Approximation algorithms for geometric intersection graphs

Thomas Erlebach

University of Leicester

Based on joint work with:
Christoph Ambühl, Klaus Jansen, Erik Jan van Leeuwen,
Matúš Mihaľák, Marc Nunkesser, Eike Seidel

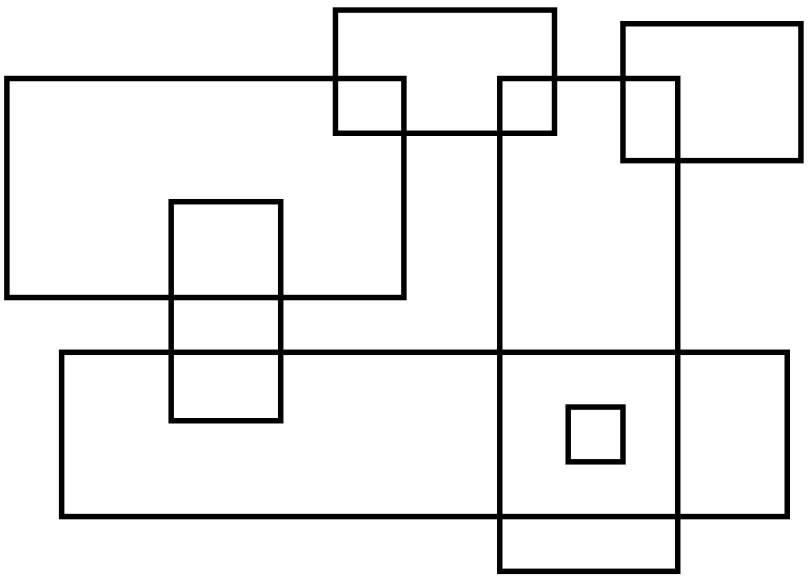
Outline

- **Introduction**
- **Independent sets in disk graphs**
  - A PTAS for unit disks
  - A PTAS for arbitrary disks
- **Dominating sets**
  - Unweighted dominating sets in unit disk graphs
  - Why the PTAS techniques don't seem to work for the weighted case or arbitrary disks
  - A constant-factor approximation algorithm for weighted dominating sets in unit disk graphs
  - Further results on dominating sets
- **Some open problems**

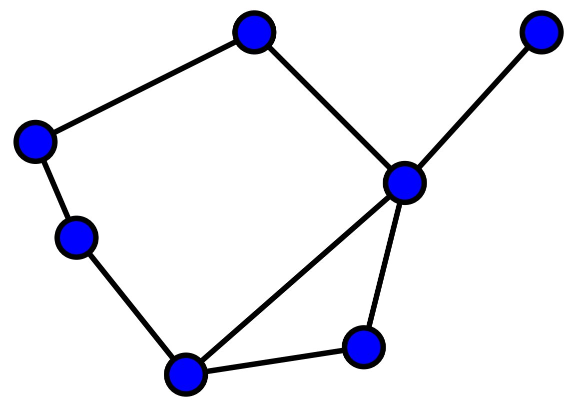
What are geometric intersection graphs?

- **vertices** = geometric objects
- **edges** = non-empty intersection between objects

Example: a rectangle intersection graph



geometric representation



intersection graph

Popular geometric intersection graphs

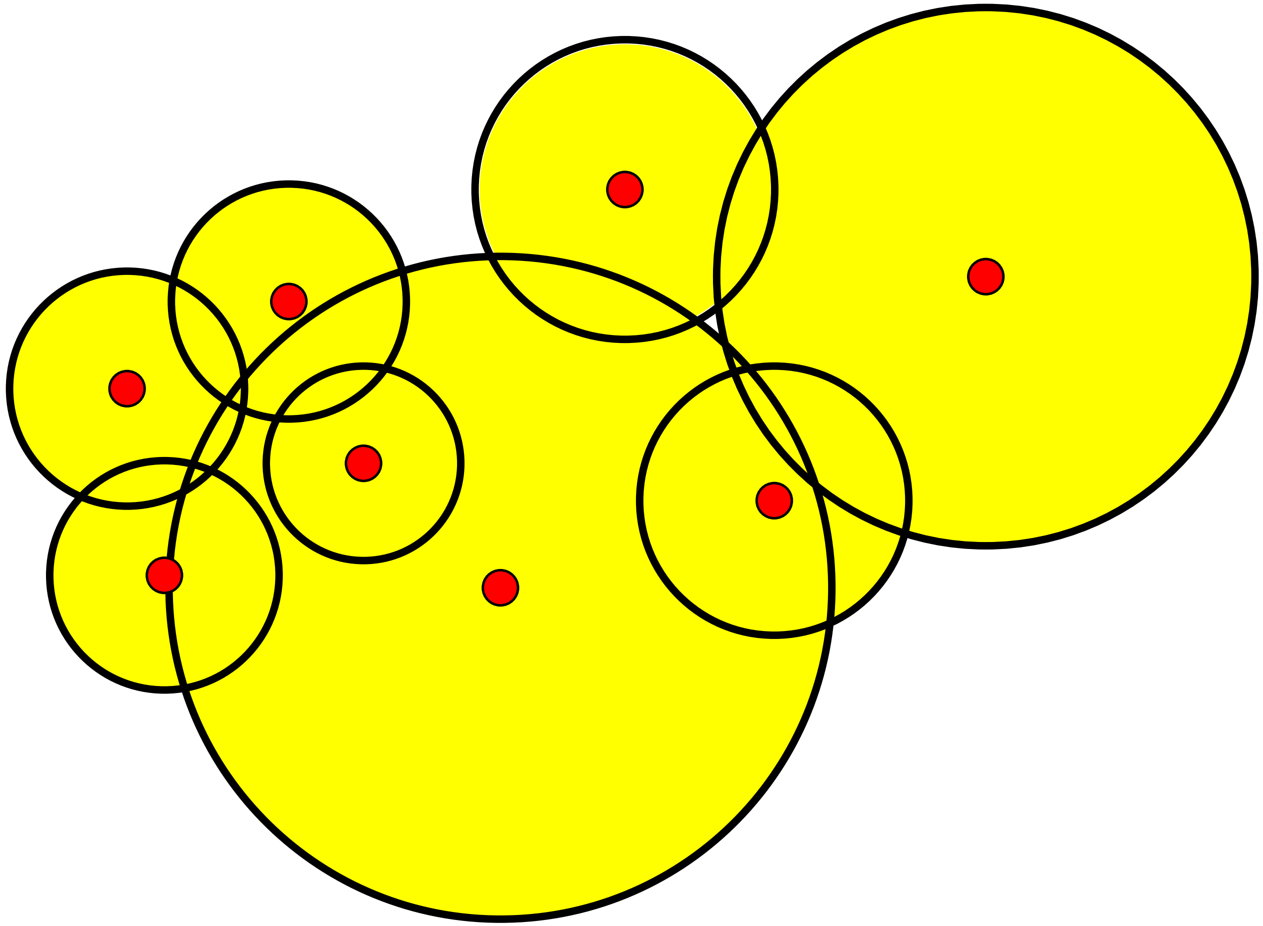
- disks (➜ **disk graphs**), squares
- "fat" objects
- ellipses, rectangles (axis-aligned), arbitrary convex objects
- line segments, curves, higher-dimensional objects

The **recognition problem is typically NP-hard**!!

Some Applications:

- ⇨ Wireless networks (frequency assignment problems)
- ⇨ Map labeling
- ⇨ Resource allocation (e.g. admission control in line networks)

