A Frog, A Mug, Gluing, and Logical Relations

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or, possibly, in the Wise words of the late Eric Morecombe:

. . . "but not necessarily," in that order . . .

Talk Outline

AMP: "Exponentials in Heyting Algebra gluing are a bit of a mystery."

The aim of this talk is to de-mystification.

- Brief review of applications of gluing.
- Artin Gluing of CCCs to $\mathcal{S}et$.
- Joke.
- A version of Logical Relations Gluing. Relating AG and LRG.
- Frog.
- Gluing of HAs to $\mathcal{P}(X)$.
- Mug.
- A 2nd version of Logical Relations Gluing. Relating it to LRG.
- Conclusions.

Roughly speaking, a glued category $\mathcal{GL}(\Gamma)$ is one built from two highly structured categories, $\mathcal{C}$ and $\mathcal{D}$, and a functor $\Gamma\colon \mathcal{D} \to \mathcal{C}$ with few structure preserving properties. What's cool is that $\mathcal{GL}(\Gamma)$ often enjoys similar structure to $\mathcal{C}$ and $\mathcal{D}$. There are many applications

- Proving the Existence and Disjunction Properties of Intuitionistic Predicate Calculus.
- Proving "similar" properties in richly typed logics of quite different flavours.
- Proving Conservative Extension Results for Equational Type Theories.
- Proving results about Normalisation ....

Let $\Gamma\colon \mathcal{D} \to \mathcal{S}et$ be a functor where $\mathcal{D}$ is a ccc. Suppose also that Γ preserves finite products. We define a category $\mathcal{GL}(\Gamma)$, the "Artin gluing of $\mathcal{D}$ to $\mathcal{S}et$ along Γ". This has objects (X, f, D) where $X \in \mathcal{S}et$, $D \in \mathcal{D}$ and $f\colon X \to D$ is a function. And morphisms

$$(\theta, \phi)\colon (X, f, D) \to (X', f', D')$$

where

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\theta"'] & \Gamma D \arrow[d, "\Gamma\phi"] \\ X' \arrow[r, "f'"'] & \Gamma D' \end{tikzcd}$$

The exponential $(X, f, D) \Rightarrow (X', f', D')$ is $(\boxed{\mathsf{P}}, p_2, D \Rightarrow D')$ given by the pullback

$$\begin{tikzcd}
\boxed{(X \Rightarrow X') \times_{X \Rightarrow \Gamma D'} \Gamma(D \Rightarrow D')} \arrow[r, "p_2"] \arrow[d, "p_1"] & \Gamma(D \Rightarrow D') \arrow[d, "\lambda(\Gamma ev \circ \cong \circ (id \times f))"] \\
X \Rightarrow X' \arrow[r, "id \Rightarrow f'"'] & X \Rightarrow \Gamma D'
\end{tikzcd}$$

where

$$\Gamma(D \Rightarrow D') \times \Gamma D \xrightarrow{\cong} \Gamma((D \Rightarrow D') \times D) \xrightarrow{\Gamma ev} \Gamma D'$$

. . . Joke . . .

“Is **everything** you do professionally, related to glue?”

“Is **everything** you do professionally, related to glue?”

The answer I gave is “yes”.

If you find this tricky to believe, I'd better give a proof ....

“Is **everything** you do professionally, related to glue?”

Proof: I spend quite a bit of time writing papers about gluing. Thus it suffices to show that, when I’m not writing such papers, I’m doing something connected with glue.

For the most part, when not writing about glue, I’m **stuck** (on a proof). Q.E.D.

We define a category $\mathcal{G}l(\Gamma)$, the "logical relations gluing of $\mathcal{D}$ to $\mathcal{S}et$ along Γ".

