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Category Theory Exercises

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Please note that these Exercises sometimes refer to the accompanying MGS 2015 slides on **Category Theory**.

If X is a set, then a **preorder** on X is a binary relation $\leq$ on X which is *reflexive* and *transitive*. If the relation $\leq$ is also **anti-symmetric**, that is for any $x, y \in X$ we have $x \leq y$ and $y \leq x$ implies $x = y$, then we call $\leq$ a **partial order**. We will often simply refer to a preorder X or partial order (poset) X, though sometimes refer to $(X, \leq)$ or $(X, \leq_X)$ if we wish to make the (name of) the set and order clear.

1 Categories

(1) For the category of sets and functions, $\mathcal{Set}$, check *in detail* that the axioms of a category hold. In the case of $\mathcal{Set}$, what *exactly* are the morphisms? What are the identities? What is morphism composition? Verify the equations for identities and associativity.

(2) If X is a set, check that $\mathcal{P}(X)$ is a *preorder* by inclusions $\subseteq$ and is hence a category, checking all of the details. (Of course $\mathcal{P}(X)$ also happens to be a partial order ...)

(3) Verify that there is a category $\mathcal{C}$ with one object $*$ and the set of morphisms $\mathcal{C}(*,*)$ consists of the algebraic terms $t ::= x_0 \mid f(t) \mid g(t,t)$ where x_0 is one given variable and f and g are two given function symbols; and composition is substitution $t[t'/x_0]$ (where "t' replaces x_0 in t is defined recursively).

(4) Verify that any monoid (M, b, e) is a single object category $\mathcal{C}$ with one object $*$ and $\mathcal{C}(*,*) \stackrel{\text{def}}{=} M$.

(5) Verify that $\mathcal{Mon}$, all monoids and all monoid homomorphisms, is a category. Make sure you are clear that this, and the last example, are related but entirely different.

(6) Choose some other examples of categories of your choice and verify the axioms.

(7) Check that X^{op} is the usual "opposite preorder" when X is a preorder regarded as a category.

(8) If X and Y are preorders check that so is the cartesian product $X \times Y$ (of underlying sets) ordered coordinate-wize.

(9) Given categories $\mathcal{C}$ and $\mathcal{D}$, the objects of the category $\mathcal{C} \times \mathcal{D}$ are pairs (A, B) of objects from $\mathcal{C}$ and $\mathcal{D}$ respectively. Convince yourself that there is an obvious category $\mathcal{C} \times \mathcal{D}$. Compare to the previous question by regarding X and Y as categories.

(10) If $f : X \to Y$ and $g : Y \to Z$ are both monotone functions between preorders, then so too is the **composition** $f \circ g : X \to Z$ defined by $(g \circ f)(x) \stackrel{\text{def}}{=} g(f(x))$ for any $x \in X$. Verify this fact, and hence that preorders and monotone functions form a category.

2 Functors

(1) Check that there is an identity functor on any category $\mathcal{C}$.

(2) Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be categories and $m : X \to Y$ a monotone function. Then m gives rise to a functor

$$M : (X, \leq_X) \to (Y, \leq_Y)$$

defined by $M(x) \stackrel{\text{def}}{=} m(x)$ on objects $x \in X$ and by $M(\leq_X) = \leq_Y$ on morphisms; since m is monotone,

$$\leq_X : x \to x' \text{ implies } M(\leq_X) : M(x) \to M(x'). \text{ Verify!}$$

(3) Check that there is a functor $F : \mathcal{S}et \to \mathcal{M}on$ defined by $FA \stackrel{\text{def}}{=} [A]$ and $Ff \stackrel{\text{def}}{=} map(f)$ (either see the notes, or go straight ahead if you know Haskell!)

(4) The diagonal functor $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ maps $f : A \to B$ to $(f, f) : (A, A) \to (B, B)$. Check that it is indeed a functor.

