Recursive Domain Equations of Filter Models

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Abstract. This paper characterizes the classes of filter models that correspond to different recursive domain equations. We will introduce four classes of type theories: *natural split*, *lazy split*, *natural equated* and *lazy equated*. Each class corresponds to a different recursive domain equation in the following sense: the type theory $\mathcal{T}$ belongs to the class iff the filter model $\mathcal{F}^{\mathcal{T}}$ satisfies the corresponding recursive domain equation. So far, in the literature, we can see only one direction: the description of D_∞ models as filter models. The introduction of the four classes of type theories allows us to turn this legendary result around and identify exactly which filter models can be described as colimits of ω-chains starting from a basic initial domain D_0 and a basic initial embedding i_0. For this result, we are extracting the essence of the specific proofs for the particular cases of intersection type theories and making one general construction that encompasses all of them. This general approach puts together all these results which may appear scattered and sometimes with incomplete proofs in the literature.

1 Introduction

It is well-known that Scott's D_∞-models are models of the lambda calculus that satisfy the recursive domain equation $D = D \to D$ [16]. Their construction depends on an initial domain D_0 and an initial embedding i_0. Variations of D_0 and i_0 define different D_∞-models. Some instances of D_0 and i_0 have given rise to some specific filter models such as $\mathcal{F}^{\mathrm{Scott}}$ and $\mathcal{F}^{\mathrm{Park}}$ [16, 14, 4]. Actually we know that any D_∞ model can be described as a filter model $\mathcal{F}^{\mathcal{T}}$ by coding the compact elements of D_0 as types and defining a subtyping relation $\mathcal{T}$ that contains both the order of D_0 and i_0 [4, 7, 8, 2]. The converse is obviously not true. Filter models are in a sense weaker structures than D_∞-models. Not all of them satisfy the recursive domain equation $D = D \to D$ for they can be non-extensional. If we restrict our attention to the extensional filter models, then some of them such as $\mathcal{F}^{\mathrm{CDZ}}$, $\mathcal{F}^{\mathrm{DHM}}$ and $\mathcal{F}^{\mathrm{HR}}$ have been described as D_∞ models by choosing an appropiate D_0 and i_0 [6, 7, 1, 9]. In spite of the fact that the non-extensional filter models do not satisfy $D = D \to D$, in some cases it is possible to find other recursive domain equations for them [1]. For instance, the non-extensional filter model $\mathcal{F}^{\mathrm{AO}}$ satisfies the equation $D = [D \to D]_\perp$ and $\mathcal{F}^{\mathrm{BCD}}$ satisfies the equation $D = B \times [D \to D]$ where B is the set of filters of types without arrows.

In this paper we want to characterize the classes of filter models that correspond to each recursive domain equation (see Fig. 1). This classification will be done by analyzing patterns of axioms in the definition of the subtyping relation inducing the filter model. In order to give a proper formalization for the study of these patterns of axioms, we will define the subtyping parametric on a set Σ of axioms. The judgements are of the form $\Sigma \vdash A \leq B$. This defines the *type theory* $\mathcal{T}^{\Sigma}$*, or just* $\mathcal{T}$*, generated by* Σ [7]. Depending on the shape of the axioms Σ, we will introduce four classes of type theories: *natural split*, *lazy split*, *natural equated* and *lazy equated* (Section 3). Each class corresponds to a different recursive

Class of type theory	Recursive domain equation
Natural split	$D = B \times [D \to D]$
Lazy split	$D = B \times [D \to D]_{\perp}$
Natural equated	$D = [D \to D]$
Lazy equated	$D = [D \to D]_{\perp}$

Fig. 1. Classification of recursive domain equations

domain equation in the following sense: the type theory $\mathcal{T}$ belongs to the class iff the filter model $\mathcal{F}^{\mathcal{T}}$ satisfies the corresponding recursive domain equation. The correspondance is more sutil than the one shown in Fig. 1. Our result will say in each case how to construct $\mathcal{F}^{\mathcal{T}}$ as a colimit of certain ω-chains (Section 4). The starting point of the ω-chains should be a basic domain $D_0^{\mathcal{T}}$ that does not mention the arrow type. This is only possible if the axioms Σ of $\mathcal{T}$ contain a partial order $\leq_{\mathcal{B}}$ on some basic set $\mathbb{B}$ of types with no arrows. In case $\mathcal{T}$ is natural equated, we can define $D_0 = \mathcal{F}^{\mathcal{B}}$ as the set of filters induced by $\leq_{\mathcal{B}}$. The rest of the axioms will be in shape of definitions which allows us to define a function $i_0 = \hat{\mathsf{def}}$ from $\mathcal{F}^{\mathcal{B}}$ to $[\mathcal{F}^{\mathcal{B}} \to \mathcal{F}^{\mathcal{B}}]$. Then, we will prove that the set $\mathcal{F}^{\mathcal{T}}$ can be constructed as the colimit of the ω-chain of the functor $\mathsf{Hom}(X) = [X \to X]$ starting from $D_0^{\mathcal{T}} = \mathcal{F}^{\mathcal{B}}$ and $i_0 = \hat{\mathsf{def}}$. We also prove the converse. Given $\rho = (\mathsf{Hom}, D_0, i_0)$ where D_0 and i_0 are arbitrary, we can construct a natural equated type theory $\mathcal{T}^{\rho}$ whose type atoms are the compact elements of D_0 and the axioms of $\mathcal{T}^{\rho}$ will contain the order of D_0 and the embedding i_0. Then, we will prove that $\mathcal{F}^{\mathcal{T}^{\rho}}$ is the colimit of the ω-chain of Hom starting from that given D_0 and i_0. Something similar we do for the other recursive domain equations and present the four proofs all together in Theorem 27 and Theorem 30. As a consequence of these two theorems, we will be able to justify the classification given above. The contribution of this paper can be summarized as follows.

1. So far, in the literature, we can see only one direction: the description of D_∞ models as filter models. The introduction of the four classes of type theories (natural split, lazy split, natural equated and lazy equated) allows us to turn

this legendary result around and identify exactly which filter models can be described as colimits of ω-chains starting from a basic initial domain D_0 and a basic initial embedding i_0.
2. In the proofs of Theorem 27 and Theorem 30, we are extracting the essence of the specific proofs for Scott, Park, CDZ, DHM and HR and making one general construction that encompasses all these particular cases. This general approach puts together all these results which may appear scattered and sometimes with incomplete proofs in the literature.

2 Domain-theoretic Preliminaries

This auxiliary section is devoted to fix notation and recall domain-theoretic definitions and results which concern lattices [12].

