Queries, Modalities, Relations, Trees, XPath
Lecture II
Preliminaries: Sets, Relations, Structures

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Sets

$$A := \{0, 1, 2, 3, 4\}$$
$$B := \{0, 1, 1, 2, 2, 3, 4, 4\}$$
$$C := \{x \in \mathrm{N}_0 \mid x \leq 4\}$$

Exercise

1. Is $A = B$?
2. Is $B = C$?
3. Is $A = C$?

Definition (Cardinality)

The number of elements of a set A is called its cardinality and denoted by $|A|$.

For infinite A, in these lectures we just write $|A| = \infty$.

(There is a theory of cardinality for infinite sets, but we do not have to enter into this)

Exercise

1. What is $|\emptyset|$?
2. What is $|\{0, 1, 2, 3, 4, 5\}|$?
3. What is $|\{x \in \mathrm{N}_0 \mid x \leq 4 \text{ and } x \text{ is even}\}|$?
4. What is $|\{x \in \mathrm{N}_0 \mid x \leq 4 \text{ and } x \text{ is odd}\}|$?

Subsets and Powersets

Definition

- $A \subseteq B$ if $\quad \forall a \in A. a \in B$ [subset]
- $\mathcal{P}(B) := \{A \mid A \subseteq B\}$ [powerset]

Exercise

1. Is $\emptyset \in \{\{\emptyset\}\}$?
2. Is $\{\emptyset\} \in \{\{\emptyset\}\}$?
3. Is $\emptyset \subseteq \{\{\emptyset\}\}$?
4. Is $\{\emptyset\} \subseteq \{\{\emptyset\}\}$?
5. Is $\emptyset \in \mathcal{P}(\{\{\emptyset\}\})$?
6. Is $\{\emptyset\} \in \mathcal{P}(\{\{\emptyset\}\})$?
7. Is $\emptyset \subseteq \mathcal{P}(\{\{\emptyset\}\})$?
8. Is $\{\emptyset\} \subseteq \mathcal{P}(\{\{\emptyset\}\})$?

Ordered Pairs

Definition (Ordered Pair)

$$\langle a, b \rangle := \{\{a\}, \{a, b\}\}$$

Exercise

1. Is $\langle 0, 1 \rangle = \langle 1, 0 \rangle$?
2. Is $\langle 0, 0 \rangle = \{0\}$?

Ordered Tuples

Definition (Ordered Tuple)

$$\langle a_0, \ldots, a_n \rangle := \langle \langle a_0, \ldots, a_{n-1} \rangle , a_n \rangle$$

Exercise

1. Is $\langle 0, 0, 1 \rangle = \langle 0, 1, 1 \rangle$?
2. Is $\langle 0, 0, 0 \rangle = \langle 0, 0 \rangle$?

Cartesian Products, Relations

Definition

- $A \times B := \{\langle a, b\rangle \mid a \in A, b \in B\}$ [cartesian product]
- $A^n := \underbrace{A \times \cdots \times A}_{n}$ [cartesian power]
- A (binary) relation between A and B: a subset of $A \times B$
- A subset of $A_1 \times \ldots A_n$ is called n-ary relation
- A n-ary relation in A: a subset of A^n
- For any set A, let $\cdot_A := \{\langle a, a\rangle \mid a \in A\}$ [identity relation]

Exercise

1. What is the cardinality of $\{0, 1\}^2$?
2. How many relations there are on $\{0, 1\}$?
3. For what A and B, $\emptyset$ is a relation from A to B?
4. Are there any other relations one can define on any A except for $\emptyset$ and $\cdot_A$?

Composition of relations

Definition

Let $R \subseteq A \times B$, $S \subseteq B \times C$. Then

$$R/S := \{\langle a, c\rangle \mid \exists b \in B.\, \langle a, b\rangle \in R \text{ and } \langle b, c\rangle \in S\}$$

[composition]

Exercise

1. Let $R \subseteq A \times B$. What is $\cdot_A/R$? $R/\cdot_B$?
2. Is $(R/S)/T = R/(S/T)$?
3. What is $R/\emptyset$?

Images, Domain and Range, Functionality

Definition

Let $R \subseteq X \times Y$, $A \subseteq X$, $B \subseteq Y$.

- $R[A] := \{y \in Y \mid \exists a \in A. aRy\}$ [image]
- $R^{-1}[B] := \{x \in X \mid \exists b \in B. xRb\}$ [inverse image]
- $\mathrm{dom}(R) := \{x \mid R[\{x\}] \neq \emptyset\}$ [domain]
- $\mathrm{ran}(R) := \{y \mid R^{-1}[\{y\}] \neq \emptyset\}$ [range]
- R is functional if for any $x \in X$, $|R[\{x\}]| \leq 1$

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With every functional relation R from A to B we can associate a partial function f_R defined as $f_R(a) = b$ iff aRb.