(5) Check that the **covariant powerset** functor $\mathcal{P} : \mathcal{S}et \to \mathcal{S}et$ which is given by

$$f : A \to B \quad \mapsto \quad \mathcal{P}(f) \equiv f_* : \mathcal{P}(A) \to \mathcal{P}(B),$$

where $f : A \to B$ is a function and f_* is defined by $f_*(A') \stackrel{\text{def}}{=} \{f(a') \mid a' \in A'\}$ where $A' \in \mathcal{P}(A)$ actually is a functor.

(6) Do the same for the **contravariant powerset** functor $\mathcal{P} : \mathcal{S}et^{op} \to \mathcal{S}et$ by setting

$$f^{op} : B \to A \quad \mapsto \quad f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A),$$

where $f : A \to B$ is a function in $\mathcal{S}et$, and the function f^{-1} is defined by

$$f^{-1}(B') \stackrel{\text{def}}{=} \{a \in A \mid f(a) \in B'\}$$

where $B' \in \mathcal{P}(B)$.

(7) Define $G : \mathcal{S}et \to \mathcal{M}on$ by $GA \stackrel{\text{def}}{=} lists(A)$ and $Gf \stackrel{\text{def}}{=} mapsq(f)$, where

$$mapsq(f) : lists(A) \to lists(B)$$

is defined by

$$mapsq(f)([a_1, \ldots, a_n]) = [f^2(a_1), \ldots, f^2(a_n)], \qquad mapsq(f)([\,]) = [\,]$$

with $[a_1, \ldots, a_n]$ any element of $lists(A)$ and $f : A \to B$ a function. Show that G is a not a functor.

(8) $*$ Let us say that a category $\mathcal{C}$ is **tiny** if the collection of objects forms a set and $\mathcal{C}$ is discrete, that is, the only morphisms are identities; prove that a category $\mathcal{C}$ is tiny iff given any category $\mathcal{D}$ with a set of objects $ob\,\mathcal{D}$ and any set function $f : ob\,\mathcal{C} \to ob\,\mathcal{D}$, then f extends uniquely to a functor $F : \mathcal{C} \to \mathcal{D}$. (Extends means that if A is an object of $\mathcal{C}$, then $FA = f(A) \in ob\,\mathcal{D}$.)

3 Natural Transformations

(1) Verify that there is an identity natural transformation for any functor $F : \mathcal{C} \to \mathcal{C}$.

(2) For the functor $F : \mathcal{S}et \to \mathcal{M}on$ above, verify in detail that there is a natural transformation $rev : F \to F$ whose components at a set A reverse lists in $[A]$.

(3) Verify in detail that there is a category $\mathcal{D}^{\mathcal{C}}$ with objects functors from $\mathcal{C}$ to $\mathcal{D}$, morphisms *natural transformations* between such functors. We can define a natural transformation $\beta \circ \alpha : F \to H$ by setting the components to be

$$(\beta \circ \alpha)_A \stackrel{\text{def}}{=} \beta_A \circ \alpha_A.$$

and this is the composition of $\mathcal{D}^{\mathcal{C}}$. **Note: this category is often denoted by** $[\mathcal{C}, \mathcal{D}]$**.**

4 Isomorphisms

(1) Suppose that a bijection f in $\mathcal{S}et$ is specified as a one-to-one and onto function (injection and surjection). Check that f is an isomorphism.

(2) Check that the relation $\cong$ of isomorphism is an equivalence relation.

(3) Find a counterexample to the following statement. A monotone function $f : X \to Y$ between posets X and Y which is a bijection on the underlying sets (an isomorphism in $\mathcal{S}et$) is necessarily an isomorphism in $\mathcal{P}ar\mathcal{S}et$.

5 Products and Coproducts

(1) Verify in detail that binary products and coproducts exist in $\mathcal{S}et$ (see the notes if required).

(2) Let X and Y be preorders and $X \times Y$ the cartesian product of the underlying sets ordered coordinate-wize. Check that there are monotone functions $\pi_X : X \times Y \to X$, $(x,y) \mapsto x$ and $\pi_Y : X \times Y \to Y$, $(x,y) \mapsto y$ where $(x,y) \in X \times Y$. Verify that given monotone

functions $f : Z \to X$ and $g : Z \to Y$ where Z is any given preorder, there is a unique monotone function $m : Z \to X \times Y$ for which $f = \pi_X \circ m$ and $g = \pi_Y \circ m$. Conclude that $\mathcal{ParSet}$ has binary products.