Definition 1. *1. Let $D = \langle D, \sqsubseteq\rangle$ be a poset. We say that D is a* complete lattice *if any subset $Z \subseteq D$ has a least upper bound $\bigsqcup Z$.*
2. *A subset $Z \subseteq D$ is* directed *if it is non-empty and for any $z, z' \in Z$ there exists $z'' \in Z$ such that $z, z' \sqsubseteq z''$.*
3. *A monotone function $f : D \to E$ is* continuous *if for any directed $Z \subseteq D$*

$$f(\bigsqcup Z) = \bigsqcup f(Z)$$

The space of continuous functions from D to E, ordered with the pointwise ordering, is denoted by $[D \to E]$.
4. *A function $f : D \to E$ is* additive *if for any $Z \subseteq D$*

$$f(\bigsqcup Z) = \bigsqcup f(Z)$$

5. *An element $d \in D$ is* compact *if for any directed $Z \subseteq D$, $d \sqsubseteq \bigsqcup Z$ implies that there exists $z \in Z$ such that $d \sqsubseteq z$. The set of compact elements of D is denoted by $\mathcal{K}(D)$.*
6. *D is an* ω-algebraic *lattice (alg) if $\mathcal{K}(D)$ is countable and moreover, for any $x \in D$,*

$$x = \bigsqcup\{d \in \mathcal{K}(D) \mid d \sqsubseteq x\}$$

ALG *is the category of ω-algebraic lattices and continuous functions.*
7. *Let D, E be ω-algebraic, and $f : \mathcal{K}(D) \to E$ be monotone. Then the* continuous extension *$\tilde{f} : D \to E$ is defined by*

$$\forall x \in D.\ \tilde{f}(x) = \bigsqcup\{f(d) \mid d \in \mathcal{K}(D)\ \&\ d \sqsubseteq x\}$$

8. *Given two ω-algebraic lattices D and E, and two compact elements $d \in \mathcal{K}(D)$, $e \in \mathcal{K}(E)$, we define the step function*

$$(d \Rightarrow e)(x) = \begin{cases} e & \textit{if } d \sqsubseteq x \\ \bot & \textit{otherwise} \end{cases}$$

We have that $(d \Rightarrow e) \sqsubseteq f$ if and only if $e \sqsubseteq f(d)$. Hence $d \Rightarrow e \sqsubseteq d' \Rightarrow e'$ if and only if $d' \sqsubseteq d$ and $e \sqsubseteq e'$. Finite sups of step functions are the compact elements in $[D \rightarrow E]$.

Lemma 2. *Let $D, E \in \mathbf{ALG}$. Then,*

1. $\mathcal{K}([D \rightarrow E]) = \{\bigsqcup_{i=1}^{n}(d_i \Rightarrow e_i) \mid n \in N \ \&\ d_i, e_i$ *compact* $\&\ 1 \leq i \leq n\}$.
2. $[D \rightarrow E]$ *is ω-algebraic.*
3. $\bigsqcup_{j=1}^{n}(c_j \Rightarrow d_j) \sqsubseteq \bigsqcup_{i=1}^{n}(a_i \Rightarrow b_i) \quad \Leftrightarrow$
   $\forall j \in \{1 \ldots n\}.d_j \sqsubseteq \bigsqcup\{b_i \mid 1 \leq i \leq n \ \&\ a_i \sqsubseteq c_j\}$.

Morever **ALG** *is a CCC (cartesian closed category) with "enough points".*

Definition 3. *Let $i : D \rightarrow E$, $j : E \rightarrow D$ be continuous functions. We say that $\iota = \langle i, j\rangle : D \rightarrow E$ is an* embedding-projection pair *(ep for short) if $j \circ i = Id_D$ and $i \circ j \sqsubseteq Id_E$.*
If $\langle i, j\rangle : D \rightarrow E$ and $\langle h, k\rangle : E \rightarrow E'$, then $\langle i, j\rangle \circ \langle h, k\rangle = \langle h \circ i, j \circ k\rangle$.
$\mathbf{ALG}^E$ *is the category of ω-algebraic lattices and ep's.*

Next lemma on ep's is very useful. Its proof can be recovered by using the results of Section 0-3 of [12] on basic properties of Galois connections.

Lemma 4. *Let $D, E \in \mathbf{ALG}$ and $\iota = \langle i, j\rangle : D \rightarrow E$ be an ep.*

1. $\forall x \in D, y \in E.\ i(x) \sqsubseteq y \ \Leftrightarrow x \sqsubseteq j(y)$.
   j is the right adjoint of i and it is often denoted by i^R.
2. *ι is completely determined by the embedding i, since j is forced to satisfy the following equality:*
$$(\dagger)\ j(y) = \bigsqcup\{x \mid i(x) \sqsubseteq y\}$$
3. *i is additive, injective and preserves compact elements.*

From now on we identify an ep $\iota = \langle i, i^R\rangle$ with its embedding i.

Definition 5. *Let $F : \mathbf{ALG}^E \rightarrow \mathbf{ALG}^E$ be a locally continuous endofunctor, $D_0 \in \mathbf{ALG}$ and $i_0 : D_0 \rightarrow F(D_0)$ an embedding in* **ALG**.

1. *The triple $\rho = \langle F, D_0, i_0\rangle$ is called a specification of a colimit.*
2. *An ω-chain of $\rho = (F, D_0, i_0)$ is*
$$D_0 \stackrel{i_0}{\rightarrow} F(D_0) \stackrel{F(i_0)}{\rightarrow} F^2(D_0) \stackrel{F^2(i_0)}{\rightarrow} F^3(D_0) \ldots$$
3. *Then, D_0 is called the* initial domain *and i_0 the* initial embedding.
4. *The colimit of ρ, denoted by colim(ρ), is the colimit of the ω-chain of ρ.*

3 Intersection type theories and Filter Models

In this section we recall the notion of intersection type theory. We will consider intersection type theories generated by a set Σ of axioms [7]. Depending on the shape of the axioms Σ, we introduce four classes of type theories: natural split, lazy split, natural equated and lazy equated. We show that many of the examples of intersection type theories that appear in the literature fit in one of these four patterns of axioms. In this section we will also show that any of these four classes are "well-behaved" in the sense that they all induce reflexive filter structures and, hence, λ-models.

Definition 6. *Let $\mathbb{A}$ be a countable set of symbols, called* atoms*.*

1. *We assume there is a special atom $\top$ in $\mathbb{A}$, called* top*.*
2. *The set $\mathbb{B}(\mathbb{A})$ (or just $\mathbb{B}$) of* basic types *over $\mathbb{A}$ is defined by $\mathbb{B} = \mathbb{A} \mid \mathbb{B} \cap \mathbb{B}$.*
3. *The set $\mathbb{T}(\mathbb{A})$ (or just $\mathbb{T}$) of* types *over $\mathbb{A}$ is defined by $\mathbb{T} = \mathbb{A} \mid \mathbb{T} \to \mathbb{T} \mid \mathbb{T} \cap \mathbb{T}$.*

Greek letters $\alpha, \beta, \ldots$ range over $\mathbb{A}$ and $A, B, \ldots$ range over $\mathbb{B}$ or $\mathbb{T}$. If $n \geq 1$, we may write $\bigcap_{i=1}^{n} A_i$ or $\bigcap\{A_1, \cdots, A_n\}$ for $A_1 \cap \cdots \cap A_n$. The case $\bigcap \emptyset$ denotes $\top$. In the literature, the top $\top$ is denoted by different symbols such as ω in [5] or Ω in [10].