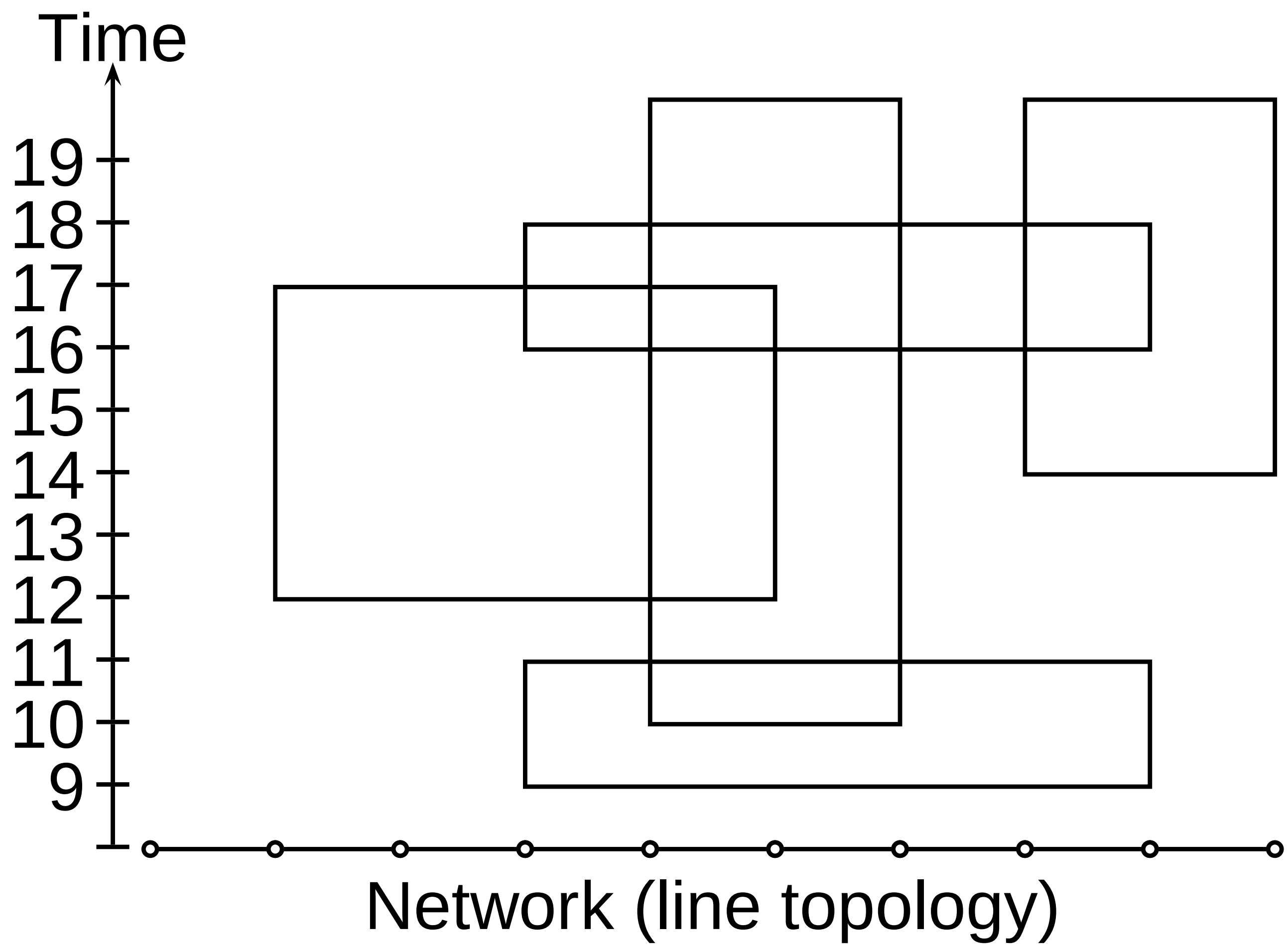
Application: Wireless networks



Application: Map labeling

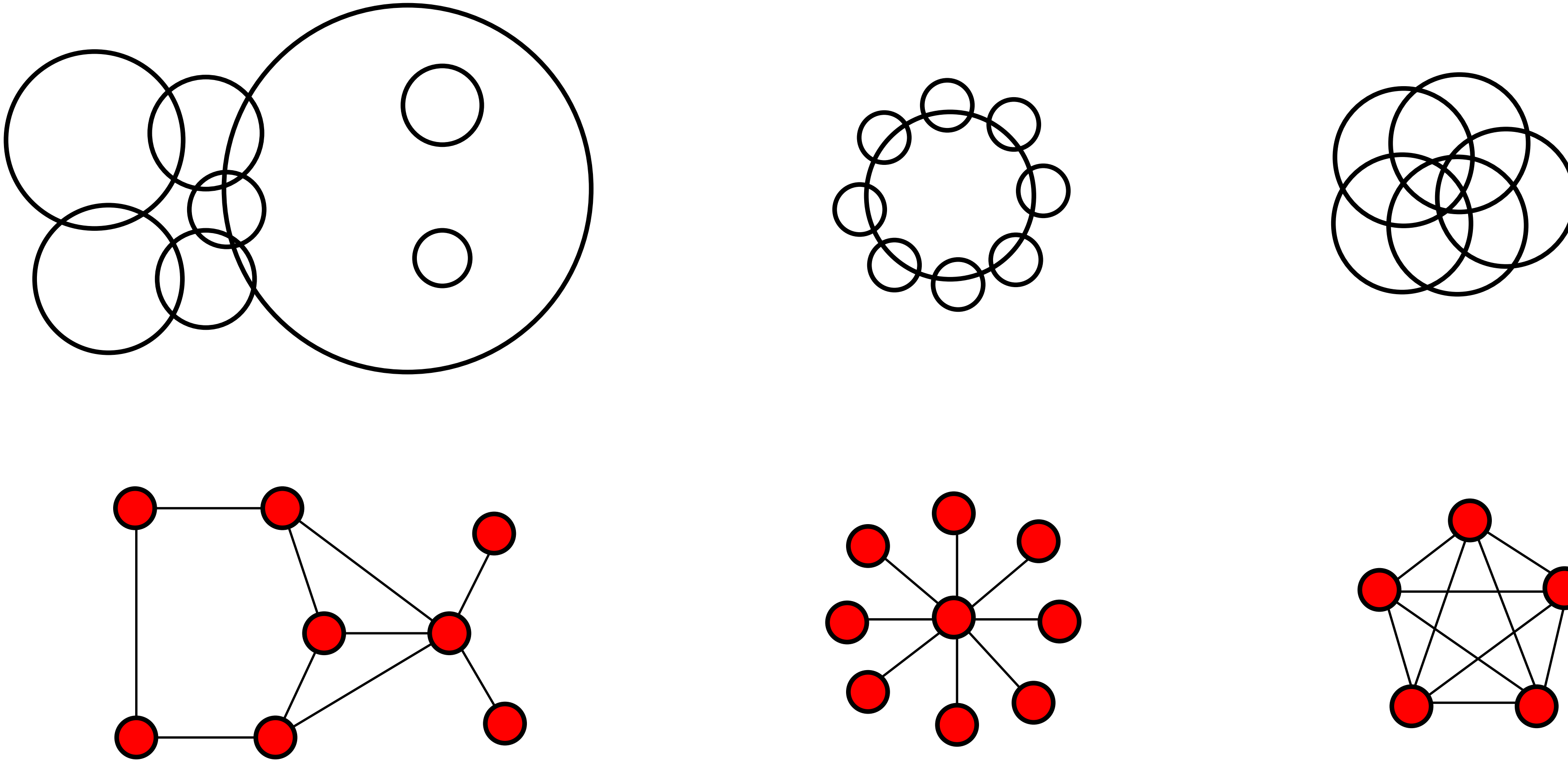
(illustration taken from a paper by van Kreveld, Strijk, Wolff)

Application: Call admission control



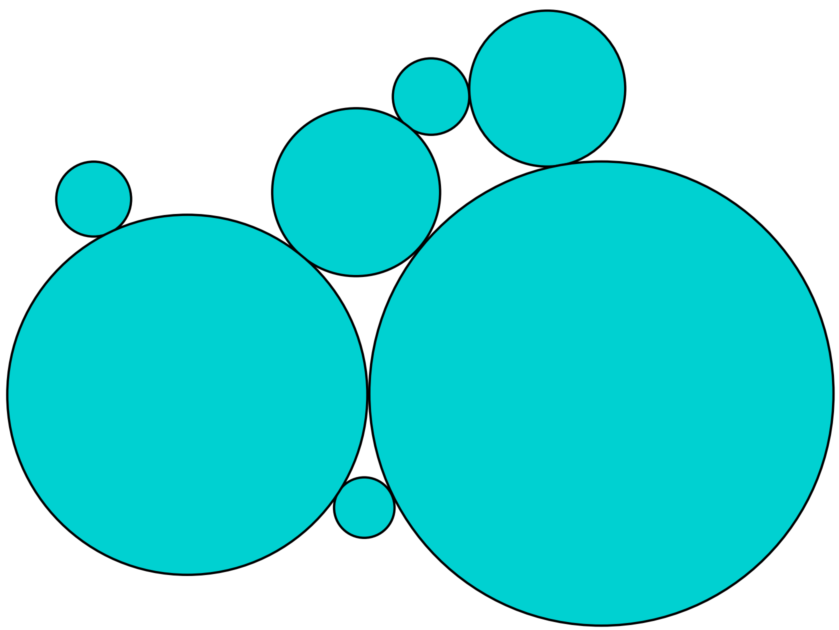
Disk graphs

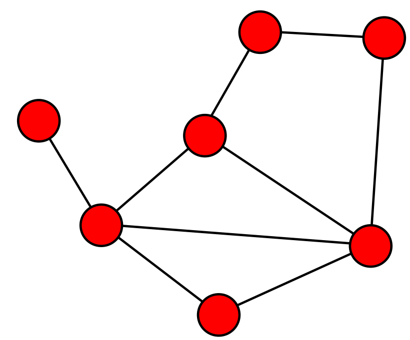
. . . are the intersection graphs of disks in the plane:



Subclasses of disk graphs

- **Unit disk graphs**: all disks have diameter 1
- **Coin graphs**: touching graphs of disks whose interiors are disjoint



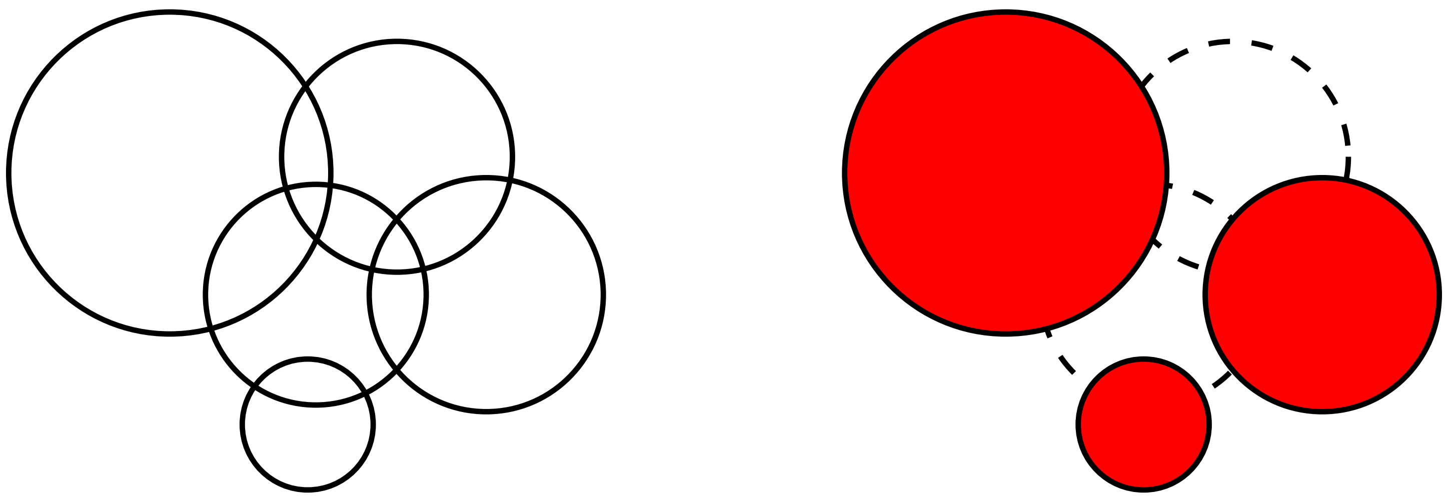


Coin graphs are exactly the planar graphs! [Koebe, 1936]

Maximum Independent Set

Maximum Independent Set (MIS)

Input: a set $\mathcal{D}$ of disks in the plane
Feasible solution: subset $A \subseteq \mathcal{D}$ of disjoint disks
Goal: maximize $|A|$



In the weighted case (MWIS), each disk is associated with a positive weight.