(3) Suppose that F_1 and F_2 are objects (that is, functors) of $\mathcal{D}^{\mathcal{C}}$ and that $\mathcal{D}$ has finite (co)products. Then both $F_1 \times F_2$ and $F_1 + F_2$ exist and are defined pointwize. Using the notes if need be, verify this in detail; *this is an important example and we will use the notation a lot in the final lecture or so*.

(4) Check that there are functors $B \times (-)$, $B + (-) : \mathcal{C} \to \mathcal{C}$ for any B so long as $\mathcal{C}$ has binary coproducts (see the notes if required).

(5) In $(\mathcal{P}(X), \subseteq)$, binary meets (products) and joins (coproducts) are given by the operations of *intersection* and *union*. Verify this. What are the top and bottom elements?

(6) Think of some simple finite posets in which meets and joins do not exist.

(7) Suppose that X is a poset. Show that meets in a poset are unique if they exist. *Hint: Suppose that, in each case, there are at least two possibilities m and m' and prove that m and m' are equal.*

(8) Show that a category $\mathcal{C}$ has *all* finite products just in case it has binary products and a terminal object.

(9) Define the partial order $\mid$ on $\mathbb{N}$ by $\forall d, n \in \mathbb{N}. d \mid n$ to mean that $(\exists k \in \mathbb{N})(n = k * d)$. With this order, binary meets and joins are given simply by *highest common factor* and *lowest common multiple* respectively. Give some informal arguments to show that this is correct (a complete answer requires some simple - undergraduate level - properties of the natural numbers, such as prime factorisation).

(10) Investigate the notion of a binary product in a category $\mathcal{C}^{op}$.

(11) Prove the coproduct of any set-indexed family of objects is unique up to isomorphism if it exists.

(12) In a category with binary (co)products,suppose that $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$. Then

$$\begin{aligned} f_1 \times f_2 &\stackrel{\text{def}}{=} \langle f_1 \circ \pi_{A_1}, f_2 \circ \pi_{A_2} \rangle : A_1 \times A_2 \to B_1 \times B_2 \\ f_1 + f_2 &\stackrel{\text{def}}{=} [\iota_{B_1} \circ f_1, \iota_{B_2} \circ f_2] : A_1 + A_2 \to B_1 + B_2 \end{aligned}$$

Convince yourself that

$$\begin{aligned} \pi_{B_i} \circ (f_1 \times f_2) &= f_i \circ \pi_{A_i} \\ (f_1 + f_2) \circ \iota_{A_i} &= \iota_{B_i} \circ f_i \end{aligned}$$

(13) Let $\mathcal{C}$ be a category with finite products and let

$$\begin{array}{lll} l : X \to A & f : A \to B & g : A \to C \\ h : B \to D & k : C \to E & \end{array}$$

be morphisms of $\mathcal{C}$. Show that $(h \times k) \circ \langle f, g \rangle = \langle h \circ f, k \circ g \rangle$ and $\langle f, g \rangle \circ l = \langle f \circ l, g \circ l \rangle$.

(14) Formulate an analogue of the previous question in terms of coproducts, and prove your conjecture.

(15) * Find an example of a functor $F : \mathcal{C} \to \mathcal{D}$ for which

$$F(A \times B) \cong FA \times FB$$

in $\mathcal{D}$ for all pairs of objects A and B in $\mathcal{C}$, but such that F does not preserve binary products. **Hint: think about countably infinite sets.**

6 More Natural Transformations and Equivalences

(1) Verify that $F_X : \mathcal{S}et \to \mathcal{S}et$ is a functor and that $ev : F_X \to id_{\mathcal{S}et}$ is a natural transformation (see slides).

(2) Given categories $\mathcal{C}$ and $\mathcal{D}$, verify that the functor category $\mathcal{D}^{\mathcal{C}}$ is indeed a category.

