A completeness result for gs-monoidal categories

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Discussions: Andrea Corradini

a roadmap

- some original motivations from graph rewriting
- an alternative presentation for cartesian categories
- a functorial characterization for partial and multi-algebras
- some facts on monoidal monads
- a completeness result for gs-monoidal cats [over semi-modules]
  - a characterization for multiset-algebras

1-slide graph transformation

- Why graph rewriting (late Sixties, early Seventies)
  - ☑ generalizes Chomsky grammars (adding data sharing)
  - ☑ used in constraint solving and data structuring (70's)
  - ☑ applied as a (visual) specification technique (80's-90's)

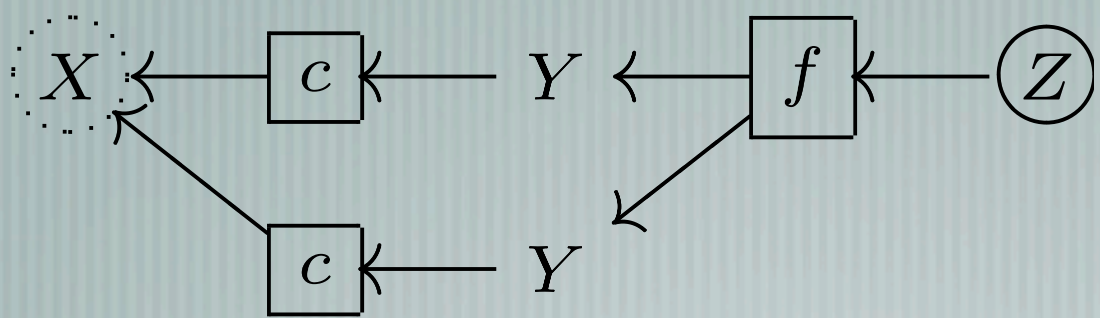
- but...
  - ☐ no (obvious) algebraic structure (no induction)
  - ☐ neither (temporal) logic nor calculus

Many data structures (HLR, adhesive...)
for the same meta-approach

addressing the syntax...

a signature

$$\Sigma = \langle\{X, Y, Z\}, \{c \in \Sigma_{X,Y}, f \in \Sigma_{YY,Z}\}\rangle$$



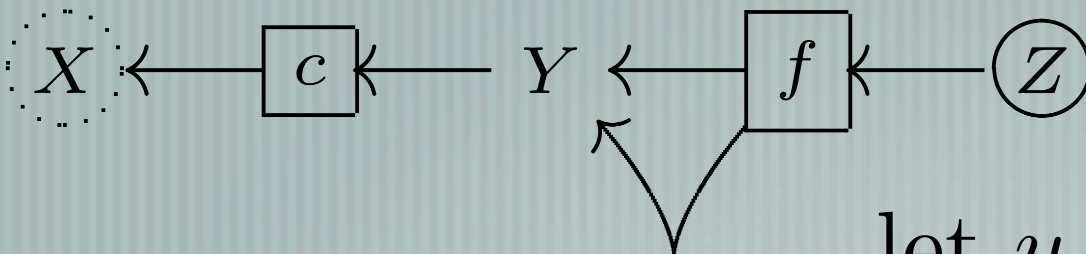
a rooted tree

a standard term $f(c(x), c(x))$

how to obtain a term-like presentation for term graphs?

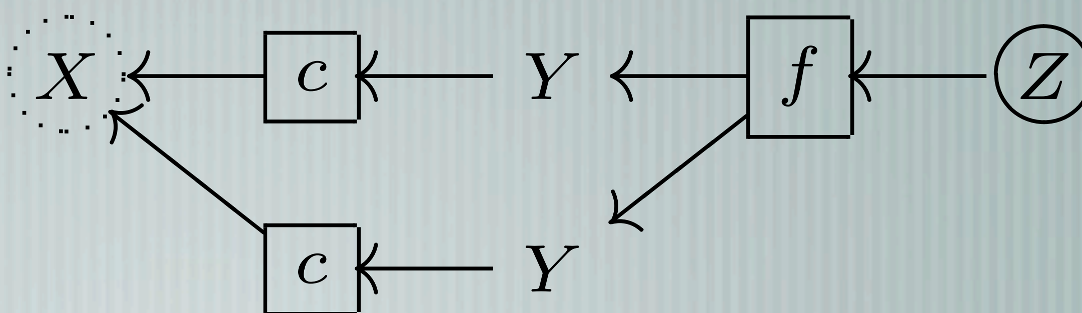
which are the associated models (algebras), if any?