Approximation algorithms for MIS

An algorithm for MIS is a **ρ-approximation algorithm** if it

- ➢ runs in **polynomial time** and
- ➢ always outputs an independent set of **size at least** OPT/ρ, where OPT is the size of the optimal independent set.

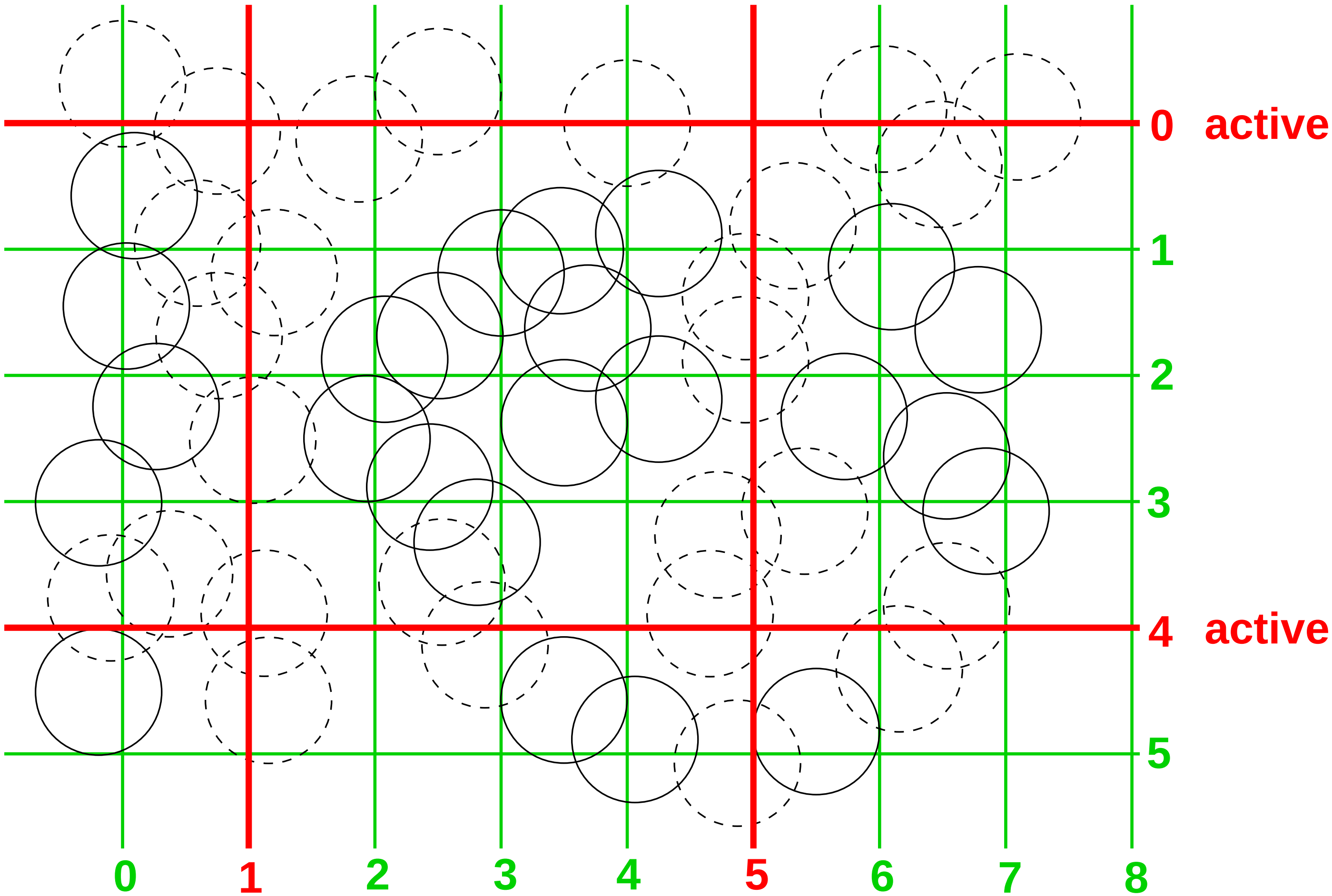
A **polynomial-time approximation scheme (PTAS)** is a family of $(1+\varepsilon)$-approximation algorithms for every constant $\varepsilon > 0$.

For MWIS, the definitions are analogous.

Known results for MIS in disk graphs

- **Unit disk graphs:**
  - $\mathcal{NP}$-hard [Clark, Colbourn, Johnson 1990].
  - Greedy gives a 5-approximation, and even a 3-approximation if applied from top to bottom [Marathe et al., 1995]
  - PTAS [Hunt III et al., 1994], based on the shifting strategy [Baker, 1984; Hochbaum and Maass, 1985]
- **Arbitrary disk graphs:**
  - PTAS [E, Jansen, and Seidel, 2001; Chan, 2001], using shifting strategy on multiple layers
- The PTASs generalize to squares, regular polygons, or, more generally, arbitrary fat objects.
- They also generalize to MWIS.

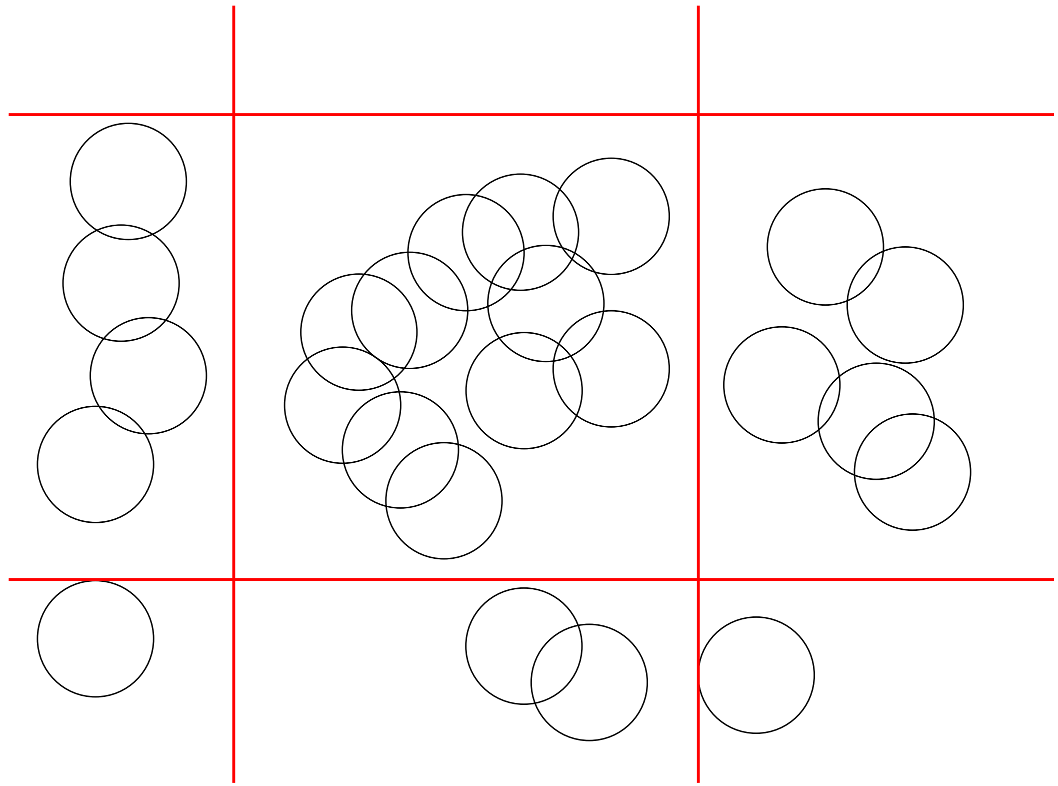
Shifting strategy for unit disk graphs



Remove disks hitting active lines (and shift active lines).

Brute-force solution to subproblems

Active lines partition the plane into squares that can be considered independently:



➥ Compute max independent set I in each square by brute-force enumeration. As $|I| = O(k^2)$, time $n^{O(k^2)}$.

PTAS for MIS in unit disk graphs

❶ For $0 \leq r, s < k$, get $\mathcal{D}(r,s)$ from $\mathcal{D}$ by deleting disks that

- ➜ hit a horizontal line equal to r modulo k or
- ➜ hit a vertical line equal to s modulo k.

❷ Compute the max independent set I_S in each $k \times k$ square S of $\mathcal{D}(r,s)$ by brute-force enumeration.

❸ The union of the sets I_S gives a maximum independent set in $\mathcal{D}(r,s)$.

❹ Output the largest independent set obtained in this way.

Running-time: $n^{O(k^2)}$ for n disks. (Can be made $n^{O(k)}$.)