Definition 7. *An* intersection type theory $\mathcal{T}$ *is a set of statements of the form $A \leq B$ (to be read: A is a subtype of B), with $A, B \in \mathbb{T}$, that satisfies the following axioms and rules.*

(refl) $A \leq A$	(trans) $\dfrac{A \leq B \quad B \leq C}{A \leq C}$	($\top$) $A \leq \top$
(incl$_L$) $A \cap B \leq A$		
(incl$_R$) $A \cap B \leq B$	(glb) $\dfrac{C \leq A \quad C \leq B}{C \leq A \cap B}$	($\top_{\text{lazy}}$) $A \to \top \leq \top \to \top$
($\to\cap$) $(A \to B) \cap (A \to C) \leq A \to B \cap C$		($\to$) $\dfrac{A' \leq A \quad B \leq B'}{A \to B \leq A' \to B'}$

Definition 8. *A* specification of axioms *is a pair $\Sigma = (<_{\mathcal{B}}, \mathsf{def})$ where $<_{\mathcal{B}}$ is a partial order on $\mathbb{B}$ such that $\cap$ is the meet and $\top$ is the top and def is a function from $\mathbb{A}$ to $\mathbb{T}$. The function def is extended from $\mathbb{A}$ to $\mathbb{T}$ by $\mathsf{def}(A \cap B) = \mathsf{def}(A) \cap \mathsf{def}(B)$ and $\mathsf{def}(A \to B) = A \to B$.*

Definition 9. *Let $\Sigma = (<_{\mathcal{B}}, \mathsf{def})$ be a specification of axioms. An* intersection type theory generated by Σ, *denoted by $\mathcal{T}^{\Sigma}$ or just $\mathcal{T}$, derives judgements of the form $A \leq B \in \mathcal{T}^{\Sigma}$ or $\Sigma \vdash A \leq B$ and it is defined as the smallest intersection type theory that contains the following two axioms.*

$$(\mathcal{B}\text{-ax})\ \frac{A \leq_{\mathcal{B}} B}{\Sigma \vdash A \leq B} \qquad (\mathsf{def}\text{-ax})\ \Sigma \vdash \alpha = \mathsf{def}(\alpha)$$

We write $\Sigma \vdash A \leq B$ if $A \leq B \in \mathcal{T}^{\Sigma}$. If there is little danger of confusion, when Σ or $\mathcal{T}$ are clear from the context, then we will just write $(A \leq B)$. We also write $A = B$ for $A \leq B \in \mathcal{T}$ and $B \leq A \in \mathcal{T}$. We extend $\leq$, $\cap$ and $\rightarrow$ to $\mathbb{T}/=$ in the obvious way. Syntactic identitiy is denoted as $A \equiv B$.
Figure 2 shows how many of the intersection type theories that appear in the literature [11] can be generated by a specification $\Sigma = (\leq_{\mathcal{B}}, \mathsf{def})$. For the $\mathcal{B}$-axioms, we do not specify the whole set $\leq_{\mathcal{B}}$. Each $\leq_{\mathcal{B}}$ is actually defined as the least partial order that contains the pairs $A \leq_{\mathcal{B}} B$ shown in the table. We also define $\mathbb{A}^{\infty} = \{\mathsf{c}_i \mid i \in N\} \cup \{\top\}$.

$\mathcal{T}$	$\mathbb{A}$	$\mathcal{B}$-axioms $\leq_{\mathcal{B}}$	def-axioms $\alpha = \mathsf{def}(\alpha)$
Scott	$\{\mathsf{0}, \top\}$	$\mathsf{0} \leq_{\mathcal{B}} \top$	$\mathsf{0} = \top \rightarrow \mathsf{0}, \top = \top \rightarrow \top$
Park	$\{\mathsf{0}, \top\}$	$\mathsf{0} \leq_{\mathcal{B}} \top$	$\mathsf{0} = \mathsf{0} \rightarrow \mathsf{0}, \top = \top \rightarrow \top$
CDZ	$\{\mathsf{0}, \mathsf{1}, \top\}$	$\mathsf{0} \leq_{\mathcal{B}} \mathsf{1} \leq_{\mathcal{B}} \top$	$\mathsf{0} = \mathsf{1} \rightarrow \mathsf{0}, \mathsf{1} = \mathsf{0} \rightarrow \mathsf{1}, \top = \top \rightarrow \top$
HR	$\{\mathsf{0}, \mathsf{1}, \top\}$	$\mathsf{0} \leq_{\mathcal{B}} \mathsf{1} \leq_{\mathcal{B}} \top$	$\mathsf{0} = \mathsf{1} \rightarrow \mathsf{0}, \mathsf{1} = (\mathsf{0} \rightarrow \mathsf{0}) \cap (\mathsf{1} \rightarrow \mathsf{1}), \top = \top \rightarrow \top$
DHM	$\{\mathsf{0}, \mathsf{1}, \top\}$	$\mathsf{0} \leq_{\mathcal{B}} \mathsf{1} \leq_{\mathcal{B}} \top$	$\mathsf{0} = \top \rightarrow \mathsf{0}, \mathsf{1} = \mathsf{0} \rightarrow \mathsf{1}, \top = \top \rightarrow \top$
BCD	$\mathbb{A}^{\infty}$	$\mathsf{c}_i \leq_{\mathcal{B}} \top$	$\mathsf{c}_i = \mathsf{c}_i, \top = \top \rightarrow \top$
AO	$\{\top\}$		$\top = \top$

Fig. 2. Intersection type theories generated by $\Sigma = (\leq_{\mathcal{B}}, \mathsf{def})$

Definition 10. *Let $\Sigma = (<_{\mathcal{B}}, \mathsf{def})$ be a specification of axioms.*

1. *We say that Σ (and also $\mathcal{T}^{\Sigma}$) is* lazy *if $\mathsf{def}(\top) = \top$.*
2. *We say that Σ (and also $\mathcal{T}^{\Sigma}$) is* natural *if $\mathsf{def}(\top) = \top \rightarrow \top$.*
3. *We say that Σ (and also $\mathcal{T}^{\Sigma}$) is* split *if the following two conditions hold.*
   (a) $\mathbb{A} - \{\top\} \neq \emptyset$.
   (b) $\mathsf{def}(\alpha) = \alpha, \ \forall \ \alpha \in \mathbb{A} - \{\top\}$.
4. *We say that Σ (and also $\mathcal{T}^{\Sigma}$) is* equated *if the following conditions hold.*
   (a) $\forall \alpha \in \mathbb{A} - \{\top\}, \exists A_1, \ldots A_n \in \mathbb{B}, B_1, \ldots B_n \in \mathbb{B} - \{\top\}$,
   $$\mathsf{def}(\alpha) \equiv \bigcap_{i=1}^{n} (A_i \rightarrow B_i);$$
   (b) $\forall A, B \in \mathbb{B} - \{\top\}$,
   $$A \leq_{\mathcal{B}} B \Leftrightarrow \forall j \in \{1 \ldots n\}. \bigcap \{B_i \mid 1 \leq i \leq m \ \& \ A_i \geq_{\mathcal{B}} C_j\} \leq_{\mathcal{B}} D_j$$
   where $\mathsf{def}(A) \equiv \bigcap_{i=1}^{m} (A_i \rightarrow B_i)$ and $\mathsf{def}(B) \equiv \bigcap_{j=1}^{n} (C_j \rightarrow D_j)$.
5. *We say that Σ (and also $\mathcal{T}^{\Sigma}$) is* disciplined *if it is either one of these four possible combinations: natural split, lazy split, natural equated or lazy equated.*

The examples of intersection type theories given in Figure 2 are classified according to the four classes defined above as follows.