(part of) running example...



$$\text{let } y \text{ be } c(x) \text{ in } f(y,y)$$

**syntax quite there...
but which underlying
(categorical) models?**



$$\text{let } \langle y_1, y_2 \rangle \text{ be } \langle c(x), c(x) \rangle \text{ in } f(y_1, y_2)$$

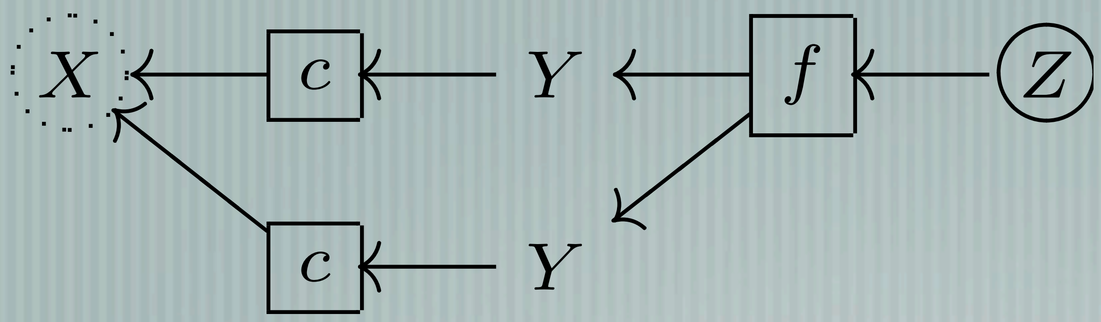
a classical presentation

- The algebraic theory $Th(\Sigma)$ is concretely defined as
  - lists of vars as objects, (tuples of) typed terms as arrows
  - term substitution as composition
- (the theory is also the free cartesian category over Σ)
- Algebras over Σ and axioms in E as functors

$$M \in [Th(\Sigma), \mathbf{Set}]^{\times}_{E}$$

- product and axioms preserving (homs as natural transfs.)

a completeness result



from objects (arrows)
to sets (functions)

$$M(X) \xrightarrow{M(c)\times M(c)} M(Y) \times M(Y) \equiv M(Y \times Y) \xrightarrow{M(f)} M(Z)$$

$\dashv$ A completeness property

$$\forall s, t \in Th(\Sigma).\{s \equiv t \iff \forall M \in [Th(\Sigma), \mathbf{Set}]_E^{\times}.M(s) = M(t)\}$$

