in \mathcal{U}$ and $\mathbf{S}MNx \notin \mathcal{U}$. By confluence, $\perp x$ and $\mathsf{skel}(\mathbf{S}MNx)$ must have a common reduct. This common reduct cannot be $\perp$ because $\mathbf{S}MNx \notin \mathcal{U}$. Then, the common reduct should be $\perp x$, i.e. $\mathsf{skel}(\mathbf{S}MNx) \twoheadrightarrow_{w\perp_\mathcal{U}} \perp x$. By postponement, $\mathsf{skel}(\mathbf{S}MNx) \twoheadrightarrow_w Q \twoheadrightarrow_\perp \perp x$. Hence, $Q = Wx$ and $W \twoheadrightarrow_\perp \perp$. By closure under expansion from $\perp$, $W \in \mathcal{U}$. Suppose $\mathsf{skel}(\mathbf{S}MNx)$ is not rootactive, then it has one of the following forms:

   a. $\mathsf{skel}(\mathbf{S}MNx)$ is $yP_1 \ldots P_k$. Then, $P_k = x$ and $yP_1 \ldots P_{k-1} \in \mathcal{U}$. It is not difficult to show that $\mathcal{U} = \mathcal{C}L^\infty$.
   b. $\mathsf{skel}(\mathbf{S}MNx)$ is $\mathbf{K}$ or $\mathbf{S}$. This case is impossible because $\mathbf{K}$ or $\mathbf{S}$ cannot reduce to $\perp x$.
   c. $\mathsf{skel}(\mathbf{S}MNx)$ is $\mathbf{K}P$ or $\mathbf{S}P$. Then $P = x$ and either $W \in \mathcal{U}$ is $\mathbf{K}$ or $\mathbf{S}$. By Lemma 7.2 we would have $\mathcal{U} = \mathcal{C}L^\infty$.
   d. $\mathsf{skel}(\mathbf{S}MNx)$ is $\mathbf{S}P_1P_2$. Then, $P_2 = x$ and $\mathbf{S}P_1 \twoheadrightarrow_w W$. By closure under w-expansion, $\mathbf{S}P_1 \in \mathcal{U}$. Similarly to case 3), we have that $\mathbf{S}P_1x \in \mathcal{U}$ and hence, $\mathbf{S}MNx \twoheadrightarrow_{w\perp_\mathcal{U}} \perp$. By closure under $w\perp$-expansion from $\perp$, $\mathbf{S}MNx \in \mathcal{U}$. This is a contradiction.
   e. $\mathsf{skel}(\mathbf{S}MNx)$ is either a head active form or a infinite left spine. So is W. Hence, $W \in \mathcal{U} \cap (\mathcal{A}^\mathcal{R} \cup \mathcal{A}^\infty)$.

◀

We do not know whether it is possible to prove the converse. However, we can prove that under the extra condition of rootactiveness the conditions necessary for confluence are also sufficient for confluence. The following lemma plays a crucial role in the proof of this result. In a similar scenario for lambda calculus this role was played by Lemma 5.5 in [16].

▶ **Lemma 7.5.** *Let $\mathcal{U} \subseteq \mathcal{C}L^\infty$ satisfy rootactiveness, closure under w-reduction, indiscernibility and weak overlap. If $M \twoheadrightarrow_{\perp_\mathcal{U}} N$ and N is a $w\perp_\mathcal{U}$-normal form, then $\mathsf{nf}_\mathcal{R}(M) \twoheadrightarrow_{\perp_\mathcal{U}} N$.*

Proof. We can suppose that $\mathcal{U} \neq \mathcal{CL}^\infty$ and $M \twoheadrightarrow^{out}_{\perp_\mathcal{U}} N$. Then, M is obtained from N by replacing some $\perp$'s by terms in $\mathcal{U}_\perp$. We consider the set of terms

$$\mathcal{W} = \{W \mid W \text{ is a maximal subterm of } M \text{ such that } W \in \mathcal{U}_\perp\}$$