Natural split	BCD
Lazy split	none
Natural equated	Scott, Park, CDZ, HR, DHM
Lazy equated	AO

Lemma 11. *In any intersection type theory $\mathcal{T}$, the following statements hold:*

1. *If $A \leq C$ and $B \leq D$ then $A \cap B \leq C \cap D$.*
2. $(A_1 \to B_1) \cap \ldots \cap (A_n \to B_n) \leq (A_1 \cap \ldots \cap A_n) \to (B_1 \cap \ldots \cap B_n)$.
3. *If $\forall j \in \{1 \ldots n\}. \bigcap\{B_i \mid 1 \leq i \leq m \ \&\ A_i \geq C_j\} \leq D_j$ then*

$$\bigcap_{i=1}^{m}(A_i \to B_i) \leq \bigcap_{j=1}^{n}(C_j \to D_j).$$

Definition 12. *We define two functions to extract the sets of outermost atoms and arrows of a type.*

$$\begin{array}{ll}
\mathsf{ats}(\alpha) = \{\alpha\} & \mathsf{ars}(\alpha) = \emptyset \\
\mathsf{ats}(A \to B) = \emptyset & \mathsf{ars}(A \to B) = \{A \to B\} \\
\mathsf{ats}(A \cap B) = \mathsf{ats}(A) \cup \mathsf{ats}(B) & \mathsf{ars}(A \cap B) = \mathsf{ars}(A) \cup \mathsf{ars}(B)
\end{array}$$

Theorem 13. (Conservativity of $\leq_{\mathcal{B}}$). *Let $A, B \in \mathbb{B}$ and Σ be disciplined. Then, $A \leq_{\mathcal{B}} B$ iff $\Sigma \vdash A \leq B$.*

For the proof of the above theorem, see the appendix. Since $\leq_{\mathcal{B}}$ is antisymmetric, we have the following:

Corollary 14. *If $\Sigma \vdash \alpha = \beta$ then $\alpha \equiv \beta$.*

The converse of Lemma 11 part 3 is an important property which is not always true. We will see later that it is a sufficient and necesary condition for having a reflexive filter structure. For this, we will define the following notion.

Definition 15. *We say that $\mathcal{T}$ is β-sound if*

$$\textstyle\bigcap_{i=1}^{m}(A_i \to B_i) \leq (C \to D) \Rightarrow \bigcap\{B_i \mid 1 \leq i \leq m \ \&\ A_i \geq C\} \leq D$$

$\forall A_i, B_i, C, D \in \mathbb{T}$ with $1 \leq i \leq m$.

A particular case of β-soundness is the invertibility of the rule $(\to)$, i.e. if $A \to B \leq C \to D$ then $A \geq C$ and $B \leq D$.

Theorem 16. *Let $\mathcal{T}$ be disciplined. If $E \leq F$ then*

1. $\bigcap \mathsf{ats}(\mathsf{def}(E)) \leq \bigcap \mathsf{ats}(\mathsf{def}(F))$
2. *$\forall j \in \{1 \ldots n\}. \bigcap\{B_i \mid 1 \leq i \leq m \ \&\ A_i \geq C_j\} \leq D_j$ where*
   $\mathsf{ars}(\mathsf{def}(E)) \equiv \{(A_i \to B_i) \mid 1 \leq i \leq m\}$ and
   $\mathsf{ars}(\mathsf{def}(F)) \equiv \{(C_j \to D_j) \mid 1 \leq i \leq n\}$.

Proof. We prove it by induction on the derivation of $E \leq F$. The interesting case is ($\mathcal{B}$-ax) which follows from Condition 4b ($\Rightarrow$) in Definition 10. $\square$

Corollary 17. *Let $\mathcal{T}$ be disciplined. Then $\mathcal{T}$ is β-sound.*

Definition 18. *Let $\mathcal{T}$ be an intersection type theory.*

1. *A* filter *(resp.* basic filter*) is a set $X \subseteq \mathbb{T}$ (resp. $X \subseteq \mathbb{B}$) such that:*
   (a) $\top \in X$;
   (b) $A \leq B$(resp.$A \leq_{\mathcal{B}} B$) and $A \in X$ imply $B \in X$;
   (c) $A \in X$ and $B \in X$ imply $A \cap B \in X$.
2. *Let $Y \subseteq \mathbb{T}$. Then, $\uparrow Y$ denotes the filter generated by Y. If $Y = \{A\}$, we write $\uparrow A$ instead of $\uparrow\{A\}$ and $\uparrow A$ is called a* principal filter*. Actually it coincides with the upper closure of A, i.e. $\uparrow A = \{B \mid A \leq B\}$. If $Y \subseteq \mathbb{B}$ the basic filter generated by Y is denoted by $\uparrow^{\mathcal{B}} Y$. Similarly, we write $\uparrow^{\mathcal{B}} A$ instead of $\uparrow^{\mathcal{B}} \{A\}$ for the principal filter.*
3. *$\mathcal{F}^{\mathcal{T}}$ (resp. $\mathcal{F}^{\mathcal{B}}$) is the set of filters (resp. basic filters) over $\mathcal{T}$, ordered by set-theoretic inclusion, and is called the* filter structure *(resp.* basic filter structure*) over $\mathcal{T}$.*

It is well-known that $\mathcal{F}^{\mathcal{T}}$ is an ω-algebraic lattice [7]. Given $\mathcal{X} \subseteq \mathcal{F}^{\mathcal{T}}$

$$\text{(fil-sup)} \bigsqcup \mathcal{X} = \uparrow \{\textstyle\bigcap_{j=1}^{n} A_j \mid n \in I\!N, \forall 1 \leq j \leq n. \exists X \in \mathcal{X}. A_j \in X\},$$

Moreover $X \sqcap Y = X \cap Y$, the bottom filter is $\uparrow\top$, the top filter $\mathbb{T}$. Compact elements in $\mathcal{F}^{\mathcal{T}}$ are the principal filters and they inherit the order $\leq^{op}$:

$$\uparrow A \subseteq \uparrow B \Leftrightarrow B \leq A$$

Definition 19. *Let $\mathcal{T}$ be an intersection type theory.*

1. *We define $\mathsf{App}^{\mathcal{T}} : \mathcal{F}^{\mathcal{T}} \to [\mathcal{F}^{\mathcal{T}} \to \mathcal{F}^{\mathcal{T}}]$ and $\mathsf{Abs}^{\mathcal{T}} : [\mathcal{F}^{\mathcal{T}} \to \mathcal{F}^{\mathcal{T}}] \to \mathcal{F}^{\mathcal{T}}$ as follows.*
$$\begin{array}{ll} \mathsf{App}^{\mathcal{T}}(X)(Y) & = \{B \mid \exists A \in Y. A \to B \in X\} \\ \mathsf{Abs}^{\mathcal{T}}(f) & = \uparrow\{A \to B \mid B \in f(\uparrow A)\} \end{array}$$
2. *We define an interpretation on λ-terms as follows [7] [1].*
$$\begin{array}{ll} [\![x]\!]_\rho & = \rho(x) \\ [\![MN]\!]_\rho & = (\mathsf{App}^{\mathcal{T}}([\![M]\!]_\rho))([\![N]\!]_\rho) \\ [\![\lambda x.M]\!]_\rho & = \mathsf{Abs}^{\mathcal{T}}(\boldsymbol{\lambda} d \in \mathcal{F}^{\mathcal{T}}.[\![M]\!]_{\rho[x/d]}) \end{array}$$

Not any filter structure $\mathcal{F}^{\mathcal{T}}$ gives rise to a lambda model $\langle \mathcal{F}^{\mathcal{T}}, \mathsf{App}^{\mathcal{T}}, \mathsf{Abs}^{\mathcal{T}}, [\![\_]\!] \rangle$.