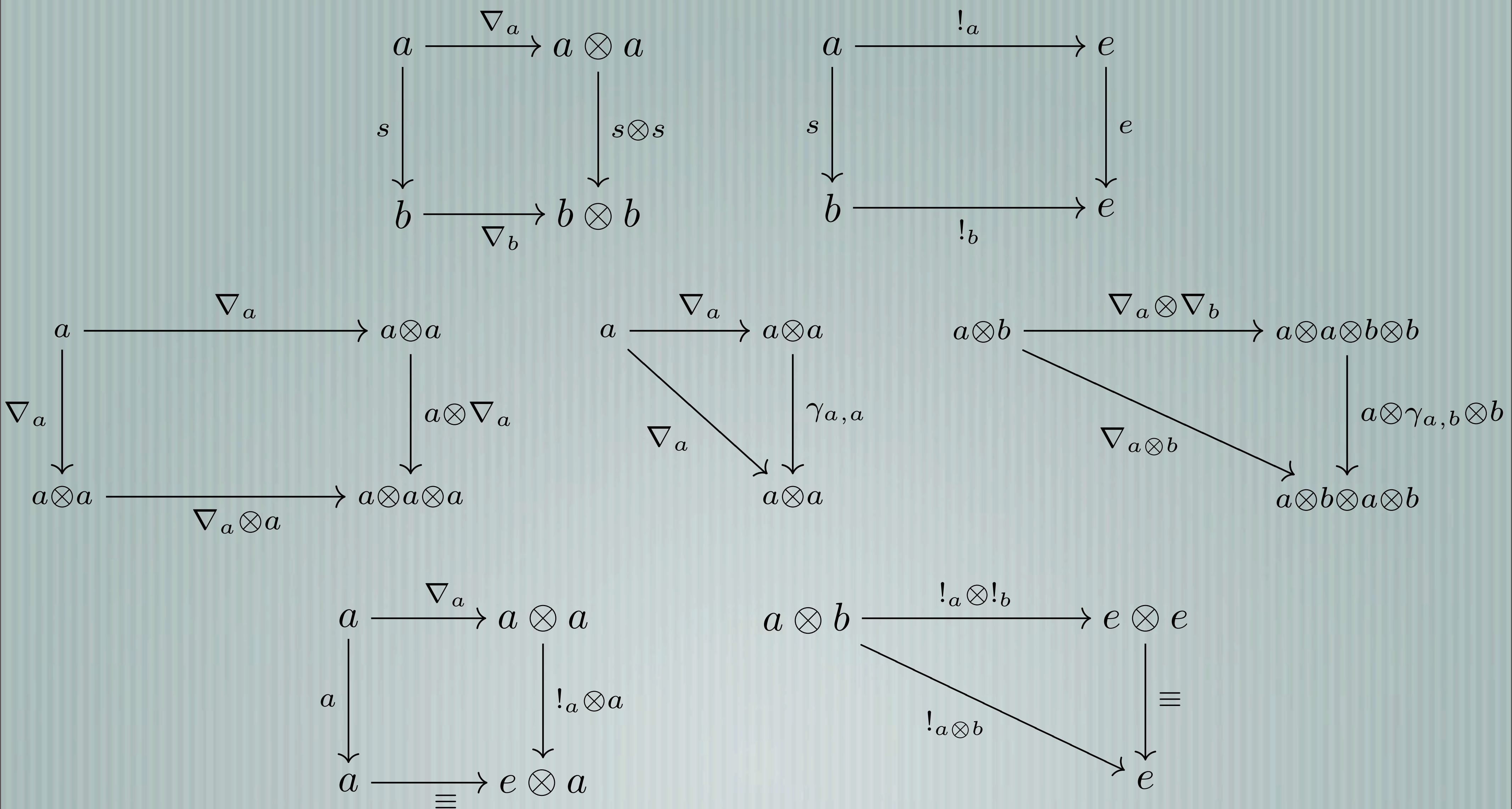
an alternative take

- *Th*(∑) is the free symmetric (strict) monoidal category equipped with symmetric monoidal natural transformations

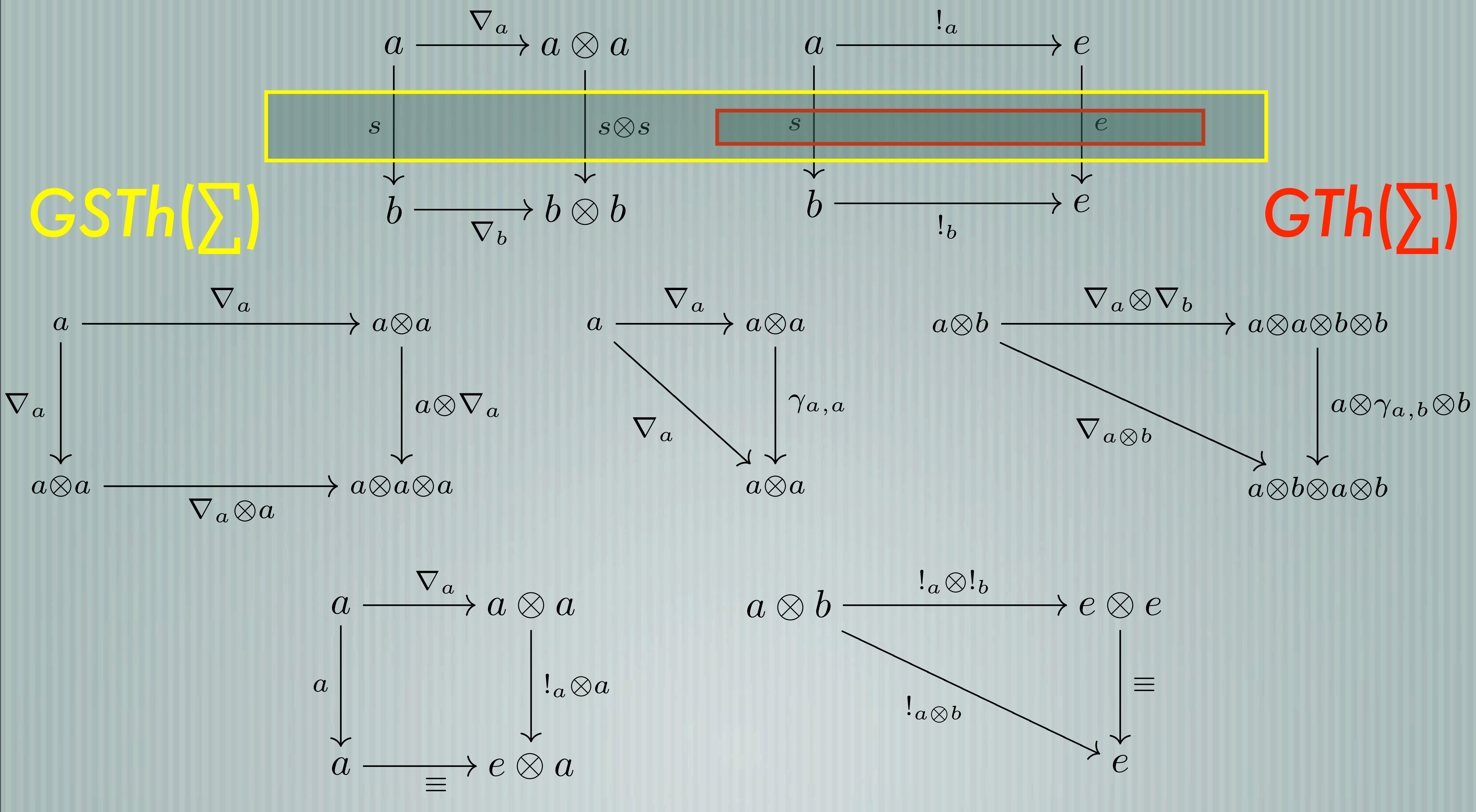
$$\nabla_a : a \to a \otimes a \qquad\qquad !_a : a \to e$$