(3) Show that any category $\mathcal{S}et^{\mathcal{C}}$ has finite products and coproducts.

(4) Given a diagram of categories and functors

$$\mathcal{C} \xrightarrow{I} \mathcal{D} \xrightarrow{F,G,H} \mathcal{E} \xrightarrow{J} \mathcal{F}$$

and natural transformations $\alpha : F \to G$ and $\beta : G \to H$, we can define $J^* : \mathcal{E}^{\mathcal{D}} \to \mathcal{F}^{\mathcal{D}}$ by $J^*(F) \stackrel{\text{def}}{=} J \circ F$ on any object F and $(J^*(\alpha))_D \stackrel{\text{def}}{=} J(\alpha_D)$ where D is an object of $\mathcal{D}$. Show that $J^*(\beta \circ \alpha) = J^*(\beta) \circ J^*(\alpha)$. There is also a functor $I_* : \mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$. Try to define I_* and show that $I_*(\beta \circ \alpha) = I_*(\beta) \circ I_*(\alpha)$.

Note: make sure you understand in which categories the compositions are defined.

(5) Let $\mathcal{S}$ be the category of non-empty sets and set functions. Define a functor $\mathcal{P} : \mathcal{S} \to \mathcal{S}$ by sending $f : X \to Y$ in $\mathcal{S}$ to the function

$$\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y) \qquad A \mapsto f(A) \stackrel{\text{def}}{=} \{f(a) \mid a \in A\}.$$

Show that there is no natural transformation $\alpha : \mathcal{P} \to id_{\mathcal{S}}$. ($\mathcal{P}(f)$ is sometimes written f_*.)

(6) $*$ Two categories are said to be equivalent, if, roughly speaking, we can write down a one to one correspondence between isomorphism classes of objects obtained from the categories. More precisely, two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\varepsilon : F \circ G \cong id_{\mathcal{D}}$ and η :

$id_{\mathcal{C}} \cong G \circ F$. We say that F is an *equivalence* with an *inverse equivalence* G and denote the equivalence by $F : \mathcal{C} \simeq \mathcal{D} : G$.

Let $\mathcal{P}art$ be the category of sets and partial functions. Write 1 for a singleton set. An object of the category $1/\mathcal{S}et$ is a function $f : 1 \to A$ where A is a set (and hence in particular A is non-empty). A morphism $m : f \to f'$ (where $f' : 1 \to A'$) is a function $m : A \to A'$ for which $m \circ f = f'$. Prove that $\mathcal{P}art \simeq 1/\mathcal{S}et$. *Hint: Note that an object* $f : 1 \to A$ *amounts to specifying an element* $a \in A$.

(7) $*$ The slice category $\mathcal{S}et/B$ is often referred to as the category of B-indexed families of sets with functions preserving the indexing. It is defined analogously to the category $1/\mathcal{S}et$. First try to work out the definition of this category.

Then to understand the description of the category, note that a function $f : X \to B$ gives rise to the family of sets $(f^{-1}(b) \mid b \in B)$, and the family of sets $(X_b \mid b \in B)$ gives rise to the function

$$f : \{(x,b) \mid x \in X_b, b \in B\} \to B$$

where $f(x,b) \stackrel{\text{def}}{=} b$.

Note that we can regard the set B as a discrete category; then there is an equivalence between the functor category $\mathcal{S}et^B$ and the slice $\mathcal{S}et/B$. Formulate this equivalence carefully and prove that your definitions really do give an equivalence.

7 Algebras

(1) There is a category $\mathcal{C}^F$ of algebras and algebra homomorphisms (details omitted) in which initial algebras are initial objects. Verify!

(2) Show that the functor $1 + (-) : \mathcal{S}et \to \mathcal{S}et$ has an initial algebra

$$[z,s] : 1 + \mathbb{N} \to \mathbb{N}$$

where $z : 1 \to \mathbb{N}$ maps $*$ to 0 and $s : \mathbb{N} \to \mathbb{N}$ adds 1. This example illustrates the paradigm of "datatypes as initial algebras".