Approximation: Solution is at least $\left(1 - \frac{2}{k}\right) \mathrm{OPT}$.

Arbitrary disk graphs

❶ Classify the disks into layers according to their sizes.

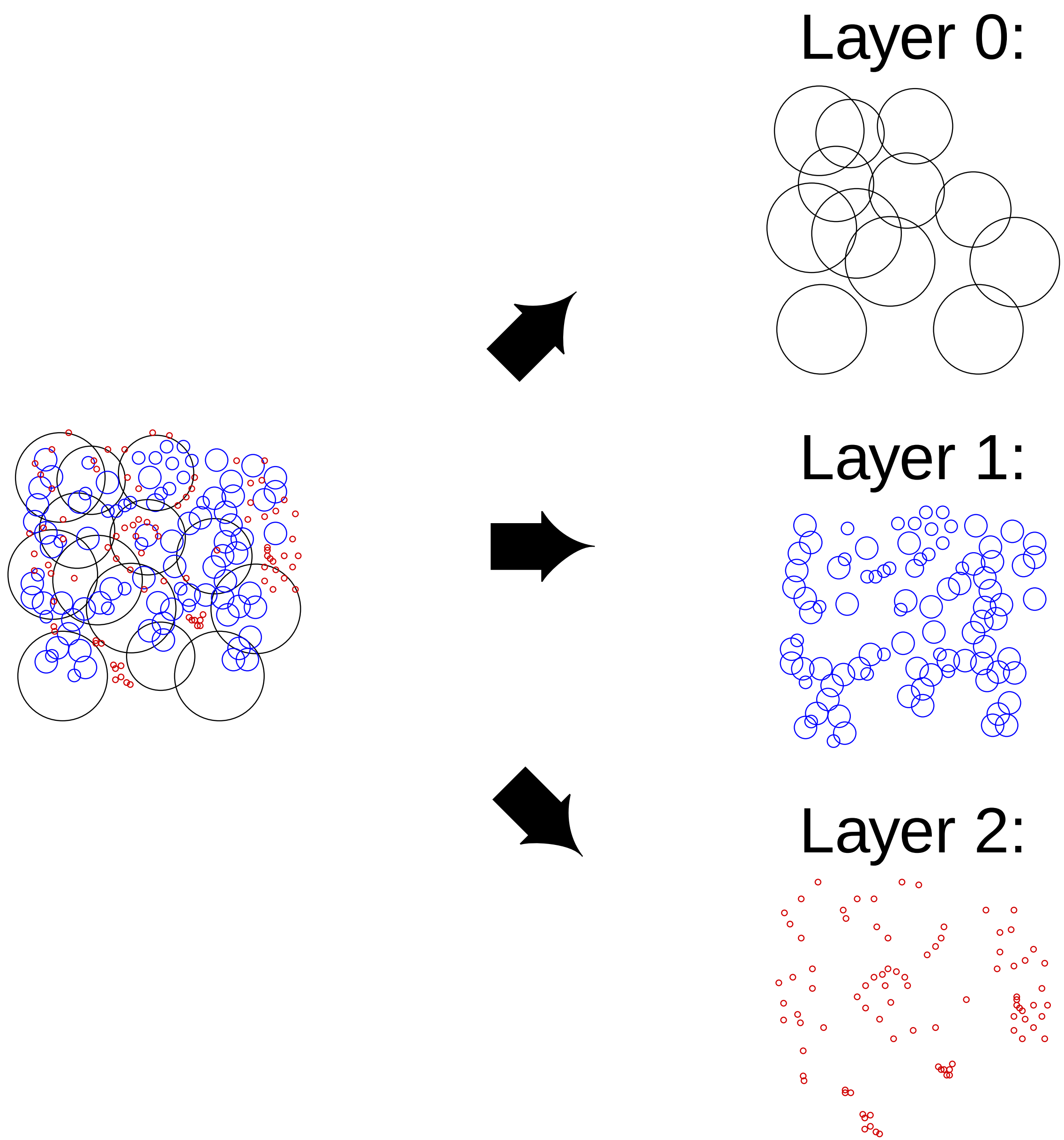
❷ Use the shifting strategy on all layers simultaneously.

❸ After removing all disks that hit active lines, use dynamic programming to compute a maximum independent set.

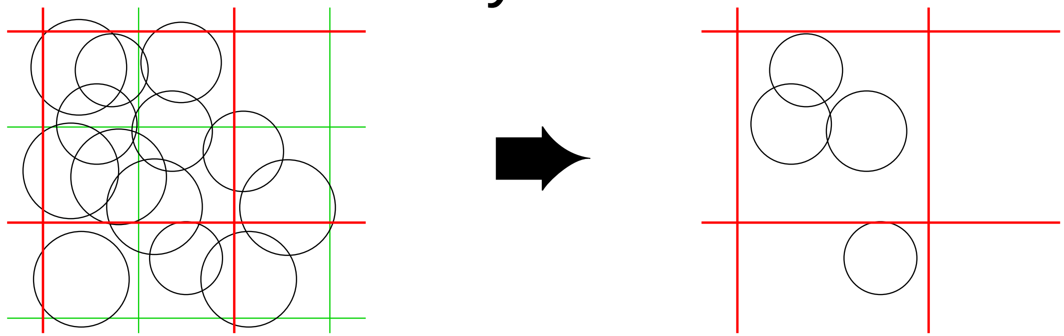
Classification into layers:

- Assume that the largest disk has diameter 1.
- Layer ℓ: disks with diameter d, $\frac{1}{(k+1)^\ell} \geq d > \frac{1}{(k+1)^{\ell+1}}$.
- Lines on layer ℓ are $\frac{1}{(k+1)^\ell}$ apart, every k-th line is active.

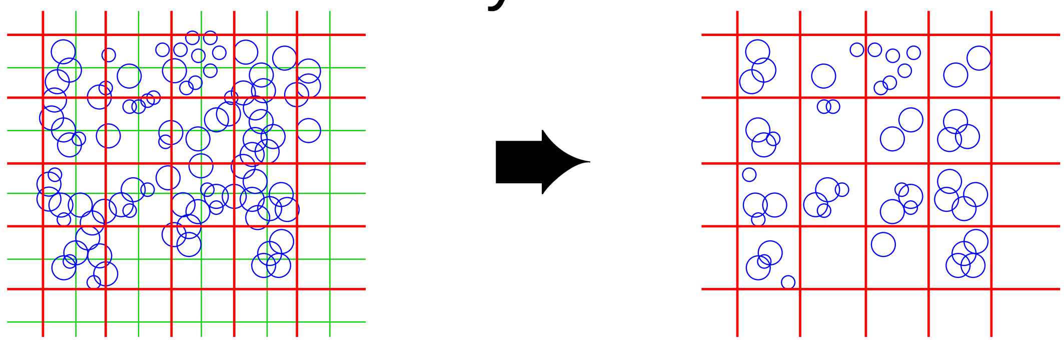
Partition into layers



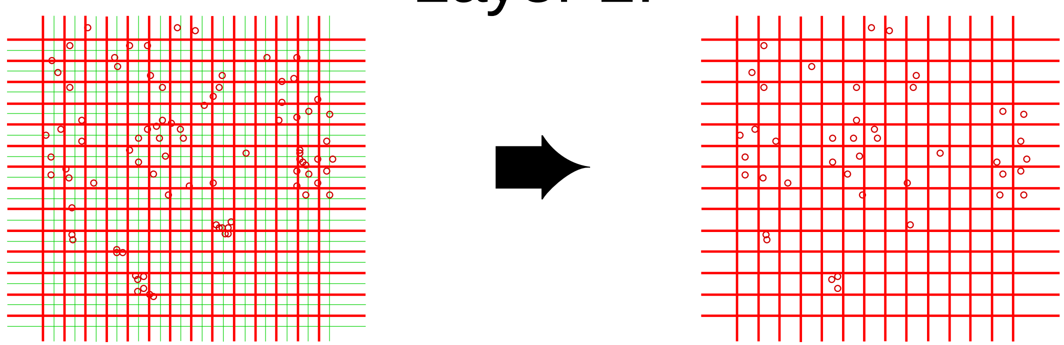
Layer 0:



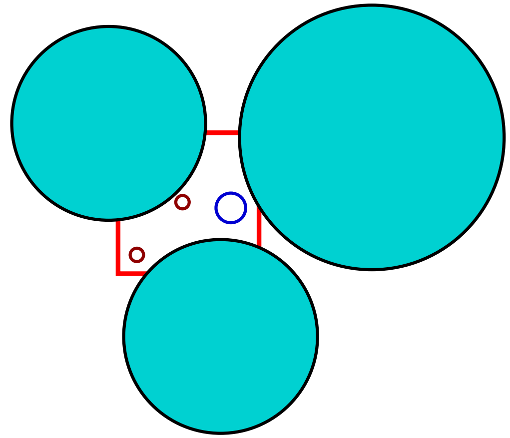
Layer 1:



Layer 2:



Dynamic programming idea



For square S on level ℓ, compute for every independent set I of larger disks intersecting S, a maximum independent set of smaller disks inside S that can be added to I. Time $n^{O(k^2)}$.

Works for weighted case, for higher dimensions, and for arbitrary disk-like or fat objects:

- All we need is: The number of disjoint large objects that can intersect a box S of side length k is bounded by a function of k.