Each subterm $W \in \mathcal{W}$ of M will be replaced by $\mathsf{nf}_\mathcal{R}(W)$ in depth-first leftmost order. We obtain a term M_0 such that $M \twoheadrightarrow_{w\perp_\mathcal{R}} M_0$. We have that $W \twoheadrightarrow_w W_0 \twoheadrightarrow_{\perp_\mathcal{R}} \mathsf{nf}_\mathcal{R}(W)$. By closure under w-reduction, $W_0 \in \mathcal{U}$. By rootactiveness, $\mathsf{nf}_\mathcal{R}(W) \preceq_\mathcal{U} W_0$. By definition of $\perp$, $\mathsf{nf}_\mathcal{R}(W) \rightarrow_{\perp_\mathcal{U}} \perp$. Hence, $M_0 \twoheadrightarrow_{\perp_\mathcal{U}} N$. Since N is a $w\perp_\mathcal{U}$-normal form, if an w-redex occurs in M_0 then it either occurs in some subterm $W \in \mathcal{W}$ or it overlaps some subterm $W \in \mathcal{W}$. By replacing W by $\mathsf{nf}_\mathcal{R}(W)$, we remove all the possible w-redexes that are in W. We may still have w-redexes that overlap a term in $\mathcal{U}$. Suppose $\mathsf{nf}_\mathcal{R}(W)$ overlaps a w-redex of the form $\mathsf{nf}_\mathcal{R}(W)N_1 \ldots N_k$ in M_0. Then $WN_1 \ldots N_k$ is a w-redex in M. We have that $\mathsf{nf}_\mathcal{R}(W)N_1 \ldots N_k \notin \mathcal{U}_\perp$ because otherwise by indiscernibility $WN_1 \ldots N_k \in \mathcal{U}_\perp$ and N would not be in $w\perp_\mathcal{U}$-normal form. By weak overlap, $\mathsf{nf}_\mathcal{R}(W)$ cannot be $\mathbf{K}$, $\mathbf{S}$, $\mathbf{K}P$ or $\mathbf{S}P$. By weak overlap, the only possibility is that $\mathsf{nf}_\mathcal{R}(W) = \mathbf{S}PQ$, $k = 1$ and $\mathbf{S}PQN_1 \twoheadrightarrow_w W_1 N_1$ where $W_1 \in \mathcal{U} \cap (\mathcal{A}^\mathcal{R} \cup \mathcal{A}^\infty)$. In this case, to remove the w-redex, we replace the term W in M by $\mathsf{nf}_\mathcal{R}(W_1)$ instead of replacing it by $\mathsf{nf}_\mathcal{R}(W)$. The fact that W is either a head active term or an infinite left spine ensures that the normal form of WP can be calculated as the application of the normal forms of W and P. In other words, we have that
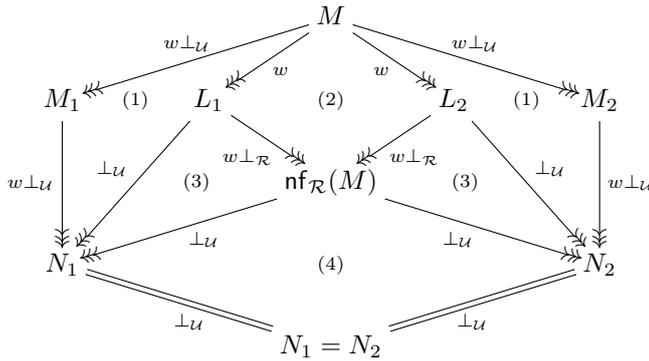
$$\mathsf{nf}_\mathcal{R}(WN_1) = \mathsf{nf}_\mathcal{R}(W_1)\mathsf{nf}_\mathcal{R}(N_1)$$

Since $\mathsf{nf}_\mathcal{R}(W_1) \preceq_\mathcal{R} W_1$, we have that $\mathsf{nf}_\mathcal{R}(W_1) \rightarrow_{\perp_\mathcal{U}} \perp$.

By replacing all terms in $\mathcal{W}$ in the fashion described above, we obtain a term M_1 in $w\perp_\mathcal{R}$-normal form such that $M \twoheadrightarrow_{w\perp_\mathcal{R}} M_0 \twoheadrightarrow_{w\perp_\mathcal{R}} M_1$ and $M_1 \twoheadrightarrow_{\perp_\mathcal{U}} N$. By confluence of $w\perp_\mathcal{R}$-reduction, we have that $M_1 = \mathsf{nf}_\mathcal{R}(M)$. ◀

▶ **Theorem 7.6** (Confluence). *Let $\mathcal{U} \subseteq \mathcal{CL}^\infty$. If $\mathcal{U}$ is satisfies rootactiveness, closure under w-reduction, indiscernibility and weak overlap then $(\mathcal{CL}^\infty_\perp, \twoheadrightarrow_{w\perp_\mathcal{U}})$ is confluent.*

Proof. The proof is described in the following diagram.