Theorem 20. *If $\mathcal{T}$ is disciplined then $\langle \mathcal{F}^{\mathcal{T}}, \mathsf{App}^{\mathcal{T}}, \mathsf{Abs}^{\mathcal{T}}, [\![\_]\!] \rangle$ is a λ-model.*

Proof. Since $\mathcal{T}$ is β-sound, it follows from [7, 15] that $\langle \mathcal{F}^{\mathcal{T}}, \mathsf{App}^{\mathcal{T}}, \mathsf{Abs}^{\mathcal{T}}, [\![\_]\!] \rangle$ is a reflexive filter structure, i.e. $\mathsf{App}^{\mathcal{T}} \circ \mathsf{Abs}^{\mathcal{T}} = Id_{[\mathcal{F}^{\mathcal{T}} \to \mathcal{F}^{\mathcal{T}}]}$. Since any reflexive filter structure is a λ-model [13, 3], we conclude that $\mathcal{F}^{\mathcal{T}}$ is also a λ-model. $\square$

[1] It coincides with the interpretation defined through the *type assignment system* [7].

4 Classification of Recursive Domain Equations

This section shows that filter structures over disciplined type theories correspond to the colimit of certain ω-chains. Definition 24 gives the correspondance between the intersection type theories $\mathcal{T}$ and the specification $\rho^{(\mathcal{T})} = \langle F^{(\mathcal{T})}, D_0^{(\mathcal{T})}, i_0^{(\mathcal{T})} \rangle$ of a colimit. In Theorem 27, we prove that $\mathcal{F}^{\mathcal{T}}$ is isomorphic to the colimit of $(F^{(\mathcal{T})}, D_0^{(\mathcal{T})}, i_0^{(\mathcal{T})})$. In Theorem 30 we prove the converse: for any triple (F, D_0, i_0) of certain class $\mathcal{C}$, it is possible to construct a displined $\mathcal{T}$ such that $\mathcal{F}^{\mathcal{T}}$ is isomorphic to the colimit of (F, D_0, i_0). As a consequence of these two theorems, we can justify the correspondance between classes and recursive domain equations shown in Figure 1.

Definition 21. *We define four different functors in* $\mathbf{ALG}^E$.

$$\begin{aligned}
\mathsf{Hom}^B(X) &= B \times [X \to X] \\
\mathsf{Hom}^B_\perp(X) &= B \times [X \to X]_\perp \\
\mathsf{Hom}(X) &= [X \to X] \\
\mathsf{Hom}_\perp(X) &= [X \to X]_\perp
\end{aligned}$$

On morphisms, we define $\mathsf{Hom}(i)(f) = i \circ f \circ j$ *for an embedding projection pair* $\langle i, j \rangle$ *and* $f : X \to X$ *a continous function.*

Definition 22. *Let* $D_0 = \{\perp\}$. *The* trivial embedding $\mathsf{trv} : D_0 \to F(D_0)$ *is defined as* $\mathsf{trv}(\perp) = \perp$.

Definition 23. *Let* Σ *be equated. We define the embedding* $\hat{\mathsf{def}} : \mathcal{F}^{\mathcal{B}} \to [\mathcal{F}^{\mathcal{B}} \to \mathcal{F}^{\mathcal{B}}]$ *on compact elements as follows.*

1. *If* $\mathsf{def}(A) \equiv \bigcap_{i=1}^m (A_i \to B_i)$ *then*
$$\hat{\mathsf{def}}(\uparrow^{\mathcal{B}} A) = \begin{cases} \bigsqcup_{i=1}^m (\uparrow^{\mathcal{B}} A_i \Rightarrow \uparrow^{\mathcal{B}} B_i) & \text{if } \Sigma \text{ is natural} \\ (\bigsqcup_{i=1}^m (\uparrow^{\mathcal{B}} A_i \Rightarrow \uparrow^{\mathcal{B}} B_i), 0) & \text{if } \Sigma \text{ is lazy} \end{cases}$$
2. *If* $\mathsf{def}(\top) = \top$ *then* $\hat{\mathsf{def}}(\uparrow^{\mathcal{B}} \top) = \perp$.

Definition 24. *Let* $\mathcal{T}$ *be disciplined. We define a specification of a colimit* $\rho^{\mathcal{T}} = (F^{(\mathcal{T})}, D_0^{(\mathcal{T})}, i_0^{(\mathcal{T})})$ *according to the following four cases.*

$\mathcal{T}$	$F^{(\mathcal{T})}$	$D_0^{(\mathcal{T})}$	$i_0^{(\mathcal{T})}$
Natural split	$\mathsf{Hom}^{\mathcal{F}^{\mathcal{B}}}$	$\{\perp\}$	trv
Lazy split	$\mathsf{Hom}_\perp^{\mathcal{F}^{\mathcal{B}}}$	$\{\perp\}$	trv
Natural equated	Hom	$\mathcal{F}^{\mathcal{B}}$	$\hat{\mathsf{def}}$
Lazy equated	$\mathsf{Hom}_\perp$	$\mathcal{F}^{\mathcal{B}}$	$\hat{\mathsf{def}}$

Finally, for any $n > 0$, *define inductively* $D_n^{(\mathcal{T})} = F^{(\mathcal{T})}(D_{n-1})$.

Definition 25. *Let $\mathcal{T}$ be disciplined. We define $\mu_n^{(\mathcal{T})} : \mathcal{K}(D_n^{(\mathcal{T})}) \to \mathbb{T}/=$ by induction as follows.*

1. – *if $\mathcal{T}$ is split, then $\mu_0^{(\mathcal{T})}(\bot) = [\top]$*
   – *if $\mathcal{T}$ is equated, then $\mu_0^{(\mathcal{T})}(\uparrow^{\mathcal{B}} A) = [A]$*
2. *Let $n > 0$. Then*
   – *if $\mathcal{T}$ is natural split, then*
   $\mu_n^{(\mathcal{T})}(\uparrow^{\mathcal{B}} A, \bigsqcup_{i=1}^m (d_i \Rightarrow e_i)) = [A] \cap (\bigcap_{i=1}^m (\mu_{n-1}^{(\mathcal{T})}(d_i) \to \mu_{n-1}^{(\mathcal{T})}(e_i))$
   – *if $\mathcal{T}$ is lazy split, then*
   $\mu_n^{(\mathcal{T})}(\uparrow^{\mathcal{B}} A, \bot) = [A]$
   $\mu_n^{(\mathcal{T})}(\uparrow^{\mathcal{B}} A, (\bigsqcup_{i=1}^m (d_i \Rightarrow e_i), 0)) = [A] \cap (\bigcap_{i=1}^m (\mu_{n-1}^{(\mathcal{T})}(d_i) \to \mu_{n-1}^{(\mathcal{T})}(e_i))$
   – *if $\mathcal{T}$ is natural equated, then*
   $\mu_n^{(\mathcal{T})}(\bigsqcup_{i=1}^m (d_i \Rightarrow e_i)) = \bigcap_{i=1}^m (\mu_{n-1}^{(\mathcal{T})}(d_i) \to \mu_{n-1}^{(\mathcal{T})}(e_i))$
   – *if $\mathcal{T}$ is lazy equated, then*
   $\mu_n^{(\mathcal{T})}(\bot) = [\top]$
   $\mu_n^{(\mathcal{T})}(\bigsqcup_{i=1}^m (d_i \Rightarrow e_i), 0) = \bigcap_{i=1}^m (\mu_{n-1}^{(\mathcal{T})}(d_i) \to \mu_{n-1}^{(\mathcal{T})}(e_i))$.