- (intuitively representing pairing tuple <x,x> and empty tuple)

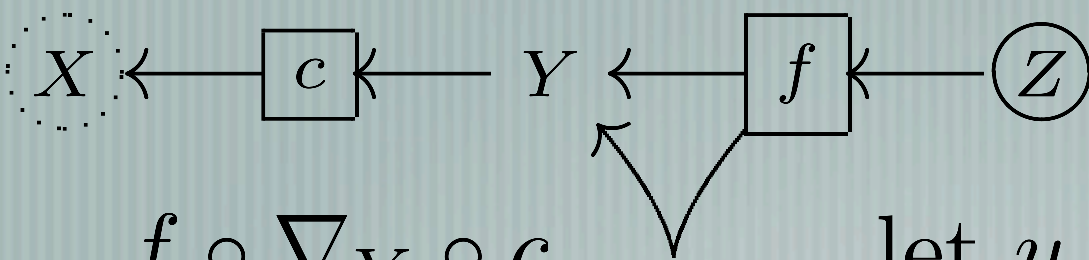
explicit definition of a theory



two alternative takes

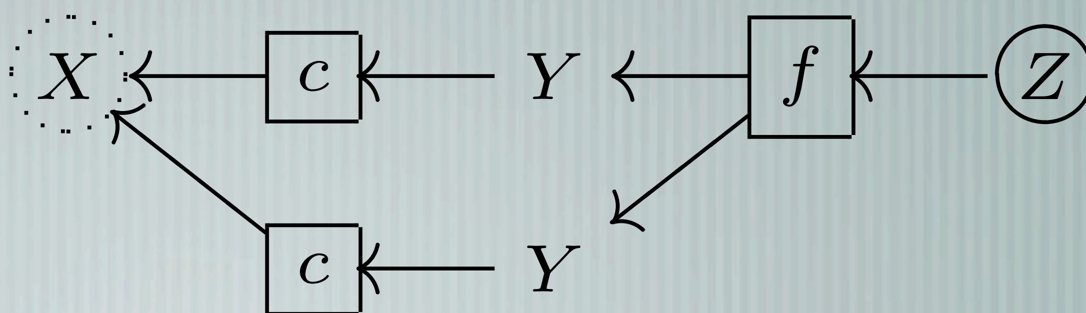


(more of) running example...



$f \circ \nabla_Y \circ c$

let y be $c(x)$ in $f(y, y)$



$f \circ (c \otimes c) \circ \nabla_X$

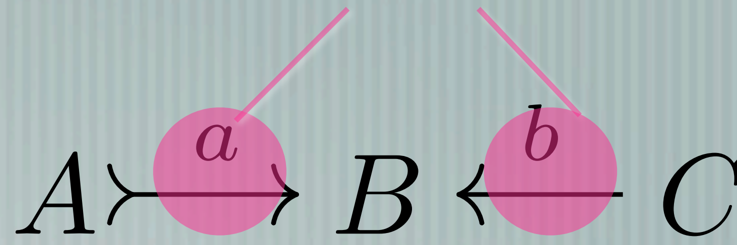
let $\langle y_1, y_2 \rangle$ be $\langle c(x), c(x) \rangle$ in $f(y_1, y_2)$

the (linear) CoSpan (bi-)category

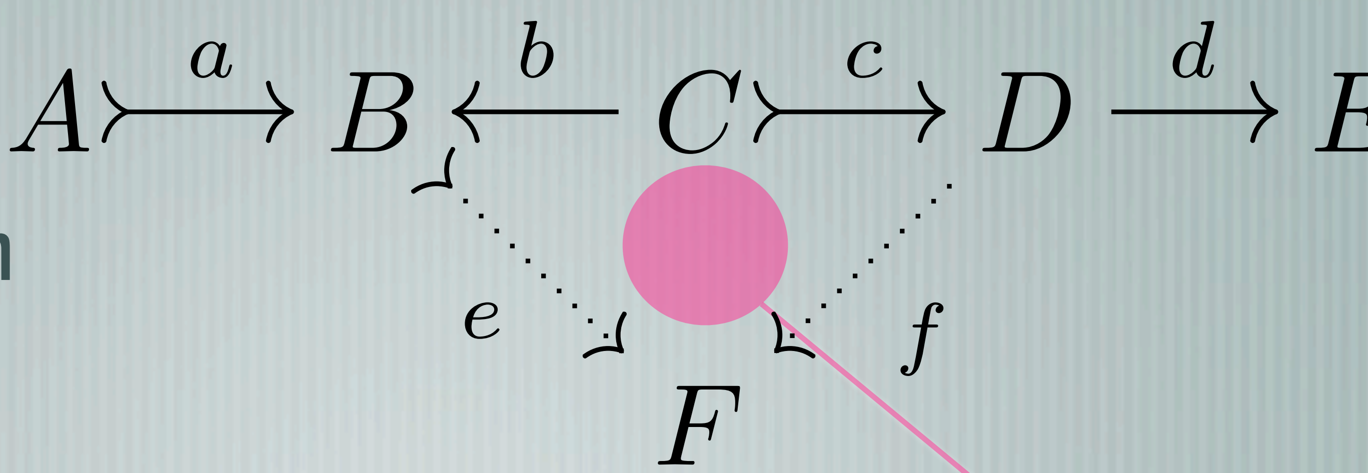
TGrp(∑) = *TGrp*↓ ∑

arrows in *TGrp(∑)*

an arrow



composition



pushout in *TGrp(∑)*

[holding for any cocomplete cat]