Suppose we have a divergence $M_1 \twoheadleftarrow_{w\perp} M \twoheadrightarrow_{w\perp} M_2$. By rootactiveness for $\mathcal{U}$, we can reduce M_1 and M_2 further to their respective $w\perp_\mathcal{U}$-normal forms N_1 and N_2 by Theorem 3.15. (1) By closure under substitution for $\mathcal{U}$ and Theorem 3.14 we find L_1 and L_2 such that $M \twoheadrightarrow_w L_1 \twoheadrightarrow_{\perp_\mathcal{U}} N_1$ and $M \twoheadrightarrow_w L_2 \twoheadrightarrow_{\perp_\mathcal{U}} N_2$. (2) By Theorems 3.16 and 3.15, we construct the reductions $L_1 \twoheadrightarrow_{w\perp_\mathcal{U}} \mathsf{nf}_\mathcal{R}(M)$ and $L_2 \twoheadrightarrow_{w\perp_\mathcal{U}} \mathsf{nf}_\mathcal{R}(M)$. (3) By Lemma 7.5 we then find the reductions $\mathsf{nf}_\mathcal{R}(L_1) \twoheadrightarrow_{\perp_\mathcal{U}} N_1$ and $\mathsf{nf}_\mathcal{R}(L_2) \twoheadrightarrow_{\perp_\mathcal{U}} N_2$. By normalisation

and confluence of $(\mathcal{CL}^\infty, \twoheadrightarrow_{w\perp_\mathcal{R}})$, we have $\mathsf{nf}_\mathcal{R}(M) = \mathsf{nf}_\mathcal{R}(L_1) = \mathsf{nf}_\mathcal{R}(L_2)$. (4) Finally Theorem 3.13 on confluence of $\perp_\mathcal{U}$ and the fact that N_1 and N_2 are by construction normal forms for $\perp_\mathcal{U}$-reduction implies that N_1 and N_2 are identical.   ◀

▶ **Example 7.7.** Let $\mathcal{U} \subseteq \mathcal{CL}^\infty$. We construct the following sets:

$$\begin{aligned}
\mathcal{U}^* \quad &= \mathcal{U} \cup \{M \in \mathcal{CL}^\infty \mid M \twoheadrightarrow_w \mathbf{S(KW)I} \text{ and } W \in \mathcal{U}\} \\
\mathcal{U}^{**} &= \mathcal{U} \cup \{M \in \mathcal{CL}^\infty \mid M \twoheadrightarrow_w \mathbf{S(S(KK)W)}N, W \in \mathcal{U} \text{ and } N \in \mathcal{CL}^\infty\}
\end{aligned}$$

If $\mathcal{U}$ is a set of meaningless terms that contains $\mathcal{R}$ and it is closed under w-expansion, then $\mathcal{U}^*$ and $\mathcal{U}^{**}$ satisfy rootactiveness, indiscernibility, closure under w-reduction, w-expansion and weak overlap. They do not satisfy overlap and hence, they are not meaningless. Yet the corresponding infinitary combinatory logics are confluent and normalising by Theorems 7.6 and 3.15. There are uncountably many such sets.

The constructions $\mathcal{U}^*$ and $\mathcal{U}^{**}$ are related to the set $\mathcal{U}^\eta$ of weakly meaningless terms defined for infinitary lambda calculus in [16] where $\mathcal{U}^\eta = \mathcal{U} \cup \{M \in \Lambda^\infty \mid M \twoheadrightarrow_\beta \lambda x.Wx \text{ and } W \in \mathcal{U}\}$. It is easy to see that $(\mathbf{S(KW)I})_\lambda$ and $(\mathbf{S(S(KK)W)}N)_\lambda$ both β-reduce to $\lambda x.W_\lambda x$.