From now on we omit the superscript $(\mathcal{T})$ on μ_n.

Proposition 26. *Let $\mathcal{T}$ be disciplined. Then for all n,*

1. *$\forall e, e' \in \mathcal{K}(D_n^{(\mathcal{T})})$. $e' \sqsubseteq e \;\Leftrightarrow \mu_n(e) \leq \mu_n(e')$.*
2. *$\forall d \in \mathcal{K}(D_n^{(\mathcal{T})}).\mu_n(d) = \mu_{n+1}(i_n(d))$ where $i_n = F^{(\mathcal{T})^{(n)}}(i_0)$.*
3. *$\forall A \in \mathbb{T}$. $\exists n \geq 0, a \in \mathcal{K}(D_n^{(\mathcal{T})})$. $\mu_n(a) = [A]$.*

For the proof of the above proposition, see the appendix.

Theorem 27. *Let $\mathcal{T}$ be disciplined. Then*

$$\mathcal{F}^{\mathcal{T}} \simeq colim(\rho^{(\mathcal{T})})$$

Proof. First define $\bar{\mu}_n : \mathcal{K}(D_n^{(\mathcal{T})}) \to \mathcal{F}^{\mathcal{T}}$, for any $n \geq 0$ and $d \in \mathcal{K}(D_n^{(\mathcal{T})})$ by

$$\bar{\mu}_n(d) = \uparrow D \text{ where } \mu_n(d) = [D]$$

$\bar{\mu}_n$ are monotone by Proposition 26(1). Then the extensions $\tilde{\mu}_n : D_n^{(\mathcal{T})} \to \mathcal{F}^{\mathcal{T}}$ of $\bar{\mu}$ defined by

$$\tilde{\mu}_n(x) = \bigsqcup\{\bar{\mu}_n(d) \mid d \in \mathcal{K}(D_n^{(\mathcal{T})}) \;\&\; d \sqsubseteq x\}$$

are continuous. It follows from Proposition 26(2) that for any $n \geq 0$, $x \in D_n^{(\mathcal{T})}$, $\tilde{\mu}_n(x) = \tilde{\mu}_{n+1}(i_n(x))$. Hence, $\mathcal{F}^{\mathcal{T}}$ together with $\tilde{\mu}_n$ is a cocone for the ω-chain of $(F^{(\mathcal{T})}, D_0^{(\mathcal{T})}, i_0^{(\mathcal{T})})$. To prove that it is initial, consider another cocone, a domain E with $\sigma_n : D_n^{(\mathcal{T})} \to E$ such that $\sigma_{n+1} \circ i_n = \sigma_n$. We define $\theta : \mathcal{F}^{\mathcal{T}} \to E$ on compact elements by $\theta(\uparrow A) = \sigma_n(a)$ where a is an element of some $\mathcal{K}(D_n^{(\mathcal{T})})$ such that $\mu_n(a) = [A]$. This element exists by Proposition 26(3). We prove that θ is monotone. Suppose $\uparrow A \subseteq \uparrow B$, i.e. $B \leq A$. By Proposition 26(2)(3) there exist

$n \geq 0$, $a \in \mathcal{K}(D_n^{(\mathcal{T})})$ and $b \in \mathcal{K}(D_n^{(\mathcal{T})})$ such that $\mu_n(a) = [A]$ and $\mu_n(b) = [B]$. It follows from Proposition 26(1) that $a \sqsubseteq b$. Hence $\sigma_n(a) \sqsubseteq \sigma_n(b)$, since σ_n are monotone. By definition of θ, $\theta(\uparrow A) \sqsubseteq \theta(\uparrow B)$. We have defined θ to have $\theta(\bar{\mu}_n(d)) = \sigma_n(d)$. The continuous extension $\tilde{\theta} : \mathcal{F}^{\mathcal{T}} \to E$ is the unique mediating morphisms such that $\forall n.\tilde{\theta} \circ \tilde{\mu}_n = \sigma_n$. □

Corollary 28.

1. *If $\mathcal{T}$ is natural split then $\mathcal{F}^{\mathcal{T}} \models D = B \times [D \to D]$.*
2. *If $\mathcal{T}$ is lazy split then $\mathcal{F}^{\mathcal{T}} \models D = B \times [D \to D]_\perp$.*
3. *If $\mathcal{T}$ is natural equated then $\mathcal{F}^{\mathcal{T}} \models D = [D \to D]$.*
4. *If $\mathcal{T}$ is lazy equated then $\mathcal{F}^{\mathcal{T}} \models D = [D \to D]_\perp$.*

As expected, we can go in the other direction, proving that a certain kind of colimits could be recovered as filter structures of disciplined intersection type theories.

Definition 29. *We define the sets of triples $\mathcal{C}_i$ ($i \in \{1, 2, 3, 4\}$) as follows:*

$$\begin{aligned}
\mathcal{C}_1 &= \{\langle \mathsf{Hom}^B, \{\perp\}, tr\rangle \mid B \in \mathbf{ALG}\} \\
\mathcal{C}_2 &= \{\langle \mathsf{Hom}_\perp^B, \{\perp\}, tr\rangle \mid B \in \mathbf{ALG}\} \\
\mathcal{C}_3 &= \{\langle \mathsf{Hom}, B, i_0\rangle \mid B \in \mathbf{ALG} \ \& \ i_0 : B \to [B \to B]\} \\
\mathcal{C}_4 &= \{\langle \mathsf{Hom}_\perp, B, i_0\rangle \mid B \in \mathbf{ALG} \ \& \ i_0 : B \to [B \to B]_\perp\}
\end{aligned}$$

We define $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$.

Theorem 30. *Let $\rho = (F, D_0, i_0) \in \mathcal{C}$. Then there exists a disciplined intersection type theory $\mathcal{T}$ such that*

$$colim(\rho) \simeq \mathcal{F}^{\mathcal{T}}$$

Proof. For each $\rho = (F, D_0, i_0) \in \mathcal{C}$, we first define a set $\mathbb{A}^\rho$ of atoms.

$$\mathbb{A}^\rho = \{\underline{d} \mid d \in \mathcal{K}(B) \ \& \ d \neq \perp\} \cup \{\top\}$$

Then, we define a preorder $\leq_\mathcal{B}$ on $\mathbb{B}(\mathbb{A}^\rho)$ as follows.

$$\begin{gathered}
\underline{d} \leq_\mathcal{B} \underline{e} \Leftrightarrow e \sqsubseteq d \\
\underline{d} =_\mathcal{B} \bigcap_{i=1}^m \underline{e_i} \Leftrightarrow d = \bigsqcup_{i=1}^m e_i \\
\underline{d} \leq \top
\end{gathered}$$

For the cases where the embeddings i_0 are non-trivial, we define def as follows.

$$\mathsf{def}(\underline{d}) = \bigcap_{i=1}^m (\underline{a_i} \to \underline{b_i}) \Leftrightarrow i_0(d) = \bigsqcup_{i=1}^m (a_i \Rightarrow b_i)$$

It is routine to check that the resulting intersection type theory $\mathcal{T}^\rho$ is disciplined in all four cases. Since $B \simeq \mathcal{F}^\mathcal{B}$, by Theorem 27 we have that $\mathcal{F}^{\mathcal{T}^\rho} \simeq colim(\rho^{\mathcal{T}^\rho}) \simeq colim(\rho)$. □

We now prove the converse of Corollary 28. These two corollaries together justify the classification shown in Figure 1.