- The objects of $\mathcal{G}l(\Gamma)$ are triples $(X, \lhd, D)$ where

$$\lhd \subseteq X \times \Gamma D$$

- A morphism $(\theta, \phi)\colon (X, \lhd, D) \to (X', \lhd', D')$ is given by function $\theta\colon X \to X'$ and a morphism $\phi\colon D \to D'$ in $\mathcal{D}$ for which given $x \in X$ and $d \in \Gamma D$

$$x \lhd d \qquad \Longrightarrow \qquad \theta x \lhd' (\Gamma\phi) d$$

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The exponential $(X, \lhd, D) \Rightarrow (X', \lhd', D')$ is

$$(X \Rightarrow X', \lhd \Rightarrow \lhd', D \Rightarrow D')$$

where given $f \in X \Rightarrow X'$ and $F \in \Gamma(D \Rightarrow D')$

$$f(\lhd \Rightarrow \lhd')F \iff$$

$$(\forall x \in X) \;\; (\forall d \in \Gamma D)$$

$$(x \lhd d \Longrightarrow fx \lhd' ((\Gamma ev) \circ \cong)(F, d)$$

$\mathcal{G}l(\Gamma)$ is a CCC

The exponential $(X, \lhd, D) \Rightarrow (X', \lhd', D')$ is

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$$f(\lhd \Rightarrow \lhd')F \iff$$

$$(\forall x \in X) \ \ (\forall d \in \Gamma D)$$

$$(x \lhd d \Longrightarrow fx \lhd' ((\Gamma ev) \circ \cong)(F, d)$$

Notice that there is an inclusion functor

$$I : \mathcal{GL}(\Gamma) \to \mathcal{G}l(\Gamma)$$

since given any (X, f, D) with $f : X \to \Gamma D$, then trivially $f \subseteq X \times \Gamma D$.

Taking (X, f, D) and (X', f', D') in $\mathcal{GL}(\Gamma)$, the exponential in $\mathcal{G}l(\Gamma)$ is

$$(X \Rightarrow X', f \Rightarrow f', D \Rightarrow D')$$

where $f \Rightarrow f' \subseteq (X \Rightarrow X') \times \Gamma(D \Rightarrow D')$, and in fact it is rather nice that

$$\boxed{f \Rightarrow f' \quad = \quad P \stackrel{\text{def}}{=} (X \Rightarrow X') \times_{X \Rightarrow \Gamma D'} \Gamma(D \Rightarrow D')}$$

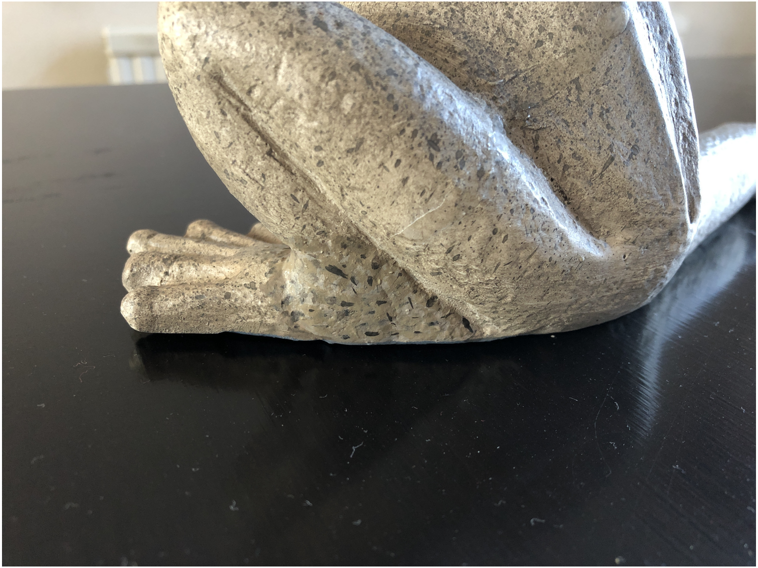
...Frog ...

Broken Frog

Epoxy Frog

Clappy Glued Frog

Happy Glued Frog



Let $\gamma\colon H \to \mathcal{P}(X)$ be a functor (monotone function) where H is a Heyting Algebra. Suppose also that γ preserves finite products (meets).