some characterization results

- arrows in *GSTh*(∑) are (isomorphic classes of linear) cospans of term graphs (typed over ∑)
  - You abstract the identity of nodes not in the interface
  - ...but this way graphs get a "standard" notion of sentence
- arrows in *GTh*(∑) are conditioned terms s | D (over ∑)
  - s a term (the functional)
  - D a sub-term closed set of terms (the domain restriction)

[garbage]

functorial characterizations

- Partial algebras with $\perp$-preserving operators, tight homomorphisms and conditioned Kleene equations
- Multialgebras with tight point-to-set operators, tight point-to-point homomorphisms and "term graph" equations

$$[GTh(\Sigma), \mathbf{Set}_\perp]^\times_E$$

$$[GSTh(\Sigma), 2^{\mathbf{Set}}]^\times_E$$

completeness & entailment

Completeness for partial algebras

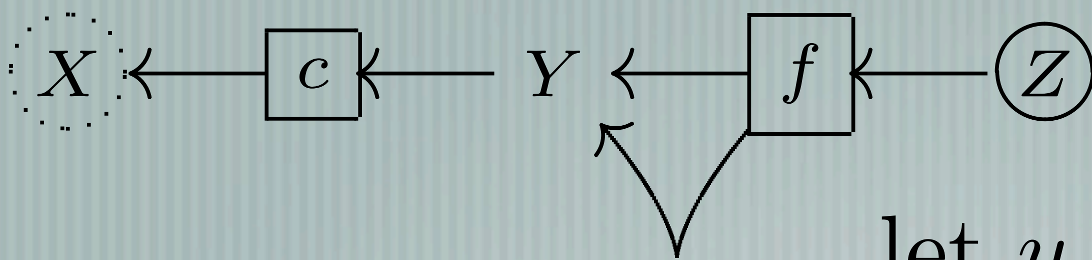
$$\forall s, t \in GTh(\Sigma).\{s \equiv t \iff \forall M \in [GTh(\Sigma), \mathbf{Set}_{\perp}]^{\times}_{E}.M(s) = M(t)\}$$

Complete entailment system for partial algebras

$$\frac{s \mid D_s \equiv t \mid D_t}{s \mid D_s \cup D \equiv t \mid D_t \cup D} \qquad \frac{u_i \mid D_u \qquad (s \mid D_s, t \mid D_t) \in E}{s[\overline{u}/\overline{x}] \mid D_s[\overline{u}/\overline{x}] \cup D_u \equiv t[\overline{u}/\overline{x}] \mid D_t[\overline{u}/\overline{x}] \cup D_u}$$

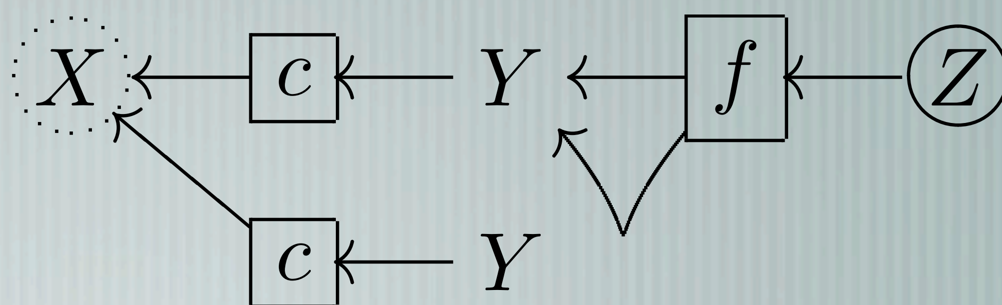
Claim: no finite complete entailment system for multi-algebras

(more of) running example...



$$\text{let } y \text{ be } c(x) \text{ in } f(y, y)$$

"garbage equivalent": same as multialgebra terms, different as [linear cospans of] term graphs



$$\text{let } \langle y_1, y_2 \rangle \text{ be } \langle c(x), c(x) \rangle \text{ in } f(y_1, y_1)$$

a semiring-based functor

- How to further generalize multialgebras?
  - First, generalize the finite power-set monad...
  - ...then, take the Kleisli category!!

$$\mathcal{S} = \langle S, 0, 1, \oplus, \otimes \rangle \qquad \mathcal{S}^{-} : \mathbf{Set} \to \mathbf{Set}$$

$$\mathcal{S}^X = [X, S] \qquad [\mathcal{S}^{f:X \to Y}(g)](y) = \bigoplus_{x \in f^{-1}(y)} g(x)$$