Corollary 31.

1. *If* $\mathcal{F}^{\mathcal{T}} \models D = B \times [D \to D]$ *then* $\exists\mathcal{T}$*' natural split such that* $\mathcal{T}/= \sim\ \mathcal{T}'/=$.
2. *If* $\mathcal{F}^{\mathcal{T}} \models D = B \times [D \to D]_\perp$ *then* $\exists\mathcal{T}$*' lazy split such that* $\mathcal{T}/= \sim\ \mathcal{T}'/=$.
3. *If* $\mathcal{F}^{\mathcal{T}} \models D = [D \to D]$ *then* $\exists\mathcal{T}$*' natural equated such that* $\mathcal{T}/= \sim\ \mathcal{T}'/=$.
4. *If* $\mathcal{F}^{\mathcal{T}} \models D = [D \to D]_\perp$ *then* $\exists\mathcal{T}$*' natural equated such that* $\mathcal{T}/= \sim\ \mathcal{T}'/=$.

Proof. We prove only the third case. The rest are similar. Take $\rho = (\mathsf{Hom}, \mathcal{F}^{\mathcal{T}}, i_0)$ where i_0 is the isomorphism from D to $[D \to D]$. By previous theorem, we have that $\mathcal{F}^{\mathcal{T}} \sim colim(\rho) \sim \mathcal{F}^{\mathcal{T}'}$ for some $\mathcal{T}'$ natural equated. Since $\mathcal{K}(\mathcal{F}^{\mathcal{T}}) \sim \mathcal{T}/=$ and $\mathcal{K}(\mathcal{F}^{\mathcal{T}'}) \sim \mathcal{T}'/=$, we conclude that $\mathcal{T}/= \sim \mathcal{T}'/=$. □

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5 Appendix

In this appendix we show the proofs ofTheorem 13 and Proposition 26.

Proof of Theorem 13.
(⇒). This is trivial. (⇐). Let Σ be disciplined. We define a minimal logic equivalent to the intersection type theory generated by Σ. The minimal logic derives judgements of the form $\Sigma \vdash_{\mathrm{ML}} A \leq B$ by the following axioms and rules.

$$(\mathcal{B}\text{-ax}) \quad \frac{A \leq_{\mathcal{B}} B}{\Sigma \vdash_{\mathrm{ML}} A \leq B}$$

$$(\top) \quad A \leq \top$$

$$(\text{unfd-}\mathsf{def}) \quad \frac{\Sigma \vdash_{\mathrm{ML}} \mathsf{def}(A) \leq \mathsf{def}(B)}{\Sigma \vdash_{\mathrm{ML}} A \leq B}$$

$$(\text{mult}\rightarrow) \quad \frac{\begin{array}{l}\forall j \in \{1 \ldots n\},\\ \Sigma \vdash_{\mathrm{ML}} \bigcap\{B_i \mid 1 \leq i \leq m \ \&\ \Sigma \vdash_{\mathrm{ML}} C_j \leq A_i\} \leq D_j\end{array}}{\Sigma \vdash_{\mathrm{ML}} \bigcap_{i=1}^{m}(A_i \to B_i) \leq \bigcap_{j=1}^{n}(C_j \to D_j)} m, n \geq 1$$

$$(\mathsf{ats} \cap \mathsf{ars}) \quad \frac{\begin{array}{l}\Sigma \vdash_{\mathrm{ML}} \bigcap \mathsf{ats}(A) \leq \bigcap \mathsf{ats}(B)\\ \Sigma \vdash_{\mathrm{ML}} \bigcap \mathsf{ars}(A) \leq \bigcap \mathsf{ars}(B)\end{array}}{\Sigma \vdash_{\mathrm{ML}} A \leq B}$$

We now prove that the minimal logic is equivalent to the intersection type theory generated by Σ. We first prove that $\Sigma \vdash_{\mathrm{ML}} A \leq B$ implies $\Sigma \vdash A \leq B$. Rule (mult→) follows from Lemma 11 part 3. Rule (unfd-def) follows from $\Sigma \vdash A = \mathsf{def}(A)$. The last one follows from $\Sigma \vdash A = \bigcap \mathsf{ats}(A) \cap \bigcap \mathsf{ars}(A)$.
The advantage of the minimal logic is that it does not contain (trans) and we can now reason by induction on the derivation in order to prove that $\Sigma \vdash_{\mathrm{ML}} A \leq B$ implies $A \leq_{\mathcal{B}} B$ for all $A, B \in \mathbb{B}$. The case worth noticing in this induction is the case of the last rule that needs Condition 4b (⇐) of Definition 10. □
Proof of Proposition 26. 1. This is proved by induction on n. Base case $n = 0$. If $\mathcal{T}$ is split, it is trivial. If $\mathcal{T}$ equated, we have that $d =\uparrow^{\mathcal{B}} A$ and $d' =\uparrow^{\mathcal{B}} B$ for some $A, B \in \mathbb{B}$. Then, $d' \sqsubseteq d \Leftrightarrow \uparrow^{\mathcal{B}} B \subseteq \uparrow^{\mathcal{B}} A \Leftrightarrow A \leq B \Leftrightarrow \mu_n(\uparrow^{\mathcal{B}} A) \leq \mu_n(\uparrow^{\mathcal{B}} B)$.

Inductive case $n > 0$. We only prove the cases of natural split and lazy equated. Assume first $\mathcal{T}$ is natural split. Let $e = (\uparrow^{\mathcal{B}} A, \bigsqcup_{i=1}^{m} a_i \Rightarrow b_i)$ and $e' = (\uparrow^{\mathcal{B}} B, \bigsqcup_{j=1}^{n} c_j \Rightarrow d_j)$.

$$\begin{aligned} e' \sqsubseteq e \Leftrightarrow\ & \uparrow^{\mathcal{B}} B \subseteq \uparrow^{\mathcal{B}} A \ \&\ \forall j \in \{1 \ldots n\}.\\ & d_j \sqsubseteq \textstyle\bigsqcup\{b_i \mid 1 \leq i \leq m \ \&\ a_i \sqsubseteq c_j\}\\ & \text{by Lemma 2(3)}\\ \Leftrightarrow\ & A \leq B \ \&\ \forall j \in \{1 \ldots n\}.\\ & \mu_{n-1}(\textstyle\bigsqcup\{b_i \mid 1 \leq i \leq m \ \&\ \mu_{n-1}(c_j) \leq \mu_{n-1}(a_i)\}) \leq \mu_{n-1}(d_j)\\ & \text{by induction hypothesis} \end{aligned}$$

$\Leftrightarrow A \leq B \;\&\; \forall j \in \{1 \ldots n\}.$
$\bigcap\{\mu_{n-1}(b_i) \mid 1 \leq i \leq m \;\&\; \mu_{n-1}(c_j) \leq \mu_{n-1}(a_i)\} \leq \mu_{n-1}(d_j)$
by definition of μ_{n-1}

$\Leftrightarrow A \leq B \;\&$
$\bigcap_{i=1}^{m} \mu_{n-1}(a_i) \to \mu_{n-1}(b_i) \leq \bigcap_{j=1}^{n} \mu_{n-1}(c_j) \to \mu_{n-1}(d_j)$
by Lemma 11 (1) (3) and Theorem 16.
$\Leftrightarrow \mu_n(e) \leq \mu_n(e')$

Assume now that $\mathcal{T}$ is lazy equated. If $e' = \bot$ or $\mu_n(e') = [\top]$ then the equivalence is trivial. If $e' = (\bigsqcup_{j=1}^{n}(c_j \Rightarrow d_j), 0)$ then $e = (\bigsqcup_{i=1}^{m}(a_i \Rightarrow b_i), 0)$, and the proof is very similar to the case of natural split.