8 Case Study: Modelling (Haskell) Algebraic Datatypes

(1) LONG EXERCISE: work through all of the details of the material presented in the notes. Try to do two things: (i) make a high level architectural picture of the main ingredients of the datatype model, including the types, expressions, categories, functors and the initial algebra; (ii) after you have a clear picture of the main ingredients,

play/calculate with the finer technical details and make sure you can manipulate the definitions with some confidence.

9 Adjunctions

(1) Let X be a preorder. If $\Delta : X \to X \times X$ is given by $\Delta(x) \stackrel{\text{def}}{=} (x,x)$, verify that there are adjoints $(\vee \dashv \Delta \dashv \wedge)$.

(2) Verify that the functions

$$\overline{(-)}_{A,M} : \mathcal{M}on(lists(A), M) \cong \mathcal{S}et(A, UM) : \widehat{(-)}_{A,M}$$

given in the slides do indeed yield a natural bijection.

(3) Verify that the diagonal functor $\Delta : \mathcal{S}et \to \mathcal{S}et \times \mathcal{S}et$ taking a function $f : A \to B$ to $(f,f) : (A,A) \to (B,B)$ has right adjoint Π taking any morphism $(f,g) : (A,A') \to (B,B')$ of $\mathcal{S}et \times \mathcal{S}et$ to $f \times g \stackrel{\text{def}}{=} \langle f \circ \pi_A, g \circ \pi_B \rangle : A \times A' \to B \times B'$.

(4) Do the same for coproducts.

(5) If categories $\mathcal{C}$ and $\mathcal{D}$ are locally small, that is, the collection $\mathcal{C}(A,B)$ of morphisms forms a set (ditto $\mathcal{D}$), then $L \dashv R$ provided that there is an isomorphism

$$\boxed{\mathcal{D}(-,+) \circ (L^{op} \times id) \stackrel{\text{def}}{=}} \quad \mathcal{D}(L-,+) \cong \mathcal{C}(-,R+) \quad \boxed{\stackrel{\text{def}}{=} \mathcal{C}(-,+) \circ (id \times R)}$$

in the functor category $\mathcal{S}et^{\mathcal{C}^{op} \times \mathcal{D}}$ where $L^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ is defined by

$$L^{op}(f : A' \to A) \stackrel{\text{def}}{=} (Lf)^{op} : LA' \to LA$$

Verify that in such a situation this is precisely the definition given in the slides.

(6) Verify in detail that a category $\mathcal{C}$ is a cartesian closed category (CCC) if and only if there is a right adjoint R to the functor $(-) \times B : \mathcal{C} \to \mathcal{C}$ for each object B of $\mathcal{C}$.

10 Colimits

(1) Show that if $\mathbb{I} \stackrel{\text{def}}{=} \{1,2\}$ is a discrete category, then a colimit for $D : \mathbb{I} \to \mathcal{C}$ is a binary coproduct.

(2) In $\mathcal{S}et$ show that a colimit object for $D : \mathbb{I} \to \mathcal{S}et$ where $ob\,\mathbb{I} \stackrel{\text{def}}{=} \{1,2\}$ and there is a single (non-identity) morphism $1 \leq 2$ is given by $D2$.

(3) What is the colimit when $\mathbb{I}$ is (the category generated by) $ob\mathbb{I} \stackrel{\text{def}}{=} \{1,\ldots,n\}$ where $i \leq i+1$ for each object i.

(4) Now let $\mathbb{I} \stackrel{\text{def}}{=} \omega$. Let U be the disjoint union of the sets $D(i)$ as i runs over the elements of ω; formally $U \stackrel{\text{def}}{=} \bigcup\{\ \{i\} \times Di \mid i \in \omega\ \}$ and a typical element of U is a pair (i,x) where $x \in D(i)$. Define a relation on U by asking that $(i,x) \sim (j,y)$ just in case there is an object k of ω where $i \leq k$, $j \leq k$ for which $D(\leq)(x) = D(\leq)(y)$ in $D(k)$.

Prove that $\sim$ is an equivalence relation.