[finite support]

A monoidal monadic detour

- The construction presented on semi-modules is an instance of the monoidal structure induced by monoidal monads!!
- First step: symmetric monoidal functor

$$F : (C, \otimes_C, e_C, \rho_C) \to (D, \otimes_D, e_D, \rho_D)$$

$$m_{a,b} : F(a) \otimes_D F(b) \to F(a \otimes_C b)$$

$$m_e : e_D \to F(e_C)$$

natural transformations
plus coherence axioms

some more coherence

- Second step: symmetric monoidal monad $\langle T, \eta, \mu \rangle$

$$\eta_a : a \rightarrow T(a)$$

$$\mu_a : T(T(a)) \rightarrow T(a)$$

natural transformations

$$\begin{array}{ccc} T(a) & \xrightarrow{T(\eta_a)} & T(T(a)) \\ \Big\downarrow{\scriptstyle \eta_{T(a)}} & & \Big\downarrow{\scriptstyle \mu_a} \\ T(T(a)) & \xrightarrow[\mu_a]{} & T(a) \end{array} \qquad \begin{array}{ccc} T(T(T(a))) & \xrightarrow{T(\mu_a)} & T(T(a)) \\ \Big\downarrow{\scriptstyle \mu_{T(a)}} & & \Big\downarrow{\scriptstyle \mu_a} \\ T(T(a)) & \xrightarrow[\mu_a]{} & T(a) \end{array}$$

some more coherence

- Second step: symmetric monoidal monad $\langle T, \eta, \mu \rangle$

$$\eta_a : a \to T(a)$$

$$\mu_a : T(T(a)) \to T(a)$$

SYMMETRIC MONOIDAL
natural transformations

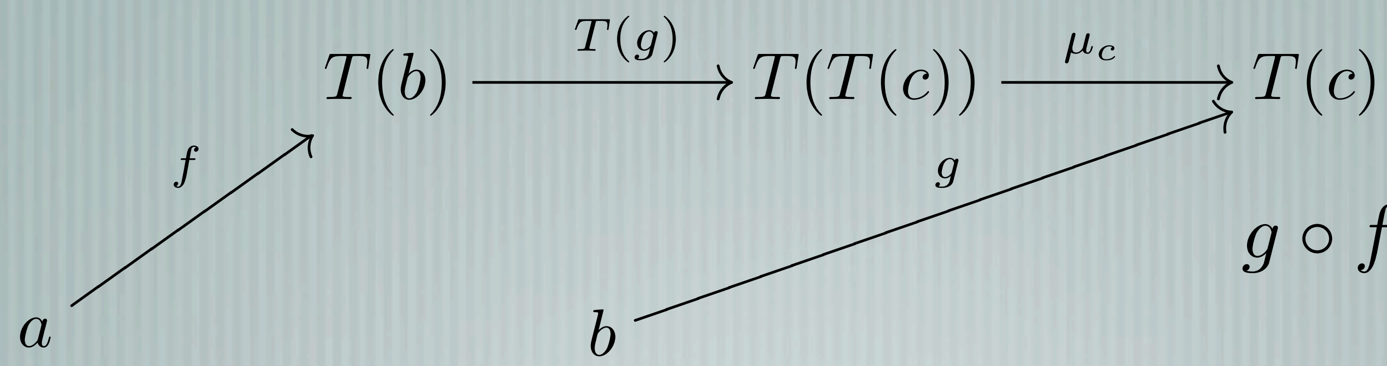
$$\begin{array}{ccc}
a \otimes b & \xrightarrow{\eta_a \otimes \eta_b} & T(a) \otimes T(b) \\
{\scriptstyle id_{a,b}} \downarrow & & \downarrow {\scriptstyle m_{a,b}} \\
a \otimes b & \xrightarrow[\eta_{a,b}]{} & T(a \otimes b)
\end{array}
\qquad
\begin{array}{ccc}
e & \xrightarrow{id_e} & e \\
{\scriptstyle id_e} \downarrow & & \downarrow {\scriptstyle m_e} \\
e & \xrightarrow[\eta_e]{} & T(e)
\end{array}$$

gsm-cats are everywhere...

- Kleisli category of a monad [aka, free algebras of a monad]

$$f : a \to b \in \mathcal{K}_T \iff f : a \to T(b) \in C$$

arrows are substitutions



- Known fact: if T is a (symmetric) monoidal monad, so is $\mathcal{K}_T$
- Less known fact: if the product is cartesian, $\mathcal{K}_T$ is gs-monoidal

a semiring-based functor

- How to further generalize multialgebras?
  - First, generalize the finite power-set monad...
  - ...then, take the Kleisli category!!

$$\mathcal{S} = \langle S, 0, 1, \oplus, \otimes \rangle \qquad \mathcal{S}^{-} : \mathbf{Set} \to \mathbf{Set}$$

$$\mathcal{S}^X = [X, S] \qquad [\mathcal{S}^{f:X \to Y}(g)](y) = \bigoplus_{x \in f^{-1}(y)} g(x)$$