2. We prove first the natural split case by induction on n. If $n = 0$, then we have $d = \bot$, so

$$\begin{aligned}\mu_0(\bot) &= [\top]\\ &= [\top \cap (\top \to \top)]\\ &= \mu_1(\uparrow^{\mathcal{B}} \top, (\bot \Rightarrow \bot))\\ &= \mu_1(i_0(\bot))\end{aligned}$$

If $n > 0$, note that $i_n(\uparrow^{\mathcal{B}} A, \bigsqcup_{i=1}^{m}(a_i \Rightarrow b_i)) = (\uparrow^{\mathcal{B}} A, \bigsqcup_{i=1}^{m}(i_{n-1}(a_i) \Rightarrow i_{n-1}(b_i)))$. Hence,

$$\begin{array}{lll} \mu_{n+1}(i_n(\uparrow^{\mathcal{B}} A, \bigsqcup_{i=1}^{m}(a_i \Rightarrow b_i))) & & \\ = & \mu_{n+1}(\uparrow^{\mathcal{B}} A, \bigsqcup_{i=1}^{m}(i_{n-1}(a_i) \Rightarrow i_{n-1}(b_i))) & \text{by above} \\ = & [A] \cap \bigcap_{i=1}^{m}(\mu_n(i_{n-1}(a_i)) \Rightarrow \mu_n(i_{n-1}(b_i))) & \\ = & [A] \cap \bigcap_{i=1}^{m} \mu_{n-1}(a_i) \Rightarrow \mu_{n-1}(b_i)) & \text{by induction} \\ = & \mu_n(\uparrow^{\mathcal{B}} A, \bigsqcup_{i=1}^{m}(a_i \Rightarrow b_i)) & \end{array}$$

Consider now the lazy equated case. We reason by induction on n. Let $n = 0$. If $d = \uparrow^{\mathcal{B}} \top$, then $\mu_0(d) = [\top] = \mu_1(\bot) = \mu_1(i_0(d))$. If $d = \uparrow^{\mathcal{B}} A$ with $A \neq \top$, then

$$\begin{aligned}\mu_1(i_0(\uparrow^{\mathcal{B}} A)) &= \mu_1(\hat{\mathsf{def}}(\uparrow^{\mathcal{B}} A), 0)\\ &= \bigcap_{i=1}^{m}(\mu_0(\uparrow^{\mathcal{B}} B_i) \to \mu_0(\uparrow^{\mathcal{B}} C_i))\\ &= \bigcap_{i=1}^{m}([B_i] \to [C_i])\\ &= [A]\\ &= \mu_0(\uparrow^{\mathcal{B}} A)\end{aligned}$$

Let $n > 0$. In this case, if $d = \bot$ the thesis follow from

$$\begin{aligned}\mu_n(d) &= [\bot]\\ &= \mu_{n+1}(\bot)\\ &= \mu_{n+1}(i_n(\bot))\end{aligned}$$

If $d = (\bigsqcup_{i=1}^{m}(a_i \Rightarrow b_i), 0)$, we have

$$\begin{aligned}\mu_n(d) &= \bigcap_{i=1}^{m}(\mu_{n-1}(a_i) \to \mu_{n-1}(b_i))\\ &= \bigsqcup_{i=1}^{m}(\mu_n(i_{n-1}(a_i)) \to \mu_n(i_{n-1}(b_i))) \text{ by induction}\\ &= \mu_{n+1}((\bigsqcup_{i=1}^{m}(i_{n-1}(a_i) \Rightarrow i_{n-1}(b_i), 0)\\ &= \mu_{n+1}(i_n(d))\end{aligned}$$

3. We consider first $\mathcal{T}$ natural split. We reason by induction on the type $A \in \mathbb{T}$. The basis cases are $\mu_0(\bot) = [\top]$ and

$$\begin{aligned}\mu_1(\uparrow^{\mathcal{B}} \alpha, \bot \Rightarrow \bot)) &= [\alpha \cap (\top \to \top)] \\ &= [\alpha]\end{aligned}$$

Let $A = B \to C$. By induction there exists n, m and $b \in \mathcal{K}(D_n^{(\mathcal{T})})$, $c \in \mathcal{K}(D_m^{(\mathcal{T})})$ such that $\mu_n(b) = [B]$ and $\mu_m(c) = [C]$. Assume $n \leq m$ (the other case is symmetric). Let $b' = i_{m-1} \circ \ldots \circ i_n(b)$ (hence $b = b'$ if $m = n$). It follows from (2) that $\mu_m(b') = \mu_n(b)$. We have

$$\begin{aligned}\mu_{m+1}(\uparrow^{\mathcal{B}} \top, (b' \Rightarrow c)) &= [\top] \cap (\mu_m(b') \to \mu_m(c)) \\ &= [\top] \cap ([B] \to [C]) && \text{by above} \\ &= [B \to C]\end{aligned}$$

Finally, let $A = B \cap C$. By induction there exists n, m and $b \in \mathcal{K}(D_n^{(\mathcal{T})})$, $c \in \mathcal{K}(D_m^{(\mathcal{T})})$ such that $\mu_n(b) = [B]$ and $\mu_m(c) = [C]$. Assume $n \leq m$. Let $b' = i_{m-1} \circ \ldots \circ i_n(b)$ (hence $b = b'$ if $m = n$). It follows from (2) that $\mu_m(b') = \mu_n(b)$. So we have

$$\begin{aligned}\mu_m(b' \sqcup c) &= \mu_m(b') \cap \mu_m(c) && \text{by definition of } \mu_m \\ &= \mu_n(b) \cap \mu_m(c) && \text{by above} \\ &= [B] \cap [C] \\ &= [A]\end{aligned}$$

Consider now $\mathcal{T}$ lazy equated. We reason again by induction on the type A. The basis cases are $\mu_0(\uparrow^{\mathcal{B}} \alpha) = [\alpha]$. Let $A = B \to C$. By induction there exists n, m and $b \in \mathcal{K}(D_n^{(\mathcal{T})})$, $c \in \mathcal{K}(D_m^{(\mathcal{T})})$ such that $\mu_n(b) = [B]$ and $\mu_m(c) = [C]$. Suppose $n \leq m$. Let $b' = i_{m-1} \circ \ldots \circ i_n(d)$ (hence $d = d'$ if $m = n$). It follows from (2)that $\mu_m(b') = \mu_n(b)$. We have

$$\begin{aligned}\mu_{m+1}((b' \Rightarrow c), 0) &= \mu_m(b') \to \mu_m(c) \\ &= [B] \to [C] && \text{by above}\end{aligned}$$

Finally, the case $A = B \cap C$ is treated similarly to the case natural split. $\square$