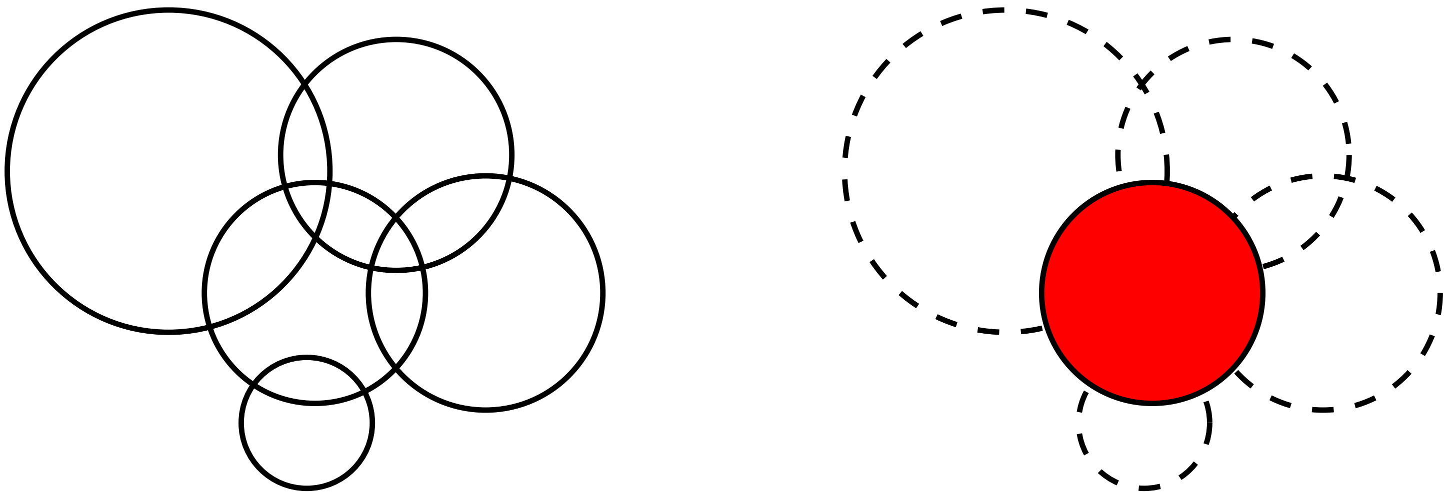
Recent related results

- [Nieberg, Hurink, Kern, 2004] PTAS for maximum weight independent set in unit disk graphs without given representation.
- [Marx, 2005] Maximum independent set in unit disk graphs is W[1]-hard. (➠ No FPT algorithm and no EPTAS unless FPT=W[1].)
- [van Leeuwen, 2005] Asymptotic FPTAS for maximum independent set (and various other problems) in unit disk graphs of bounded density.

Minimum Dominating Set

Minimum Dominating Set (MDS)

Input: a set $\mathcal{D}$ of disks in the plane
Feasible solution: subset $A \subseteq \mathcal{D}$ that dominates all disks
Goal: minimize $|A|$



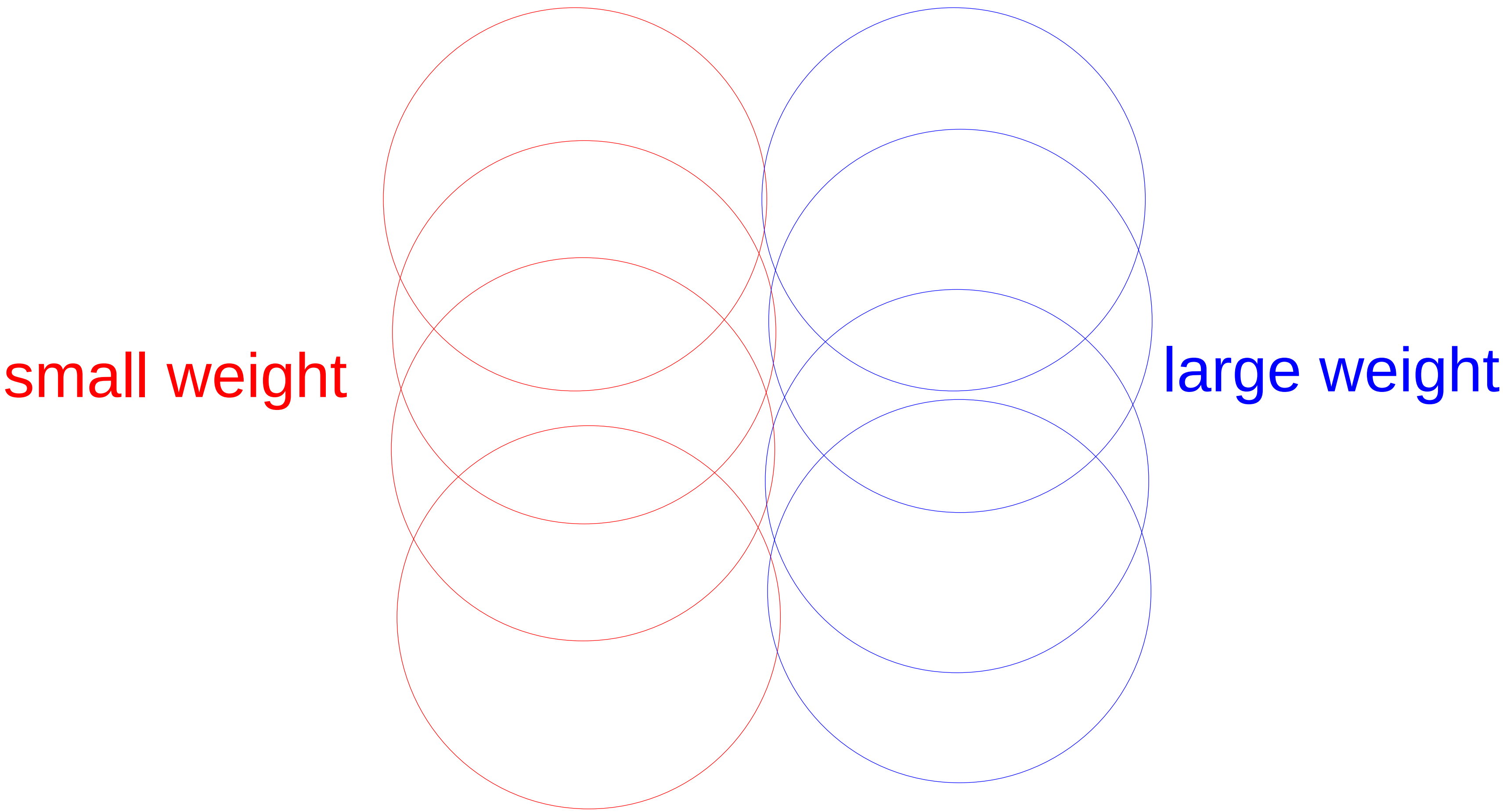
In the weighted case (MWDS), each disk is associated with a positive weight.

Known results for MDS

- In **arbitrary graphs**, ratio $\Theta(\log n)$ is best possible (unless $P = NP$). [Feige '96; Arora and Sudan '97]
- For **MDS in unit disk graphs**, a PTAS can be obtained using the shifting strategy [Hunt III et al., 1994]:
  - Any maximal independent set is a dominating set.
  - Therefore, the smallest dominating set in a square can be found in polynomial time by enumeration.
  - Special treatment of disks on square boundaries.
- **Questions:**
  - What about MWDS in unit disk graphs? (backbone formation in wireless ad-hoc networks, [Wang&Li '05])
  - What about MDS (and MWDS) in arbitrary disk graphs (or intersection graphs of fat objects)?

MWDS can be arbitrarily large . . .

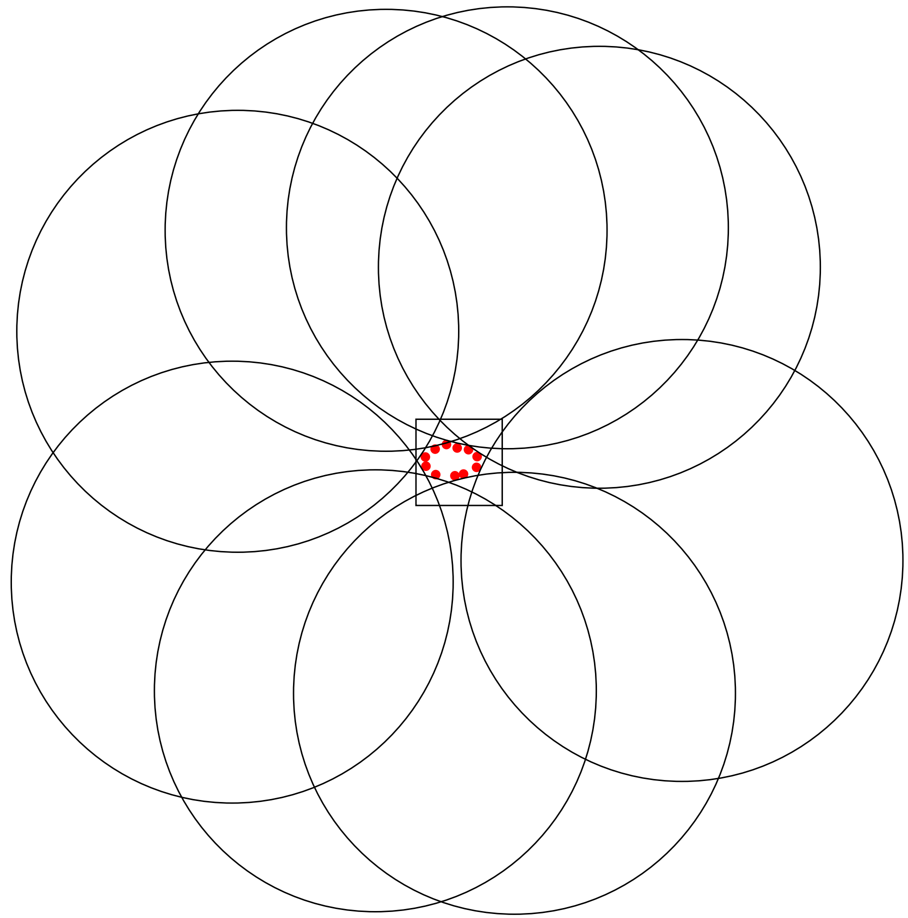
. . . for unit disks in an area of constant size:



➠ Brute-force enumeration does no longer work.