Set $col_i D(i) \stackrel{\text{def}}{=} U/\sim$ and define a function $\iota_i : D(i) \to col_i D(i)$ by $x \mapsto [x]$ where $x \in D(i)$.

Prove that $(\iota_i : D(i) \to col_i D(i) \mid i \in \mathbb{I})$ is a colimit for D.

(5) In the light of the last question, does a "finite analogue" of the result for $\underline{n} \stackrel{\text{def}}{=} \{0,1,\ldots,n-1\}$ in place of ω, which you should try to formulate, yield a colimit that corresponds to the ones of the previous questions?

(6) Verify that if $D : \mathbb{I} \to \mathcal{C}$, $L : \mathcal{C} \to \mathcal{D}$ and $L \dashv R$ for some R, then

$$L(col_I DI) \cong col_I LDI$$

is witnessed by $[L(\iota_{DI}) \mid I \in \mathbb{I}] : col_I LDI \to L(col_I DI)$.

(7) In fact $\mathcal{S}et$ has all colimits. Let $D : \mathbb{I} \to \mathcal{S}et$ be a diagram. Again let U be the disjoint union of the DI and define a relation R on U by asking that $(I,x)\ R\ (J,y)$ just in case there is a morphism $\alpha : I \to J$ in $\mathbb{I}$ for which $y = D\alpha(x)$. Let $\sim$ be the equivalence relation generated by R, write $col_I DI \stackrel{\text{def}}{=} U/\sim$, and let $\iota_I : DI \to col_I DI$ map elements to their equivalence classes. Prove that this gives rise to a colimit for D. Try to tie up this construction of a general colimit with the previous questions.

(8) Suppose that X is a poset viewed as a category. A colimit for $\Delta : \omega \times \omega \to X$ exists if and only if a colimit for $\Delta' : \omega \to X$ where $\Delta'(\xi) \stackrel{\text{def}}{=} \Delta(\xi,\xi)$ exists, and when they exist they are isomorphic. Such colimits are in fact given by joins, namely

$$\bigvee_{i,j} x_{(i,j)} \text{ and } \bigvee_{k,k} x_{(k,k)} \text{ and } \bigvee_i (\bigvee_j x_{(i,j)}) \text{ and } \bigvee_j (\bigvee_i x_{(j,i)})$$

where we write $x_{(i,j)}$ for $\Delta(i,j)$. Prove this fact. *Hint: Do this simply by making use of the definition of joins.*

(9) Recall from the slides that a colimit for $\Delta : \omega \times \omega \to \mathcal{C}$ exists if and only if a colimit for $\Delta' : \omega \to \mathcal{C}$ where $\Delta'(\xi) \stackrel{\text{def}}{=} \Delta(\xi,\xi)$ exists, and when they exist they are isomorphic, that is

$$col_k \Delta'(k) \cong col_{(i,j)} \Delta(i,j)$$

Further (exercise: what does $col_j\Delta(i,j)$ mean ...)

$$col_i(col_j\Delta(i,j)) \cong col_j(col_i\Delta(j,i))$$

Prove this. *Hint: Do this simply by making use of the definition of colimit.*

(10) Review the final slides which cover the existence of initial algebra: Suppose that F preserves colimits of the form $D : \omega \to \mathcal{C}$, that $\mathcal{C}$ has an initial object 0, and a colimit for D where $D(i \leq i+1) \stackrel{\text{def}}{=} F^i !_X : F^i 0 \to F^{i+1} 0$ for $i \in \omega$ exists. Then $\mathsf{I} \stackrel{\text{def}}{=} col_i Di$ is an initial algebra for F.

(11) $*$ Read up on *limits* and prove that $\mathcal{S}et$ has all limits ... OR can you work out for yourself what the construction is? Think about $D : \mathbb{I} \to \mathcal{S}et$ where $ob\,\mathbb{I} \stackrel{\text{def}}{=} \{1,2\}$ and there is a single (non-identity) morphism $\alpha : 1 \to 2$; in general, the construction is based around a "form of" cartesian product, and in this special case the product is a certain subset of $D1 \times D2$ which you should try to work out.