[finite support]

looking at the monad structure

Recasting finite power-sets via boolean algebra $\langle \{tt, ff\}, ff, tt, \vee, \wedge \rangle$

[free dioid]

$$\eta_X : X \to [X, S]$$

$$\eta_X(x) = \iota_x$$ [the injection function: 1 for x, 0 elsewhere]

$$\mu_X : [[X, S], S] \to [X, S]$$

$$[\mu_X(\lambda : [X, S] \to S)](x) = \bigoplus_{f:X \to S} \lambda(f) \otimes f(x)$$

a monoidal monad

$$m_{X,Y} : \mathcal{S}^X \times \mathcal{S}^Y \to \mathcal{S}^{X\times Y}$$

$$[m_{X,Y}(\langle f, g\rangle)](\langle x, y\rangle) \mapsto f(x) \otimes g(y)$$

$$\begin{array}{ccc} X \times Y & \xrightarrow{\iota_- \times \iota_-} & \mathcal{S}^X \times \mathcal{S}^Y \\ {\scriptstyle id_{X,Y}}\downarrow & & \downarrow{\scriptstyle -\otimes=} \\ X \times Y & \xrightarrow[\iota_{\langle -,=\rangle}]{} & \mathcal{S}^{X\times Y} \end{array} \qquad \iota_x \otimes \iota_y = \iota_{x,y}$$

$$m_\emptyset : \emptyset \to \mathcal{S}^\emptyset \qquad m_\emptyset = \eta_\emptyset = id_\emptyset$$

the associated Kleisli cat

$$[X, S] = \{n_1 \cdot x_1 \oplus \ldots \oplus n_k \cdot x_k \mid n_i \in S, x_i \in X\}$$

Kleisli cat: sets as objects, multiset relations as arrows!!

$$f : X \to [Y, S]$$

$$\forall x \in X. \, g \circ f(x) = \bigoplus_{n \cdot y \in f(x)} n \cdot g(y)$$

$$\forall \langle x, w \rangle \in X \times W. \, h \otimes k(\langle x, w \rangle) = \bigoplus_{n \cdot y \in h(x), m \cdot z \in k(w)} nm \cdot \langle y, z \rangle$$

the Kleisli category is gs-monoidal

(use of) running example...

$$f(y_i, y_j) = z_{ij}$$

$$c(x) = \bigoplus_{i=1,2} n_i \cdot y_i$$

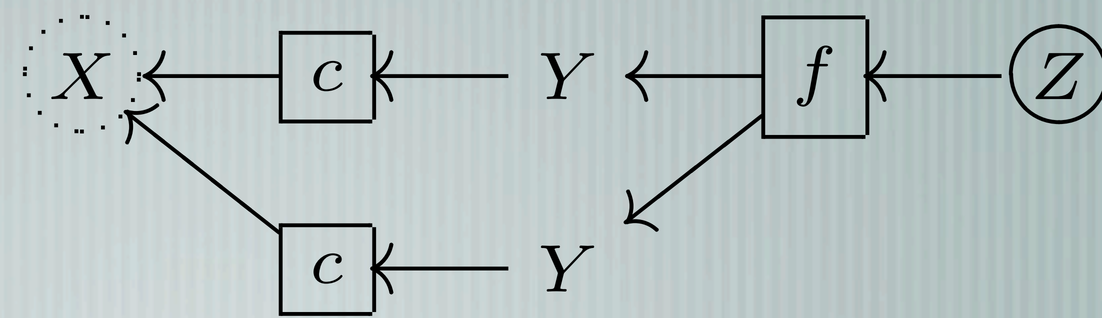
$$(c \otimes c)(\langle x, x \rangle) = \bigoplus_{i,j=1,2} n_i n_j \cdot \langle y_i, y_j \rangle$$

$$Z = \{z_{ij} \mid i, j = 1, 2\}$$

$$Y = \{y_1, y_2\}$$

$$X = \{x\}$$

$$f \circ (c \otimes c) \circ \nabla_X$$



$$x \mapsto \bigoplus_{i,j=1,2} n_i n_j \cdot z_{ij}$$

(use of) running example...

$$f(y_i, y_j) = z_{ij}$$

$$c(x) = \bigoplus_{i=1,2} n_i \cdot y_i$$

$$(c \otimes c)(\langle x, x \rangle) = \bigoplus_{i,j=1,2} n_i n_j \cdot \langle y_i, y_j \rangle$$