MDS can use arbitrarily many . . .

. . . larger disks intersecting a square:



➠ Dynamic programming table size is no longer polynomial.

MWDS in unit disk graphs

Theorem. [Ambühl, E, Mihaľák, Nunkesser, 2006]
There is a constant-factor approximation algorithm for MWDS in unit disk graphs.

Ideas:

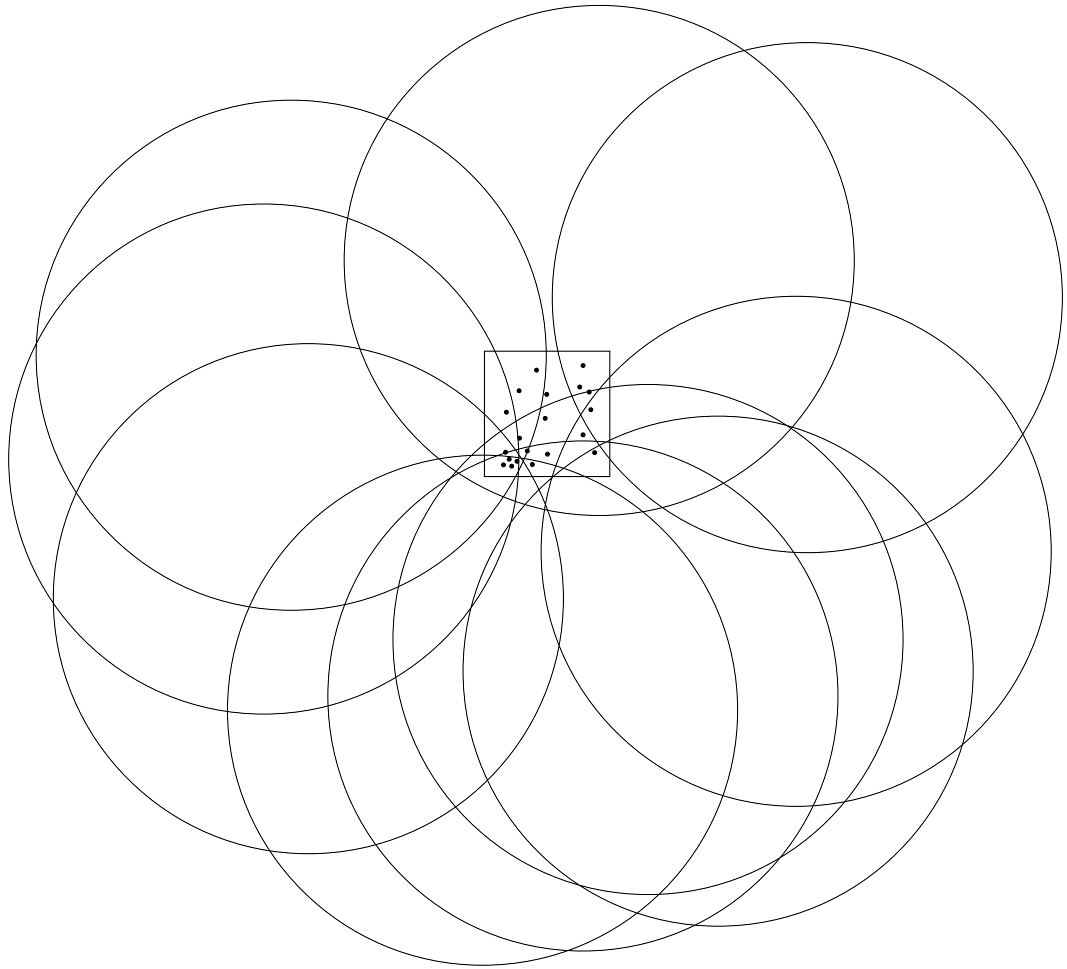
- Partition the plane into squares and solve the problem for each square separately.
- In each square, reduce the problem to the problem of covering points with weighted disks.
- Use enumeration techniques (guess properties of OPT) and dynamic programming to solve the latter problem.

The constant factor is currently 72.

The subproblem for each square

- Find a dominating set for the square:
  - Let $\mathcal{D}_S$ denote the set of disks with center in a 1×1 square S.
  - Let $N(\mathcal{D}_S)$ denote the disks in $\mathcal{D}_S$ and their neighbors.
  - **Task:** Find a minimum weight set of disks in $N(\mathcal{D}_S)$ that dominates all disks in $\mathcal{D}_S$.
- Reduces to covering points in a square with weighted disks:
  - Let P be a set of points in a $\frac{1}{2} \times \frac{1}{2}$ square S.
  - Let $\mathcal{D}$ be a set of weighted unit disks covering P.
  - **Task:** Find a minimum weight set of disks in $\mathcal{D}$ that covers all points in P.

Covering points by weighted disks



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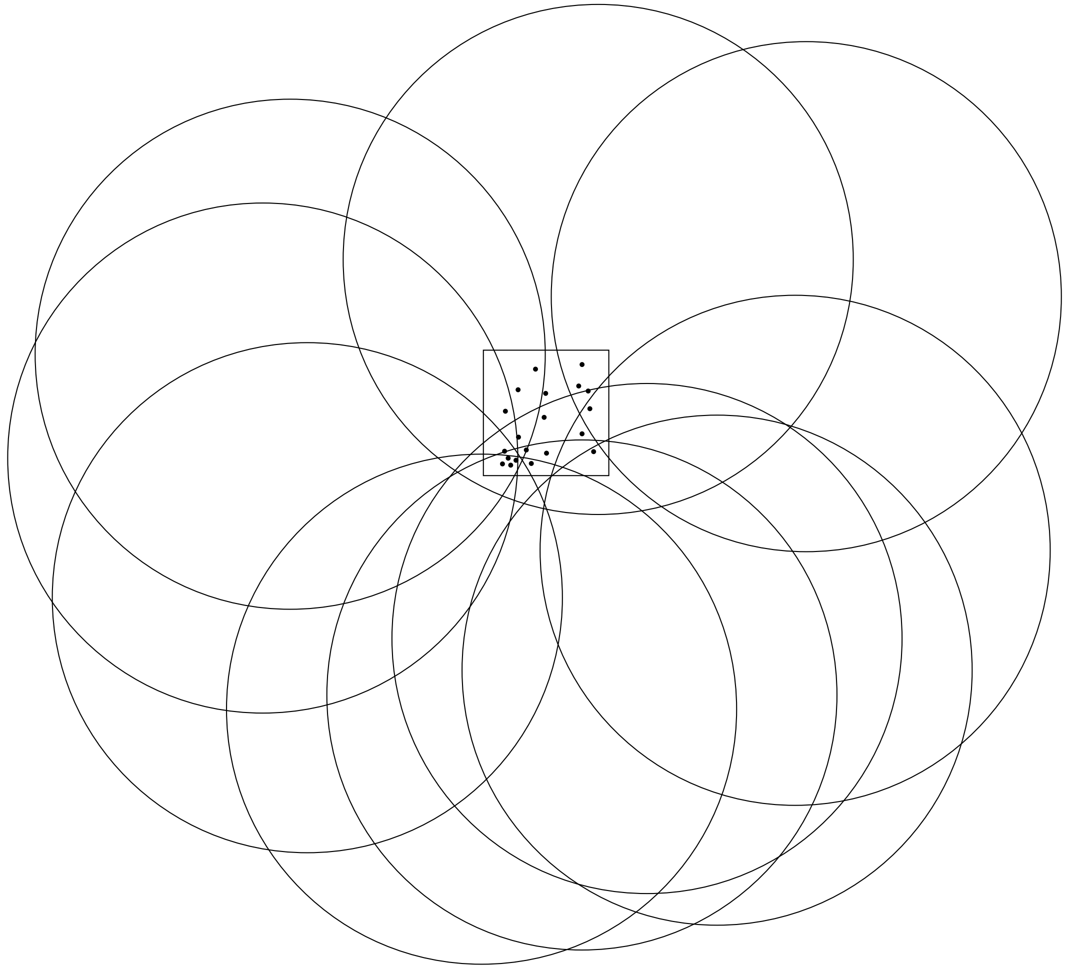
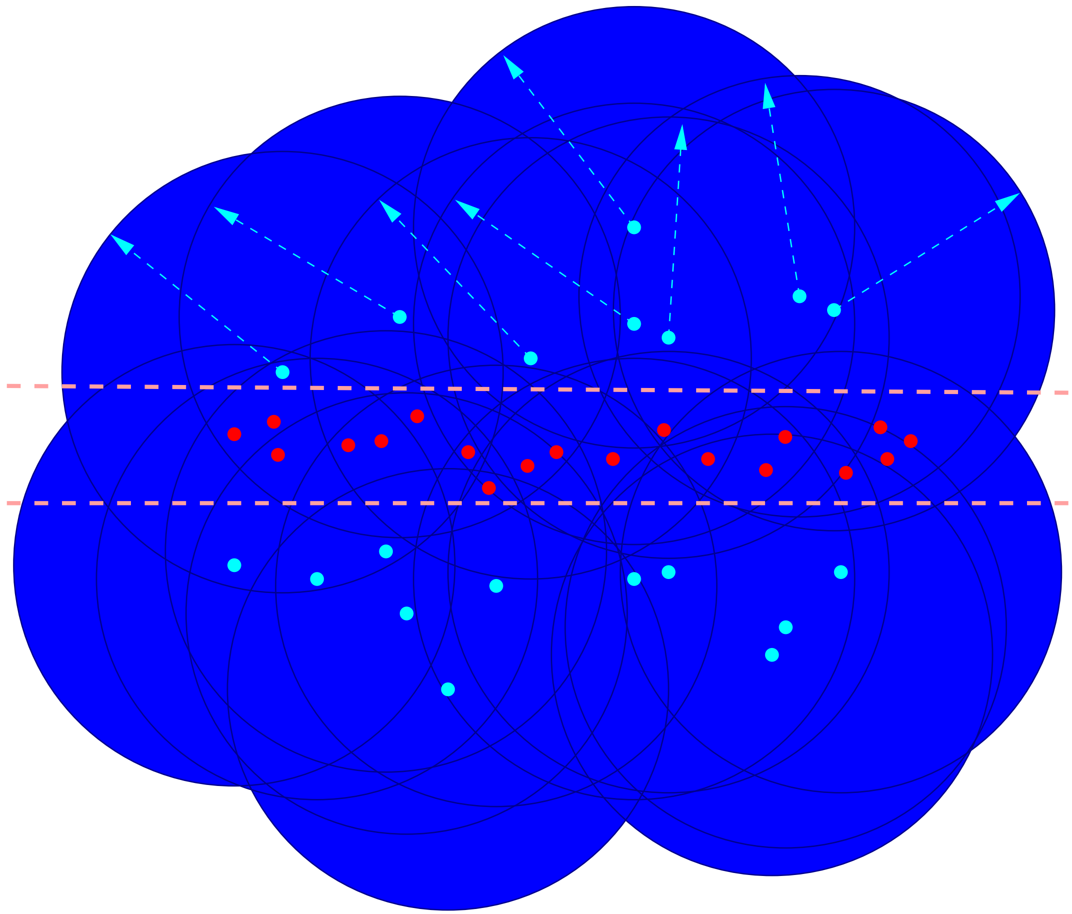