We define a poset $\mathcal{GL}(\gamma)$, the "gluing of H to $\mathcal{S}et$ along γ". This has elements (k, h) where $k \in \mathcal{P}(X)$, $h \in H$ and $k \leq_{\mathcal{P}(X)} \gamma h$. And order instances

$$(k, h) \leq (k', h')$$

just in case

$$\begin{tikzcd}
k \arrow[d, "\leq_{\mathcal{P}(X)}" description] \arrow[r, phantom, "\leq_{\mathcal{P}(X)}"] & \gamma h \arrow[d, "\leq_{\mathcal{P}(X)}" description] \\
k' \arrow[r, phantom, "\leq'_{\mathcal{P}(X)}"] & \gamma h'
\end{tikzcd}$$

The exponential $(k, h) \Rightarrow (k', h')$ is

$$((k \Rightarrow k') \wedge \gamma(h \Rightarrow h'), h \Rightarrow h')$$

Don't forget that $k \leq_{\mathcal{P}(X)} \gamma h$ is simply the subset $k \subseteq \gamma h$.

And that in $\mathcal{P}(X)$, the exponential $k \Rightarrow k'$ is $(X \setminus k) \cup k'$.

The meet is of course $\bigcap$ in $\mathcal{P}(X)$.

. . . Mug . . .

Broken Mug


2011A
Araldite
www.araldite-adhesives.com
2011B
Araldite
130g
www.araldite-adhesives.com
EVO-STIK
SERIOUS GLUE
Araldite
Rapid
Araldite

Fixed Mug


not absurd,
for it.
ALBERT EINSTEIN
max planck institut
informatik

I always thought of the (k, h) as being pairs satisfying a property, namely $k \subseteq \gamma h$.

I spent a little while thinking about a generalisation with objects (k, Φ, h) where $\Phi(k, h)$ is a property.

But the right thing to do is to note that we have an inclusion *function* $\iota\colon k \to \gamma h$.

And further that this is the *relation* $\mathbf{\Delta}_{k} \colon k \rightarrow \gamma h$ where $\mathbf{\Delta}_{k}$ is the diagonal on k.

We define a category $\mathcal{G}l(\gamma)$, the "logical relations gluing of H to $\mathcal{S}et$ along γ".

- The objects of $\mathcal{G}l(\gamma)$ are triples $(k, \lhd, h)$ where $\lhd \subseteq k \times \gamma h$.
- The order $(k, \lhd, h) \leq (k', \lhd', h')$ is pointwise. The pointwise order on the first and third components makes the order on the second components type-correct.

$\mathcal{G}l\Gamma$ is a Heyting Algebra

The exponential $(k, \lhd, h) \Rightarrow (k', \lhd', h')$ is

$$(k \Rightarrow k', \lhd \Rightarrow \lhd', h \Rightarrow h')$$

where given $x \in k \Rightarrow k'$ and $y \in \gamma(h \Rightarrow h')$

$$x (\lhd \Rightarrow \lhd') y \quad \iff \quad (x \lhd y \quad \Longrightarrow \quad x \lhd' y)$$

Notice that there is an inclusion functor

$$I \colon \mathcal{GL}(\gamma) \to \mathcal{G}l(\gamma)$$

since given any (k, h) with $k \leq \gamma h$, then trivially $\Delta_k \subseteq k \times \gamma h$. So $I(k, h) \stackrel{\text{def}}{=} (k, \Delta_k, h)$.

Taking (k, h) and (k', h') in $\mathcal{GL}(\Gamma)$, the exponential (of the images) in $\mathcal{G}l(\gamma)$ is

$$(k \Rightarrow k', \Delta_k \Rightarrow \Delta_{k'}, h \Rightarrow h')$$

where $\Delta_k \Rightarrow \Delta_{k'} \subseteq (k \Rightarrow k') \times \gamma(h \Rightarrow h')$ and in fact I think it is definitely rather nice that

$$\boxed{\Delta_k \Rightarrow \Delta_{k'} = (k \Rightarrow k') \wedge \gamma(h \Rightarrow h')}$$

- Artin exponential $\cong$ logical relation: appears in my Applied Categorical Structures 1996 paper, *On Fixpoint Objects and Gluing Constructions* ($\dagger$).
- There are many "gluing techniques" results. While they have overlaps, to me they often feel disconnected.
- It would be good to write a paper that presents the theory of gluing in a single framework.
- How general can the target categories for Γ be while still facilitating logical relations?
  - In ($\dagger$) it is $\omega\mathcal{CPO}^{\mathcal{C}}$.