$$Z = \{z_{ij} \mid i, j = 1, 2\}$$

$$Y = \{y_1, y_2\}$$

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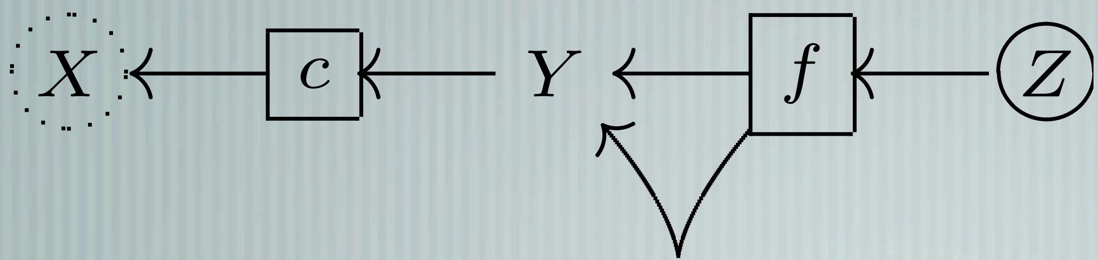


$$f \circ \nabla_Y \circ c$$

$$x \mapsto \bigoplus_{i=1,2} n_i \cdot z_{ii}$$

(use of) running example...

$$f(y_i, y_j) = z_{ij}$$

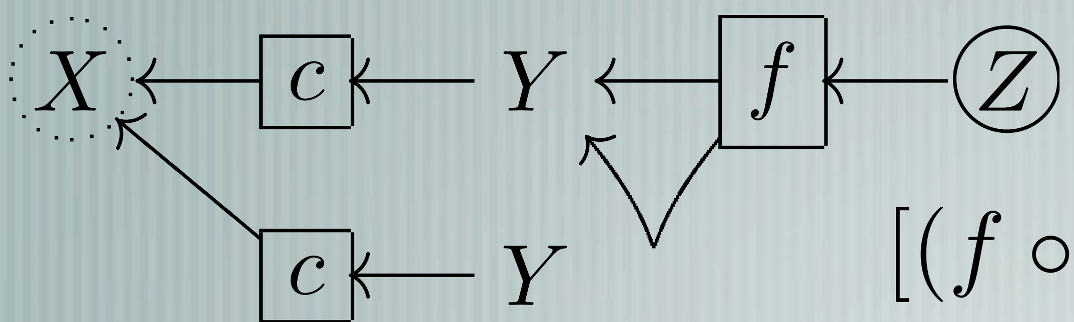
$$c(x) = \bigoplus_{i=1,2} n_i \cdot y_i$$

$$(c \otimes c)(\langle x, x \rangle) = \bigoplus_{i,j=1,2} n_i n_j \cdot \langle y_i, y_j \rangle$$

$$Z = \{z_{ij} \mid i, j = 1, 2\}$$

$$Y = \{y_1, y_2\}$$

$$X = \{x\}$$



$$[(f \circ \nabla_Y \circ c) \otimes (!_Y \circ c)] \circ \nabla_X$$

$$x \mapsto \bigoplus_{i,j=1,2} n_i n_j \cdot z_{ii}$$

functorial characterizations

- Multiset-algebras with tight point-to-multiset operators, tight point-to-point homomorphisms and "term graph" equations

$$[GSTh(\Sigma), \mathcal{S}^{\mathbf{Set}}]^{\times}_{E}$$

- Multiset-algebras: each operator is a multiset relation (hence, in relational algebras jargon, tight and point-to-multiset)...
- Multiset-homomorphisms: since natural transformations must preserve ∇_X and $!_X$, homs are just functions (again, tight and point-to-point)

power-sets vs. natural numbers

If the semiring is the free dioid $\langle \{tt, ff\}, ff, tt, \vee, \wedge \rangle$

$x \mapsto \bigoplus_{i,j=1,2} n_i n_j \cdot z_{ii}$ and $x \mapsto \bigoplus_{i=1,2} n_i \cdot z_{ii}$ coincide...

Stronger claim: no finitary complete entailment system for multi-algebras

a novel completeness

If the semiring is the free semiring $\langle \{0, 1, 2, \ldots\}, 0, 1, +, \times \rangle$

and at most one coefficient (among $\{n_1, n_2\}$) is not zero

$$x \mapsto \bigoplus_{i,j=1,2} n_i n_j \cdot z_{ii} \quad \text{and} \quad x \mapsto \bigoplus_{i=1,2} n_i \cdot z_{ii} \quad \text{do not coincide...}$$

$$x \mapsto \bigoplus_{i,j=1,2} n_i n_j \cdot z_{ii} \quad \text{and} \quad x \mapsto \bigoplus_{i,j=1,2} n_i n_j \cdot z_{ij} \quad \text{coincide...}$$

[recovering partial functions]

a new completeness result

If the semiring is the free semiring $\langle\{0,1,2,\ldots\},0,1,+,\times\rangle$

the three multiset relations are different

$$\forall s,t \in GSTh(\Sigma).\{s \equiv t \iff \forall M \in [GSTh(\Sigma),\mathbb{N}^-]^{\times}_{E}.M(s)=M(t)\}$$

to be addressed...

- Algebraic issues
  - tackling hyper-graphs and hyper-signatures
    - (singular vs plural) interpretation for hyper-operators
  - considering cospans of cocomplete categories
    - free construction for suitable algebraic varieties
- Coalgebraic issues
  - analyzing trace equivalence [via Hasuo-Jacobs]