Remark. $O(1)$-approximation algorithms are known for unweighted disk cover [Brönninmann and Goodrich, 1995].

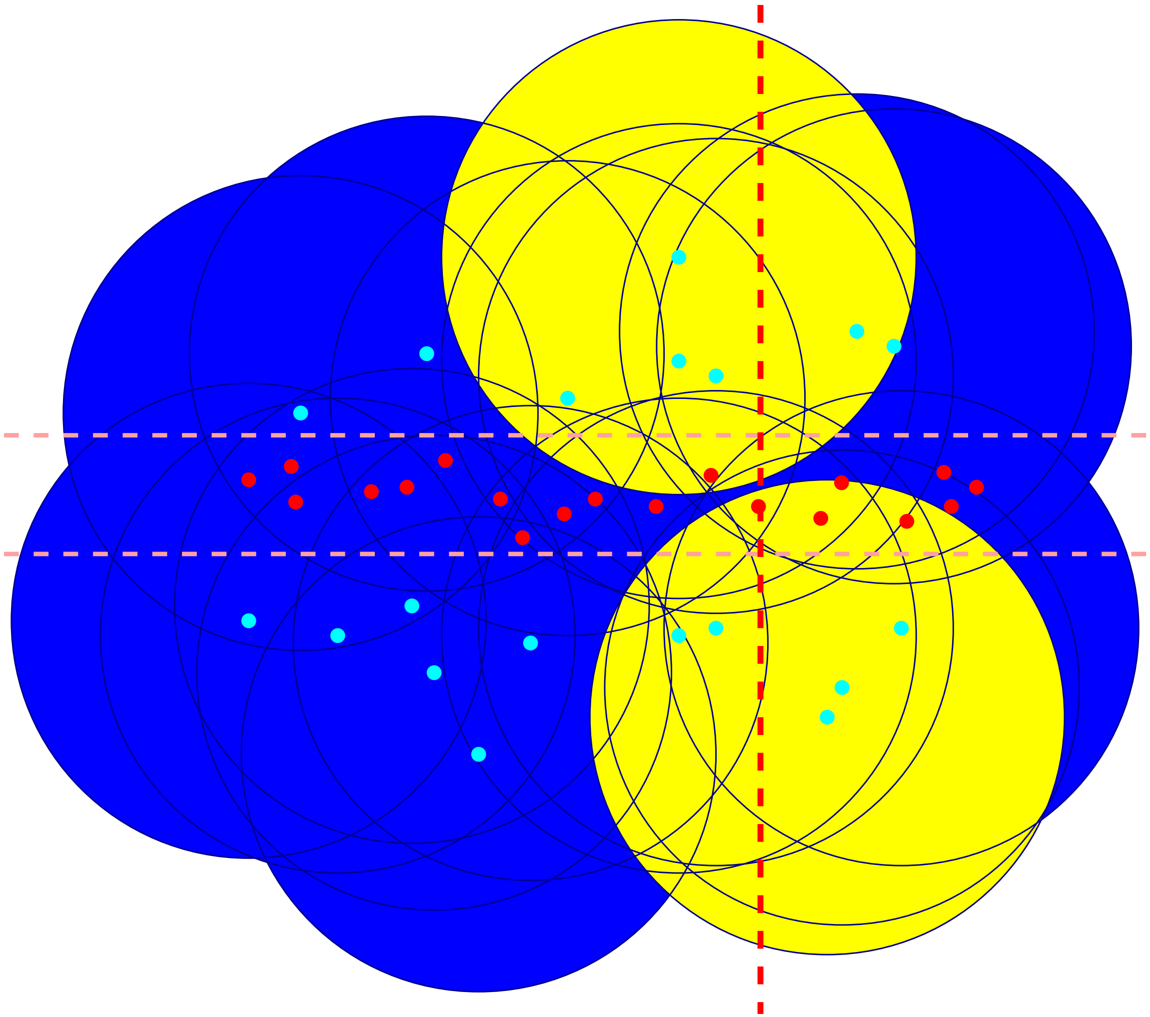
Polynomial-time solvable subproblem

- Given a set of points in a strip, and a set of weighted unit disks with centers outside the strip, compute a minimum weight set of disks covering the points.



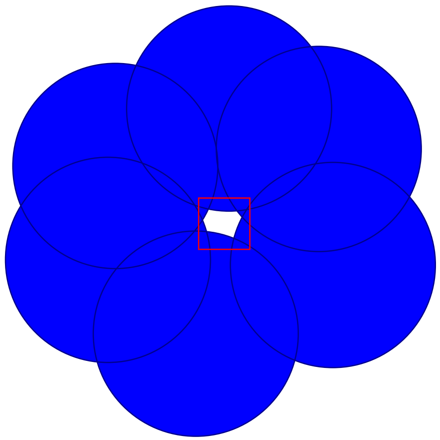
Dynamic programming

- Vertical sweepline, table entry for every pair of disks that could be on the lower and upper envelope:

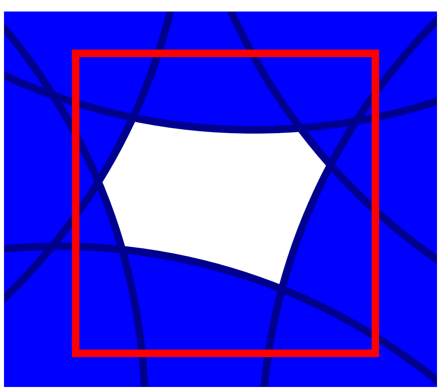


Main cases: One hole or many holes

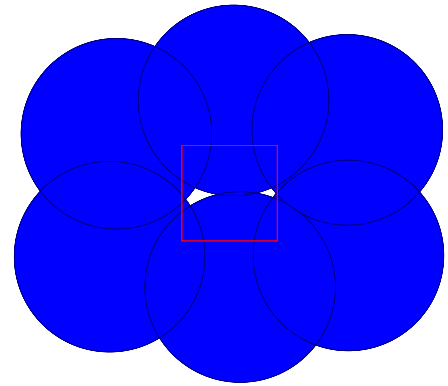
One-hole case:



Enlarged:



Many-holes case:



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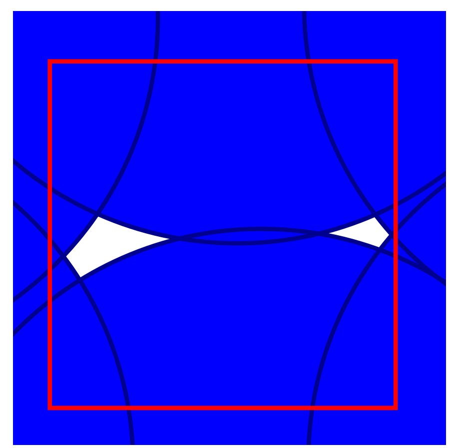
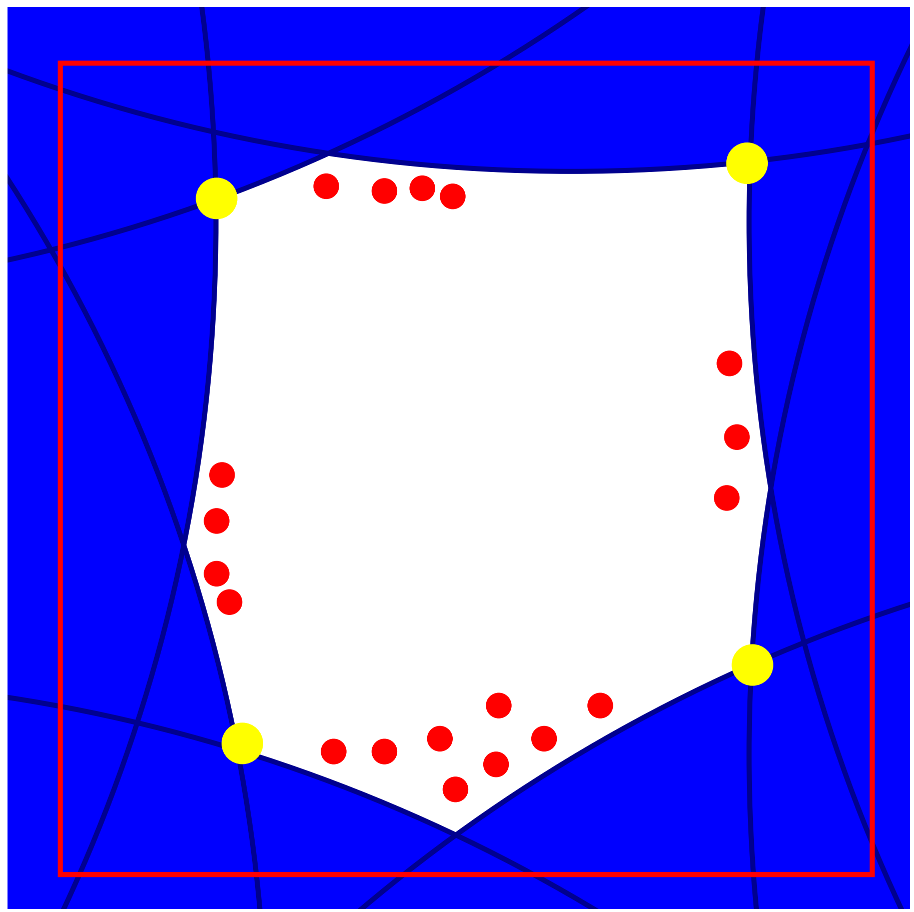


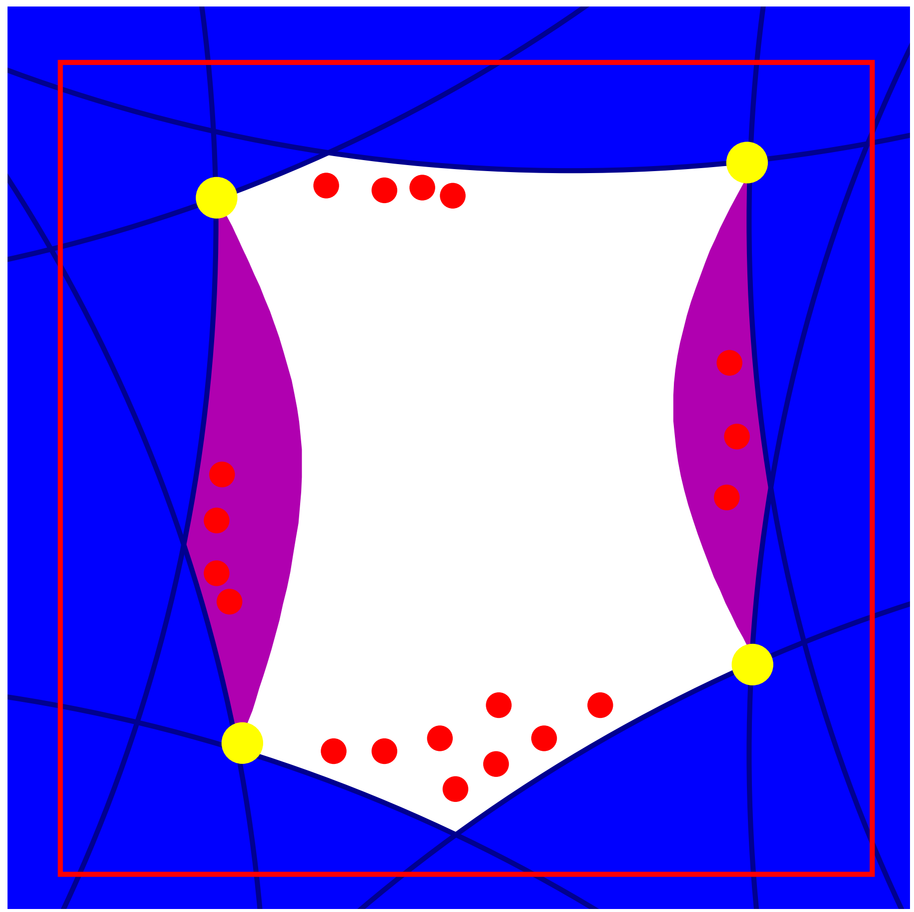
Sketch of the one-hole case

Step 1: Guess the four “corner points” of the optimal solution (each of them is defined by two disks).



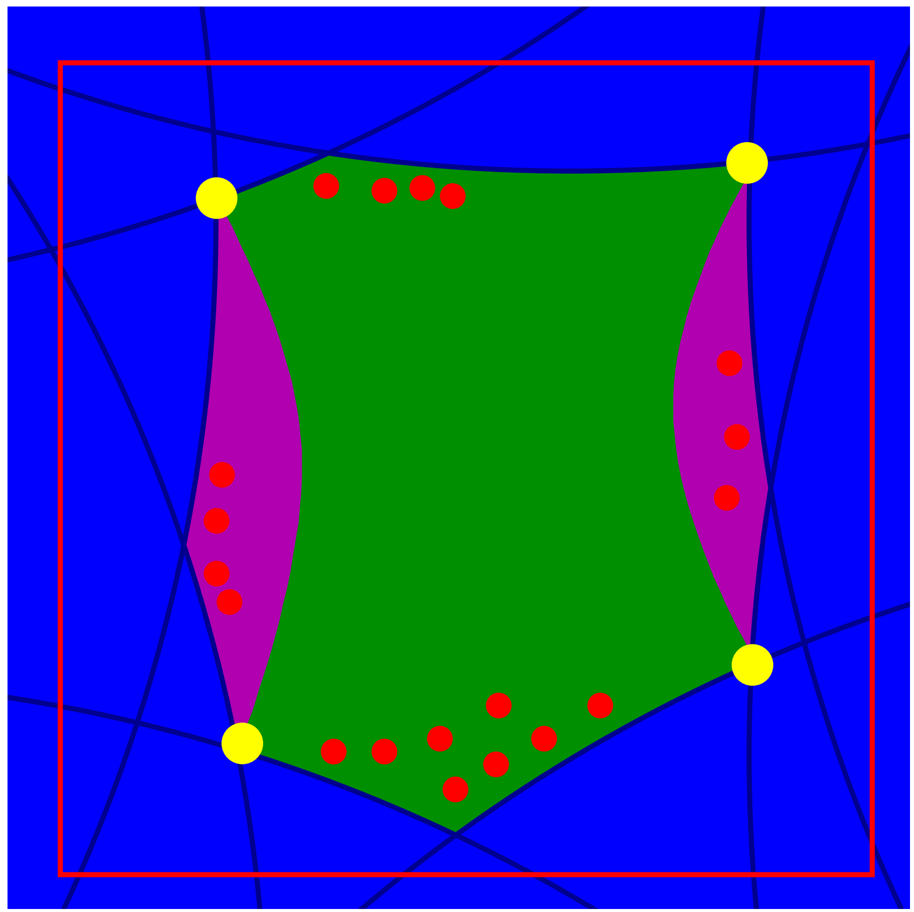
Sketch of the one-hole case

Step 2: Two regions that can only be covered with disks whose centers are to the left or right of the square.



Sketch of the one-hole case

Step 3: Remaining area can only be covered with disks whose centers are above or below the square.



Summary: MWDS in unit disk graphs

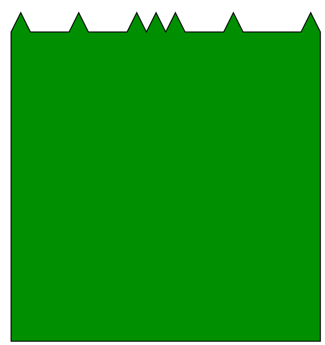
- For each square, reduce the weighted dominating set problem to a weighted disk cover problem.
- Distinguish one-hole case and many-holes case.
- In both cases, we have a 2-approximation algorithm for covering points in a square with weighted unit disks.
- This implies the constant-factor approximation algorithm for MWDS in unit disk graphs.
- The algorithm extends to the minimum weight connected dominating set problem in unit disk graphs.

Further results on MDS and MWDS

Theorem. [E, van Leeuwen 2006] For disk graphs with bounded ply, there is a $(3+\varepsilon)$-approximation algorithm for MWDS.

Theorem. [E, van Leeuwen 2006] For rectangle intersection graphs, MDS is APX-hard.

Theorem. [E, van Leeuwen 2006] For intersection graphs of “squares with bumps”, MDS cannot be approximated within $o(\log n)$ unless $P = NP$.



Open Problems

Disk graphs

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- What is the best possible approximation ratio for minimum dominating set in disk graphs:

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  - Is the problem APX-hard?

Disk graphs

- Is there a PTAS for disk graphs with bounded ply?
- What is the best possible approximation ratio for minimum dominating set in disk graphs:
  - Is there an $O(1)$-approximation algorithm or even a PTAS?
  - Is the problem APX-hard?
- What is the complexity of the maximum clique problem in disk graphs?
  (polynomial for unit disk graphs [Clark et al., 1990], NP-hard for ellipses [Ambühl, Wagner 2002])

Rectangle intersection graphs

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- Can we achieve approximation ratio $o(\log n)$ for MDS and MWDS?
- Can rectangle intersection graphs be **colored** with $O(\omega)$ colors, where ω is the clique number?
  (best known upper bound: $O(\omega^2)$ colors [Asplund and Grünbaum, 1960])

Thank you!