(partial meaning possibly undefined for some elements of A)

In the same way, with every partial function f we can associate a partial function R_f the graph of f

$$\mathrm{dom}(f) := \mathrm{dom}(R_f)$$

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Functions

Definition

- A partial function f from A to B is called a [total] function from A to B iff $\mathrm{dom}(f) = A$
  We abbreviate it as $f : A \mapsto B$.
- Let $f : A \mapsto B$, $g : B \mapsto C$. Then $g \circ f : A \mapsto C$ is defined as [composition]

$$\forall a \in A. \qquad (g \circ f)(a) = g(f(a))$$

Exercise

1. Is $R_{g \circ f}$ composition of R_f and R_g?
2. Is $\cdot_A$ the graph of a total function on A?

In set theory, we usually identify a function with its graph

Indexed families

Definition

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Indexed families of sets are used to define infinitary operations

Unions, infinite unions

Unions, infinite unions

Definition

- Let A, B be sets. [finite union]

$$A \cup B := \{x \mid x \in A \text{ or } x \in B\}$$

- Let $\{A_i\}_{i \in I}$ be an indexed family of sets. [infinite union]

$$\bigcup_{i \in I} A_i := \{x \mid \exists i \in I. x \in A_i\}$$

Exercise

1. Is $R/(S \cup T) = (R/S) \cup (R/T)$?
2. Is $R/(\bigcup_{i \in I} T_i) = \bigcup_{i \in I}(R/T_i)$?
3. Is $A \times (B \cup C) = (A \times B) \cup (A \times C)$?

Transitive Relations, Chain

Transitive Relations, Chain

Definition

- $R \subseteq W \times W$ is transitive iff for every element $x, y, z \in W$, xRy and yRz implies xRz
- A transitive $R \subseteq W \times W$ is a chain iff for $x \neq y$ either xRy or yRx holds

Transitive Closure

Definition

Let R be a relation on A. Then we define the transitive closure of R as

$$R^{+} = \bigcup_{n \in \mathrm{N}_1} \underbrace{R/\ldots/R}_{n}$$

Reflexive-and-transitive closure of R is

$$R^{*} = \cdot_A \cup R^{+}$$

Fact

The transitive closure of R is the smallest transitive relation containing it

Exercise

1. Let $S = \{\langle n, n+1\rangle \mid n \in \mathrm{N}_0\}$. What is S^+?
2. What is S^*?
3. What is S/S^+?

Intersections, infinite intersections

Intersections, infinite intersections

Definition

- Let A, B be sets. [finite intersection]

$$A \cap B := \{x \mid x \in A \text{ and } x \in B\}$$

- Let $\{A_i\}_{i \in I}$ be an indexed family of sets. [infinite intersection]

$$\bigcap_{i \in I} A_i := \{x \mid \forall i \in I. x \in A_i\}$$

Exercise

1. Is $R/(S \cap T) = (R/S) \cap (R/T)$?
2. Is $A \times (B \cap C) = (A \times B) \cap (A \times C)$?
3. Is $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$?
4. Is $A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$?

Relational Structures

Relational Structures

Definition

- A relational signature Σ is any collection of relation symbols $\{R, S, T, R_1, S_1, \ldots\}$.
- The basic signature consists of a single relation symbol R
- A relational structure/Kripke frame for Σ is a pair $\mathfrak{F} = \langle W, \{S^{\mathfrak{F}}\}_{S \in \Sigma} \rangle$, where each $S^{\mathfrak{F}} \subseteq W \times W$. W is called the carrier of $\mathfrak{F}$ and denoted as $\underline{\mathfrak{F}}$.
- The class of all relational structures for Σ will be denoted as $\mathsf{Str}(\Sigma)$

Labelled Relational Structures

Labelled Relational Structures

Definition

- We fix a collection of labels $\Pi = \{P, P_1, P_2, \ldots\}$
- A Π-labelled relational structure is a pair $\mathfrak{W} = \langle \mathfrak{F}, \Lambda \rangle$, where
  - $\mathfrak{F} \in \mathsf{Str}(\Sigma)$ and
  - $\Lambda : \Pi \mapsto \mathcal{P}(\underline{\mathfrak{F}})$
- $\underline{\mathfrak{W}} := \underline{\mathfrak{F}}$

Well-founded relations

Definition (Well-foundedness)

A relation $R \subseteq W \times W$ is well-founded iff for every non-empty $S \subseteq W$ there is $s \in S$ s.t. for no $s' \in S$, $s' R s$.

Fact

- *A relation is well-founded iff its transitive closure is.*
- *A relation is well-founded iff it contains no infinite descending chains:* $\ldots x_3 R x_2 R x_1 R x_0$

Exercise

1. Is $\langle \mathrm{N}_0, \leq \rangle$ well-founded?
2. Is $\langle \mathrm{N}_0, < \rangle$ well-founded?
3. Is $\langle \mathrm{N}_0, \geq \rangle$ well-founded?
4. Is $\langle \mathrm{N}_0, > \rangle$ well-founded?

Trees

Definition

A relational structure $\langle W, R^{\mathfrak{W}} \rangle$ is a tree if

- for every $w \in W$, $(R^{\mathfrak{W}})^{-1}[\{w\}]$ is a well-ordered chain
- there is $r \in W$ s.t. $R^{\mathfrak{W}}[\{r\}] = W$. Such r is called a root

Exercise

1. Examples of trees and non-trees: on the whiteboard

Sibling-ordered Trees

Definition

A relational structure $\langle W, \downarrow^{\mathfrak{W}}, \rightarrow^{\mathfrak{W}} \rangle$ is a sibling-ordered tree if

- $\langle W, (\downarrow^{\mathfrak{W}})^{+} \rangle$ is a tree and
- $\rightarrow^{\mathfrak{W}}$ is the successor relation of some linear ordering between siblings