▶ Remark. Note that Theorem 7.6 could not be proved using the schema of the proof of Theorem 3.18 since commutation of $\perp$-reduction and w-reduction outside $\mathcal{U}$ may not hold. To see this, take $\mathcal{R}^{**}$ defined in Example 7.7 and $M = (\mathbf{S(KK)}W)$. Then, the only way to join $\perp P \leftarrow_{\perp_\mathcal{U}} \mathbf{S}MNP \rightarrow_w MP(NP)$ is by $w\perp$-reducing $MP(NP)$ to $\perp P$.

8   Related and Future Work

Sufficient and Necessary Condition for Confluence. In [16], we define a notion of weak meaningless set for infinitary lambda calculus and prove that this is a sufficient and necessary condition for confluence of $\beta\perp$-reduction . In the case of infinitary combinatory logic, it remains open to give a sufficient and necessary condition for confluence of $w\perp$-reduction. We think that hypercollapseness, closure under w-reduction, weak overlap and indiscernibility should be sufficient condition for confluence besides of being necessary (Theorem 7.4) but also sufficient. In other words, it remains open to prove Theorem 7.6 assuming only hypercollapseness instead of rootactiveness.

Sufficient and Necessary Condition for Normalisation. In [16] we show that normalisation implies rootactiveness for infinitary lambda calculus with $\beta\perp$-reduction. This does not hold in the setting of combinatory logic with $w\perp$-reduction as witness by the following example of normalising infinitary combinatory logic that does not satisfy rootactiveness. Let $\mathbf{D}[\ ] = \mathbf{S(K}[\ ])\mathbf{I}$. Note that $\mathbf{D}^\omega P$, the infinite nesting of such contexts applied to P is rootactive and reduces to itself for all P. Define:

$$\mathcal{D} = \{M \in \mathcal{R} \mid \forall P \in \mathcal{CL}^\infty \cdot M \not\twoheadrightarrow_w \mathbf{D}^\omega P\} \cup \{M \in \mathcal{CL}^\infty \mid M \twoheadrightarrow_w \mathbf{D}^\omega\}.$$

To construct the normal form of a term $M \in \mathcal{CL}_\perp^\infty$ for $w\perp_\mathcal{D}$-reduction we first $\rightarrow_w$-reduce M to its skeleton N. Next in the depth-first left-most order, we replace rootactive subterms that can not reduce to a term of the form $\mathbf{D}^\omega P$ by $\perp$; and we replace rootactive subterms that reduce to a term of the form $\mathbf{D}^\omega P$ by $\perp P$. Then we repeat the procedure *ad infinitum* on these latter terms P that still may contain untreated rootactive terms. The resulting reduction is strongly converging and its limit is a normal form in $\mathcal{CL}_{\perp_\mathcal{U}}^\infty$.

It remains open to find a sufficient and necessary condition for having a normalising infinitary combinatory logic $(\mathcal{CL}_\perp^\infty, \twoheadrightarrow_{w\perp_\mathcal{U}})$. This condition is necessarily weaker than rootactiveness.

Generalisation to Infinitary Term Rewriting and Combinatory Reduction Systems. A further next step following the explorations in lambda calculus and combinatory logic would be to investigate sufficient and necessary conditions for having confluence and normalisation in the wider context of orthogonal term rewriting systems and combinatory reduction systems.

Combinatory Algebras. Bethke, Klop and de Vrijer show that not every partial combinatory algebra can be completed [2]. We could define a partial combinatory algebra from the set of w-normal forms in $\mathcal{CL}^\infty$. The interpretation of a term M is N if $M \twoheadrightarrow_w N$ and N in w-normal form. The fact that this is a partial combinatory algebra follows from the normal form property [6]. This partial combinatory algebra is made complete by adding $\bot$. Since there are many sets of meaningless terms, we have many ways of completing it.

Selinger shows that the standard term algebra as a combinatory algebra cannot be ordered, i.e. every compatible partial order on it is trivial [13]. Lusin and Salibra show that there exists a wide class of combinatory algebras that admit extensions with a non-trivial compatible partial order [10]. Salibra shows that there is a continuum of unorderable lambda models [12]. In [14] we study orderability on the lambda models induced by the infinitary lambda calculus. It will also be interesting to study orderability on the combinatory algebras induced by the infinitary combinatory logic. This may shed new light on Plotkin's conjecture saying that an absolutely unorderable combinatory algebra exists, i.e. it cannot be embedded in any combinatory algebra admitting a non-trivial partial order [11].

Acknowledgements

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